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# The fixed point alternative and Hyers-Ulam stability of generalized additive set-valued functional equations

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**Abstract**

We define generalized additive set-valued functional equations, which are related with the following generalized additive functional equations:
$f(x_1 + \cdots + x_l) = (l-1)f(\frac{x_1+\cdots+x_{l-1}}{l-1}) + f(x_l)$,

$$f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1} + x_l\right) + f\left(\frac{x_1 + \cdots + x_{l-2} + x_l}{l-1} + x_{l-1}\right) + \cdots + f\left(\frac{x_2 + \cdots + x_l}{l-1} + x_1\right) = 2\big[f(x_1) + f(x_2) + \cdots + f(x_l)\big]$$

for a fixed integer $l$ with $l > 1$, and they prove the Hyers-Ulam stability of the generalized additive set-valued functional equations by using the fixed point method.

**MSC:** Primary 39B52; 54C60; 91B44

**Keywords:** Hyers-Ulam stability; generalized additive set-valued functional equation; closed and convex set; cone

## 1 Introduction and preliminaries

After the pioneering papers were written by Aumann [1] and Debreu [2], set-valued functions in Banach spaces have been developed in the last decades. We can refer to the papers by Arrow and Debreu [3], McKenzie [4], the monographs by Hindenbrand [5], Aubin and Frankowska [6], Castaing and Valadier [7], Klein and Thompson [8] and the survey by Hess [9]. The theory of set-valued functions has been much related with the control theory and the mathematical economics.

Let $Y$ be a Banach space. We define the following:

$2^Y$: the set of all subsets of $Y$;

$C_b(Y)$: the set of all closed bounded subsets of $Y$;

$C_c(Y)$: the set of all closed convex subsets of $Y$;

$C_{cb}(Y)$: the set of all closed convex bounded subsets of $Y$;

$C_{cc}(Y)$: the set of all closed compact subsets of $Y$.

We can consider the addition and the scalar multiplication on $2^Y$ as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by

$$C \oplus C' = \overline{C + C'}.$$

We can easily check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C,$$

where $C, C' \in 2^Y$ and $\lambda, \mu \in \mathbb{R}$. Furthermore, when $C$ is convex, we obtain

$$(\lambda + \mu)C = \lambda C + \mu C$$

for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are, respectively, defined by

$$d(x, C) = \inf\{\|x - y\| : y \in C\}, \quad x \in Y,$$
$$s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}, \quad x^* \in Y^*.$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between $C$ and $C'$ by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, C' \subseteq C + \lambda B_Y\},$$

where $B_Y$ is the closed unit ball in $Y$.

The following proposition is related with some properties of the Hausdorff distance.

**Proposition 1.1** *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*:
(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;
(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance $h$. Since $Y$ is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [7]). Debreu [2] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

**Lemma 1.2** [2] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on $B_{Y^*}$ endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties*:
(a) $j(A \oplus B) = j(A) + j(B)$;
(b) $j(\lambda A) = \lambda j(A)$;
(c) $h(A, B) = \|j(A) - j(B)\|_u$;
(d) *$j(C_{cb}(Y))$ is closed in $C(B_{Y^*})$*
*for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.*

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space $(\Omega, \Sigma, \nu)$ into $C_{cb}(Y)$. Then $f$ is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [10]). In this case, the Debreu integral of $f$ in $\Omega$ is the unique element

$(D)\int_\Omega f\,d\nu \in C_{cb}(Y)$ such that $j((D)\int_\Omega f\,d\nu)$ is the Bochner integral of $j\circ f$. The set of Debreu integrable functions from $\Omega$ to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f+g)(\omega) = f(\omega)\oplus g(\omega)$ for all $f,g \in D(\Omega, C_{cb}(Y))$. Then we find that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

The stability problem of functional equations originated from a question of Ulam [11] concerning the stability of group homomorphisms. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [13] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [12, 14, 16–20]).

Let $X$ be a set. A function $d : X\times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Note that the distinction between the generalized metric and the usaul metric is that the range of the former includes the infinity.

Let $(X,D)$ be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant $L$ if there exists a constant $L \ge 0$ such that $d(Tx,Ty) \le Ld(x,y)$ for all $x,y\in X$. If the Lipschitz constant is less than 1, then the operator $T$ is called a strictly contractive operator. We recall a fundamental result in the fixed point theory.

**Theorem 1.3** [21, 22] *Let $(X,d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x\in X$, either*

$$d\bigl(J^n x, J^{n+1}x\bigr) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x) < \infty$, $\forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y,y^*) \le \frac{1}{1-L} d(y,Jy)$ *for all $y \in Y$.*

In 1996, Isac and Rassias started to use the fixed point theory for the proof of stability theory of functional equations. Afterwards the stability problems of several functional equations by using the fixed point methods have been extensively investigated by a number of authors [19, 20, 23].

Set-valued functional equations have been studied by a number of authors and there are many interesting results concerning this problem (see [24–31]). In this paper, we define generalized additive set-valued functional equations and prove the Hyers-Ulam stability of generalized additive set-valued functional equations by using the fixed point method.

Throughout this paper, let $X$ be a real vector space and $Y$ a Banach space.

## 2 Stability of a generalized additive set-valued functional equation

**Definition 2.1** Let $f : X \to C_{cb}(Y)$ be a set-valued function. The generalized additive set-valued functional equation is defined by

$$f(x_1 + \cdots + x_l) = (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l) \tag{2.1}$$

for all $x_1, \ldots, x_l \in X$. Every solution of the generalized additive set-valued functional equation is called a *generalized additive set-valued mapping*.

**Theorem 2.2** *Let $\varphi : X^l \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, \ldots, x_l) \le \frac{L}{l}\varphi(lx_1, \ldots, lx_l) \tag{2.2}$$

*for all $x_1, \ldots, x_l \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h\left(f(x_1 + \cdots + x_l), (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l)\right) \le \varphi(x_1, \ldots, x_l) \tag{2.3}$$

*for all $x_1, \ldots, x_l \in X$. Then there exists a unique generalized additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that*

$$h\big(f(x), A(x)\big) \le \frac{L}{l(1-L)}\varphi(x, \ldots, x) \tag{2.4}$$

*for all $x \in X$.*

*Proof* Let $x_1 = \cdots = x_l = x$ in (2.3). Since $f(x)$ is a convex set, we get

$$h\big(f(lx), lf(x)\big) \le \varphi(x, \ldots, x) \tag{2.5}$$

and if we replace $x$ by $\frac{x}{l}$ in (2.6), then we obtain

$$h\left(f(x), lf\left(\frac{x}{l}\right)\right) \le \varphi\left(\frac{x}{l}, \ldots, \frac{x}{l}\right) \le \frac{L}{l}\varphi(x, \ldots, x) \tag{2.6}$$

for all $x \in X$. Consider

$$S := \big\{g : g : X \to C_{cb}(Y), g(0) = \{0\}\big\}$$

and introduce the generalized metric on $X$,

$$d(g,f) = \inf\big\{\mu \in (0,\infty) : h\big(g(x), f(x)\big) \le \mu\varphi(x, \ldots, x), x \in X\big\},$$

where, as usual, $\inf\varphi = +\infty$. It is easy to show that $(S,d)$ is complete (see [23], Theorem 2.5). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := lg\left(\frac{x}{l}\right)$$

for all $x \in X$. Let $g,f \in S$ be given such $d(g,f) = \varepsilon$. Then

$$h\big(g(x),f(x)\big) \leq \varepsilon\varphi(x,\dots,x)$$

for all $x \in X$. Hence

$$h\big(Jg(x),Jf(x)\big) = h\left(lg\left(\frac{x}{l}\right),lf\left(\frac{x}{l}\right)\right) = lh\left(g\left(\frac{x}{l}\right),f\left(\frac{x}{l}\right)\right) \leq \varepsilon L\varphi(x,\dots,x)$$

for all $x \in X$. So $d(g,f) = \varepsilon$ implies the $d(Jg,Jf) \leq L\varepsilon$. This means that

$$d(Jg,Jf) \leq Ld(g,f)$$

for all $g,f \in S$. Furthermore we can have $d(f,Jf) \leq \frac{L}{l}$ from (2.6). By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, *i.e.*,

$$A\left(\frac{x}{l}\right) = \frac{1}{l}A(x) \tag{2.7}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \big\{g \in S : d(f,g) < \infty\big\}.$$

This implies that $A$ is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0,\infty)$ satisfying

$$h\big(f(x),A(x)\big) \leq \mu\varphi(x,\dots,x)$$

for all $x \in X$;

(2) $d(J^n f,A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} l^n f\left(\frac{x}{l^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leq \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,A) \leq \frac{L}{l-lL}.$$

This implies that the inequality (2.4) holds. By (2.3),

$$\begin{aligned} &h\left(l^n f\left(\frac{x_1}{l^n}+\frac{x_2}{l^n}+\cdots+\frac{x_l}{l^n}\right), l^n(l-1)f\left(\frac{x_1+x_2+\cdots+x_{l-1}}{l^n(l-1)}\right) \oplus l^n f\left(\frac{x_l}{l^n}\right)\right) \\ &\quad \leq l^n\varphi\left(\frac{x_1}{l^n},\frac{x_2}{l^n},\dots,\frac{x_l}{l^n}\right) \leq L^n\varphi(x_1,x_2,\dots,x_l), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x_1,x_2,\dots,x_l \in X$. Thus

$$A(x_1+x_2+\cdots+x_l) = (l-1)A(x_1+\cdots+x_{l-1}) \oplus A(x_l),$$

as desired. □

**Corollary 2.3** *Let $1 > p > 0$ and $\theta \geq 0$ be real numbers, and let $X$ be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h\left(f(x_1 + \cdots + x_l), (l-1)f\left(\frac{x_1 + \cdots + x_{l-1}}{l-1}\right) \oplus f(x_l)\right) \leq \theta \sum_{j=1}^{l} \|x_j\|^p \tag{2.8}$$

*for all $x_1, \ldots, x_l \in X$. Then there exists a unique generalized additive set-valued mapping $A : X \to Y$ satisfying*

$$h\bigl(f(x), A(x)\bigr) \leq \frac{l\theta}{l - l^p} \|x\|^p$$

*for all $x \in X$.*

*Proof* The proof follows from Theorem 2.2 by taking

$$\varphi(x_1, \ldots, x_l) := \theta \sum_{j=1}^{l} \|x_j\|^p$$

for all $x_1, \ldots, x_l \in X$. □

**Theorem 2.4** *Let $\varphi : X^l \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, \ldots, x_l) \leq lL\varphi\left(\frac{x_1}{l}, \ldots, \frac{x_l}{l}\right) \tag{2.9}$$

*for all $x_1, \ldots, x_l \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying* (2.3). *Then there exists a unique generalized additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that*

$$h\bigl(f(x), A(x)\bigr) \leq \frac{L}{1 - L} \varphi(x, \ldots, x)$$

*for all $x \in X$.*

*Proof* It follows from (2.5) that

$$h\left(\frac{1}{l} f(lx), f(x)\right) \leq L\varphi\left(\frac{x}{l}, \ldots, \frac{x}{l}\right) \tag{2.10}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5** *Let $p > 1$ and $\theta \geq 0$ be real numbers, and let $X$ be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying* (2.8). *Then there exists a unique generalized additive set-valued mapping $A : X \to Y$ satisfying*

$$h\bigl(f(x), A(x)\bigr) \leq \frac{l\theta}{l^p - l} \|x\|^p$$

*for all $x \in X$.*

*Proof* The proof follows from Theorem 2.4 by taking

$$\varphi(x_1,\dots,x_l) := \theta \sum_{j=1}^{l} \|x_j\|^p$$

for all $x_1,\dots,x_l \in X$. □

## 3 Stability of a generalized Cauchy-Jensen type additive set-valued functional equation

**Definition 3.1** Let $f : X \to C_{cb}(Y)$ be a set-valued function. The generalized Cauchy-Jensen type additive set-valued functional equation is defined by

$$\begin{aligned} &f\left(\frac{x_1+\cdots+x_{l-1}}{l-1}+x_l\right) \oplus f\left(\frac{x_1+\cdots+x_{l-2}+x_l}{l-1}+x_{l-1}\right) \oplus \cdots \oplus \\ &\quad f\left(\frac{x_2+\cdots+x_l}{l-1}+x_1\right) = 2\bigl[f(x_1)\oplus f(x_2)\oplus\cdots\oplus f(x_l)\bigr] \end{aligned} \tag{3.1}$$

for all $x_1,\dots,x_l \in X$. Every solution of the generalized Cauchy-Jensen type additive set-valued functional equation is called a *generalized Cauchy-Jensen type additive set-valued mapping*.

**Theorem 3.2** *Let $\phi : X^l \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1,x_2,\dots,x_l) \le \frac{L}{2}\varphi(2x_1,2x_2,\dots,2x_l)$$

*for all $x_1,x_2,\dots,x_l \in X$. Suppose that $f : X \to (C_{cb}(Y),h)$ is a mapping satisfying*

$$\begin{aligned} &h\biggl(f\left(\frac{x_1+\cdots+x_{l-1}}{l-1}+x_l\right) \oplus f\left(\frac{x_1+\cdots+x_{l-2}+x_l}{l-1}+x_{l-1}\right) \oplus \cdots \oplus \\ &\quad f\left(\frac{x_2+\cdots+x_l}{l-1}+x_1\right), 2\bigl[f(x_1)\oplus f(x_2)\oplus\cdots\oplus f(x_l)\bigr]\biggr) \le \varphi(x_1,x_2,\dots,x_l) \end{aligned} \tag{3.2}$$

*for all $x_1,x_2,\dots,x_l \in X$. Then*

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

*exists for each $x \in X$ and defines a unique generalized Cauchy-Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y),h)$ such that*

$$\begin{aligned} &A\left(\frac{x_1+\cdots+x_{l-1}}{l-1}+x_l\right) \oplus A\left(\frac{x_1+\cdots+x_{l-2}+x_l}{l-1}+x_{l-1}\right) \oplus \cdots \oplus \\ &\quad A\left(\frac{x_2+\cdots+x_l}{l-1}+x_1\right) = 2\bigl[A(x_1)\oplus A(x_2)\oplus\cdots\oplus A(x_l)\bigr] \end{aligned} \tag{3.3}$$

*and*

$$h\bigl(f(x),A(x)\bigr) \le \frac{L}{2l-2lL}\varphi(x,\dots,x) \tag{3.4}$$

*for all $x \in X$.*

*Proof* Let $x_1 = \cdots = x_l$ in (3.2). Since $f(x)$ is a convex set, we get

$$h\bigl(lf(2x), 2lf(x)\bigr) \le \varphi(x, \dots, x) \tag{3.5}$$

and so

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \le \frac{1}{l}\varphi\left(\frac{x}{2}, \dots, \frac{x}{2}\right) \le \frac{L}{2l}\varphi(x, \dots, x) \tag{3.6}$$

for all $x \in X$. Consider

$$S := \bigl\{g : g : X \to C_{cb}(Y), g(0) = \{0\}\bigr\}$$

and introduce the generalized metric on $X$,

$$d(g,f) = \inf\bigl\{\mu \in (0, \infty) : h\bigl(g(x), f(x)\bigr) \le \mu\varphi(x, \dots, x), x \in X\bigr\},$$

where, as usual, $\inf \varphi = +\infty$. Then $(S, d)$ is complete. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$. Let $g, f \in S$ be given such that $d(g,f) = \varepsilon$. Then

$$h\bigl(g(x), f(x)\bigr) \le \varepsilon\varphi(x, \dots, x)$$

for all $x \in X$. Hence

$$h\bigl(Jg(x), Jf(x)\bigr) = h\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \le L\varepsilon\varphi(x, \dots, x)$$

for all $x \in X$. So $d(g,f) = \varepsilon$ implies the $d(Jg, Jf) \le L\varepsilon$. This means that

$$d(Jg, Jf) \le Ld(g,f)$$

for all $g, f \in S$. It follows from (3.6) that $d(f, Jf) \le \frac{L}{2l}$. By Theorem 1.3, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, *i.e.*,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.7}$$

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \bigl\{g \in S : d(f, g) < \infty\bigr\}.$$

This implies that $A$ is a unique mapping satisfying (3.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$h\bigl(f(x), A(x)\bigr) \le \mu\varphi(x, \dots, x)$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2l - 2lL}.$$

This implies that the inequality (3.4) holds. By (3.3),

$$\begin{aligned} &h\Bigg(2^n f\left(\frac{x_1 + x_2 + \cdots + x_{l-1}}{2^n(l-1)} + \frac{x_l}{2^n}\right) \oplus 2^n f\left(\frac{x_1 + \cdots + x_{l-2} + x_l}{2^n(l-1)} + \frac{x_{l-1}}{2^n}\right) \oplus \cdots \oplus \\ &\quad 2^n f\left(\frac{x_2 + \cdots + x_l}{2^n(l-1)} + \frac{x_1}{2^n}\right), 2^{n+1}\left[f\left(\frac{x_1}{2^n}\right) \oplus f\left(\frac{x_2}{2^n}\right) \oplus \cdots \oplus f\left(\frac{x_l}{2^n}\right)\right]\Bigg) \\ &\quad \leq 2^n \varphi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_l}{2^n}\right), \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x_1, x_2, \ldots, x_l \in X$. Thus we can have

$$\begin{aligned} &A\left(\frac{x_1 + \cdots + x_{l-1}}{l-1} + x_l\right) \oplus A\left(\frac{x_1 + \cdots + x_{l-2} + x_l}{l-1} + x_{l-1}\right) \oplus \cdots \oplus \\ &\quad A\left(\frac{x_2 + \cdots + x_l}{l-1} + x_1\right) = 2\big[A(x_1) \oplus A(x_2) \oplus \cdots \oplus A(x_l)\big] \end{aligned} \tag{3.8}$$

as desired. □

**Corollary 3.3** *Let* $1 > p > 0$ *and* $\theta \geq 0$ *be real numbers, and let* $X$ *be a real normed space. Suppose that* $f : X \to (C_{cb}(Y), h)$ *is a mapping satisfying*

$$\begin{aligned} &h\Bigg(f\left(\frac{x_1 + x_2 + \cdots + x_{l-1}}{(l-1)} + x_l\right) \oplus f\left(\frac{x_1 + \cdots + x_{l-2} + x_l}{(l-1)} + x_{l-1}\right) \oplus \cdots \oplus \\ &\quad f\left(\frac{x_2 + \cdots + x_l}{(l-1)} + x_1\right), 2\big[f(x_1) \oplus f(x_2) \oplus \cdots \oplus f(x_l)\big]\Bigg) \leq \theta \sum_{j=1}^{l} \|x_j\|^p \end{aligned} \tag{3.9}$$

*for all* $x_1, \ldots, x_l \in X$. *Then there exists a unique generalized Cauchy-Jensen type additive set-valued mapping* $A : X \to Y$ *satisfying* (3.3) *and*

$$h\big(f(x), A(x)\big) \leq \frac{l\theta}{l - l^p} \|x\|^p$$

*for all* $x \in X$.

*Proof* The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, \ldots, x_l) := \theta \sum_{j=1}^{l} \|x_j\|^p$$

for all $x_1, \ldots, x_l \in X$. □

**Theorem 3.4** *Let $\varphi : X^l \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x_1, x_2, \dots, x_l) \leq 2L\varphi\left(\frac{x_1}{2}, \frac{x_2}{2}, \dots, \frac{x_l}{2}\right) \tag{3.10}$$

*for all $x_1, \dots, x_l \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying* (3.2). *Then there exists a unique generalized Cauchy-Jensen type additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ satisfying* (3.3) *and*

$$h\bigl(f(x), A(x)\bigr) \leq \frac{L}{l - lL}\varphi(x, \dots, x)$$

*for all $x \in X$.*

*Proof* It follows from (3.10) that

$$h\left(\frac{1}{2}f(2x), f(x)\right) \leq \frac{L}{l}\varphi\left(\frac{x}{2}, \dots, \frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Corollary 3.5** *Let $p > 1$ and $\theta \geq 0$ be real numbers and let $X$ be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying* (3.2). *Then there exists a unique generalized Cauchy-Jensen type additive set-valued mapping $A : X \to Y$ satisfying* (3.3)

$$h\bigl(f(x), A(x)\bigr) \leq \frac{l\theta}{l^p - l}\|x\|^p$$

*for all $x \in X$.*

*Proof* The proof follows from Theorem 3.4 by taking

$$\varphi(x_1, \dots, x_l) := \theta \sum_{j=1}^{l} \|x_j\|^p$$

for all $x_1, \dots, x_l \in X$. □

**Competing interests**
The author declares that they have no competing interests.

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