Gaboury and Tremblay *Advances in Difference Equations* 2014, 2014:126
http://www.advancesindifferenceequations.com/content/2014/1/126

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Remark on certain transformations for multiple hypergeometric functions

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Abstract
In this paper, we provide many new general transformations for multiple hypergeometric functions. These transformations can be viewed as generalizations of some of those obtained recently by Wei *et al.* (Adv. Differ. Equ. 2013:360, 2013). We obtain these transformations by using the fractional calculus method which is a more general method than the beta integral method.
MSC: 26A33; 33C20; 33C05

Keywords: fractional derivatives; Appell functions; Srivastava function; beta integral; multiple hypergeometric series

1 Introduction

The largely investigated generalized hypergeometric function ${}_pF_q$ with p numerator parameters $a_1, \ldots, a_p$ such that $a_j \in \mathbb{C}$ $(j = 1, \ldots, p)$ and q denominator parameters $b_1, \ldots, b_q$ such that $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(j = 1, \ldots, q;\ \mathbb{Z}_0^- := \mathbb{Z} \cup \{0\} = \{0, -1, -2, \ldots\})$ is defined by (see, for example [1, Chapter 4]; see also [2, pp.71-72])

$$
{}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix}\ z\right] = {}_pF_q[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \tag{1.1}
$$

$$
\bigl(p \leq q \text{ and } |z| < \infty; p = q + 1 \text{ and } |z| < 1; p = q + 1, |z| = 1 \text{ and } \operatorname{Re}(\omega) > 0\bigr),
$$

where

$$
\omega := \sum_{j=1}^{q} b_i - \sum_{j=1}^{p} a_i
$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$
(\alpha)_n := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + n - 1) & (n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ 1 & (n = 0; \alpha \in \mathbb{C} \setminus \{0\}). \end{cases}
$$

Multi-variable hypergeometric functions and their reduction formulas have also been largely investigated (for example, see [3]). Let us recall the general definition of the double hypergeometric function given by Srivastava and Panda [4, p.423, Eq. (26)]. Let (H_h) denotes the sequence of parameters $(H_1, H_2, \ldots, H_h)$, and let nonnegative integers define the

Pochhammer symbol $((H_h))_n = (H_1)_n(H_2)_n \cdots (H_h)_n$. Then the generalized version of the Kampé de Fériet function is defined as follows:

$$F^{h:a;b}_{g:c;d}\left[\begin{matrix}(H_h): & (A_a); & (B_b); \\ (G_g): & (C_c); & (D_d);\end{matrix}\, x, y\right] = \sum_{m,n\geq 0} \frac{((H_h))_{m+n}((A_a))_m((B_b))_n}{((G_g))_{m+n}((C_c))_m((D_d))_n} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{1.2}$$

For the numerous conditions of convergence for this function, the reader is referred to [4].

Some special cases of hypergeometric function of two variables are the Appell functions [3, 5–7] defined as

$$F_1[a;b_1,b_2;c;x,y] := \sum_{m,n\geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad \bigl(|x| < 1, |y| < 1\bigr), \tag{1.3}$$

$$F_2[a;b_1,b_2;c_1,c_2;x,y] := \sum_{m,n\geq 0} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad \bigl(|x| + |y| < 1\bigr), \tag{1.4}$$

$$F_3[a_1,a_2;b_1,b_2;c;x,y] := \sum_{m,n\geq 0} \frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!} \quad \bigl(|x| < 1, |y| < 1\bigr), \tag{1.5}$$

$$F_4[a;b;c_1,c_2;x,y] := \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \quad \bigl(|x|^{1/2} + |y|^{1/2} < 1\bigr). \tag{1.6}$$

Other interesting special cases of hypergeometric functions of two variables are Horn's functions G_1 and G_2 studied in [6, 8] and defined as follows:

$$G_1(\alpha;\beta_1,\beta_2;x,y) = \sum_{m,n\geq 0} \frac{(\alpha)_{m+n}(\beta_1)_{n-m}(\beta_2)_{m-n}}{m!n!} x^m y^n \quad \bigl(|x| + |y| < 1\bigr), \tag{1.7}$$

$$G_2(\alpha_1,\alpha_2;\beta_1,\beta_2;x,y) = \sum_{m,n\geq 0} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_{n-m}(\beta_2)_{m-n}}{m!n!} x^m y^n \quad \bigl(|x|, |y| < 1\bigr). \tag{1.8}$$

For the purpose of this work, we need to introduce Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$ [3, p.44] defined by

$$\begin{aligned} F^{(3)}[x,y,z] &= F^{(3)}\left[\begin{matrix}(a) :: & (b);(b');(b'') : & (c);(c');(c''); \\ (e) :: & (g);(g');(g'') : & (h);(h');(h'');\end{matrix}\, x,y,z\right] \\ &= \sum_{m,n,p\geq 0} \Lambda(m,n,p) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \end{aligned} \tag{1.9}$$

where, for convenience,

$$\begin{aligned} \Lambda(m,n,p) &= \frac{\prod_{j=1}^{A}(a_j)_{m+n+p} \prod_{j=1}^{B}(b_j)_{m+n} \prod_{j=1}^{B'}(b'_j)_{n+p} \prod_{j=1}^{B''}(b''_j)_{m+p}}{\prod_{j=1}^{E}(e_j)_{m+n+p} \prod_{j=1}^{G}(g_j)_{m+n} \prod_{j=1}^{G'}(g'_j)_{n+p} \prod_{j=1}^{G''}(g''_j)_{m+p}} \\ &\quad \cdot \frac{\prod_{j=1}^{C}(c_j)_m \prod_{j=1}^{C'}(c'_j)_n \prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m \prod_{j=1}^{H'}(h'_j)_n \prod_{j=1}^{H''}(h''_j)_p}, \end{aligned} \tag{1.10}$$

and (a) abbreviates the array of A parameters $a_1, \ldots, a_A$ with similar interpretations for (b), (b'), (b''), and so on.

Finally, we also require two special cases of hypergeometric function of three variables given by Srivastava [9–11]:

$$H_A(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x,y,z)=\sum_{m,n,p\geq 0}\frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_{n+p}m!n!p!}x^m y^n z^p \tag{1.11}$$

$$\bigl(|x|=r<1,|y|=s<1,|z|=t<(1-r)(1-s)\bigr),$$

$$H_B(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)=\sum_{m,n,p\geq 0}\frac{(\alpha)_{m+p}(\beta_1)_{m+n}(\beta_2)_{n+p}}{(\gamma_1)_m(\gamma_2)_n(\gamma_3)_p m!n!p!}x^m y^n z^p \tag{1.12}$$

$$\bigl(|x|=r,|y|=s,|z|=t;r+s+t+2\sqrt{rst}<1\bigr).$$

Recently, many authors [12–14] obtained several transformations formulas involving hypergeometric functions as well as their multi-variable analogs by using the so-called beta integral method. The beta function $B(\alpha,\beta)$ is defined by the following integral representation:

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\quad\bigl(\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0\bigr). \tag{1.13}$$

The so-called beta integral method consists essentially of integral from 0 to 1 expressions which contain terms in the form $z^a(1-z)^b$ to obtain new transformations formulas.

The aim of this paper is to present many new general transformations for multiple hypergeometric functions. These transformations can be viewed as generalizations of some of those obtained recently by Wei *et al.* [14]. All these transformations are obtained by using a fractional calculus operator based on the Pochhammer contour integral. In Section 2, we give the representation of the fractional derivatives based on the Pochhammer contour of integration. Section 3 is devoted to the fractional calculus operator ${}_zO_\beta^\alpha$ introduced by Tremblay [15]. Finally, in Section 4, we present the several transformations involving multi-variable hypergeometric functions.

2 Pochhammer contour integral representation for fractional derivative and a new generalized Leibniz rule

The use of a contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [16–18], that is,

$$D_z^\alpha z^p f(z)=\frac{1}{\Gamma(-\alpha)}\int_0^z f(\xi)\xi^p(\xi-z)^{-\alpha-1}\,d\xi, \tag{2.1}$$

which is valid for $\mathrm{Re}(\alpha)<0$, $\mathrm{Re}(p)>1$ and where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by parts m times, we obtain

$$D_z^\alpha z^p f(z)=\frac{d^m}{dz^m}D_z^{\alpha-m}z^p f(z). \tag{2.2}$$

This allows one to modify the restriction $\mathrm{Re}(\alpha)<0$ to $\mathrm{Re}(\alpha)<m$ [18]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula

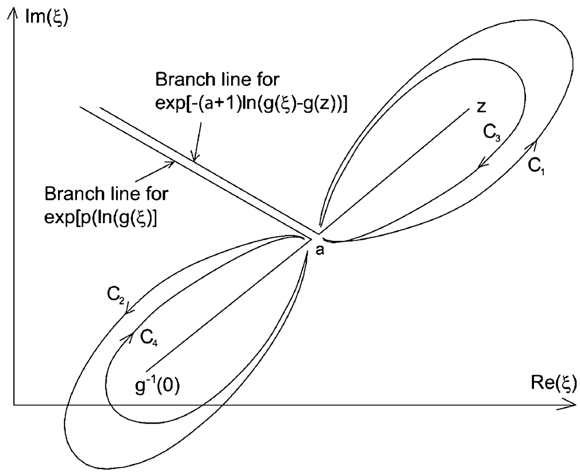

Figure 1 Pochhammer's contour.

widely used by Osler [19–22]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is the Pochhammer contour definition introduced in [15, 23] (see also [24–28]).

Definition 2.1 Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Then if α is not a negative integer, p is not an integer, and z is in $\mathcal{R} - \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$D^{\alpha}_{g(z)} g(z)^p f(z) = \frac{\mathrm{e}^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)g(\xi)^p g'(\xi)}{(g(\xi)-g(z))^{\alpha+1}}\, d\xi. \tag{2.3}$$

For non-integer α and p, the functions $g(\xi)^p$ and $(g(\xi)-g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi = z$ and $\xi = g^{-1}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1. $F(a)$ denotes the principal value of the integrand in (2.3) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2.2 In Definition 2.1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However, it is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and Equation (2.3) would still be valid.

Remark 2.3 The Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (2.3), then we know that the integral is analytic for all p and for all α and for z in $\mathcal{R} - \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D^{\alpha}_{g(z)} g(z)^p f(z)$ are $\alpha = -1, -2, \ldots$, and $p = 0, \pm 1, \pm 2, \ldots$ which can directly be identified from the coefficient of the integral (2.3). However, integrating by parts N times the integral in (2.3) by two different ways, we can show that $\alpha = -1, -2, \ldots$, and $p = 0, 1, 2, \ldots$ are removable singularities (see [23]).

It is well known that [29, p.83, Equation (2.4)]

$$D_z^\alpha z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \quad \bigl(\mathrm{Re}(p) > -1\bigr), \tag{2.4}$$

but adopting the Pochhammer-based representation for the fractional derivative this last restriction becomes p not a negative integer.

3 The well poised fractional calculus operator ${}_zO_\beta^\alpha$

In this section, we recall some of the important properties of the fractional calculus operator ${}_zO_\beta^\alpha$ introduced by Tremblay [15] as

$${}_zO_\beta^\alpha := \frac{\Gamma(\beta)}{\Gamma(\alpha)} z^{1-\beta} D_z^{\alpha-\beta} z^{\alpha-1} \quad (\beta \text{ not a negative integer}). \tag{3.1}$$

We choose to simply list them since the proofs are readily obtainable.

(1) Linearity

$${}_zO_\beta^\alpha \bigl\{\lambda_1 f(z) + \lambda_2 g(z)\bigr\} = \lambda_{1z} O_\beta^\alpha f(z) + \lambda_{2z} O_\beta^\alpha g(z). \tag{3.2}$$

(2) Identity

$${}_zO_\alpha^\alpha = I. \tag{3.3}$$

(3) Reductions

$${}_zO_\beta^\alpha {}_zO_\gamma^\beta = {}_zO_\gamma^\alpha, \tag{3.4}$$

$${}_zO_\beta^\alpha {}_zO_\alpha^\gamma = {}_zO_\beta^\gamma. \tag{3.5}$$

(4) Elementary cases

$${}_zO_\beta^\alpha 1 = 1, \tag{3.6}$$

$${}_zO_\beta^\alpha z^n = \frac{(\alpha)_n}{(\beta)_n} z^n. \tag{3.7}$$

(5) Useful cases

$${}_zO_\beta^\alpha z^\lambda f(z) = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)} z^\lambda {}_zO_{\beta+\lambda}^{\alpha+\lambda} f(z), \tag{3.8}$$

$${}_zO_\beta^\alpha (w-z)^\theta f(z)|_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)} z^\theta {}_zO_{\beta+\theta}^\alpha f(z), \tag{3.9}$$

$${}_zO_\beta^\alpha z^\lambda (w-z)^\theta f(z)|_{w=z} = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)\Gamma(\beta-\alpha+\theta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)\Gamma(\beta+\theta+\lambda)} z^{\theta+\lambda} {}_zO_{\beta+\lambda+\theta}^{\alpha+\lambda} f(z). \tag{3.10}$$

It is worthy to mention that operator ${}_zO_\beta^\alpha$ has a lot more interesting properties and applications. Tremblay introduced this operator in order to deal with special functions more efficiently and to facilitate the obtention of new relations such as hypergeometric transformations.

For this work, the most important property of the operator ${}_zO_\beta^\alpha$ is given by the following relation:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}\, {}_zO_\beta^{\alpha+\beta} z^\gamma \Big|_{z=1}. \tag{3.11}$$

This relation shows, in fact, that the so-called beta integral method consists in a fractional derivative evaluated at the point $z = 1$.

4 Main results

In this section, we apply the fractional calculus operator ${}_zO_\beta^\alpha$ to certain transformations involving multi-variable hypergeometric functions in order to obtain new transformations more general than those obtained by means of the beta integral method. Many special cases are also computed.

Theorem 4.1 *Let b_1 and b_2 be two nonpositive integers or α be a nonpositive integer and let $c, \beta \neq 0, -1, -2, \ldots$. Then the following transformation*

$$\begin{aligned} &F_{2:0;0}^{1:2;2}\left[\begin{matrix} a: & b_1,\alpha; & b_2,\beta-\alpha; \\ c,\beta: & -; & -; \end{matrix}\; z, z\right] \\ &\quad = \sum_{m,n,k,j\geq 0} \frac{(c-a)_{m+n}(b_1)_m(b_2)_n(b_1+m)_k(b_2+n)_j(\alpha)_{m+k}(\beta-\alpha)_{n+j}}{(c)_{m+n}(\beta)_{m+n+k+j}} \frac{(-z)^m}{m!}\frac{(-z)^n}{n!}\frac{z^k}{k!}\frac{z^j}{j!} \end{aligned} \tag{4.1}$$

holds true.

Proof We start from the following transformation of Appell function F_1 [7, p.217, Eq. (8.3.2)]:

$$F_1[a;b_1,b_2;c;x,y] = (1-x)^{-b_1}(1-y)^{-b_2}F_1\left[c-a;b_1,b_2;c;\frac{x}{x-1},\frac{y}{y-1}\right]. \tag{4.2}$$

By making the substitutions $x \mapsto z$ and $y \mapsto w - z$ in (3.3), we obtain

$$\begin{aligned} &F_1[a;b_1,b_2;c;z,w-z] \\ &\quad = (1-z)^{-b_1}(1-w+z)^{-b_2}F_1\left[c-a;b_1,b_2;c;\frac{z}{z-1},\frac{w-z}{w-z-1}\right]. \end{aligned} \tag{4.3}$$

Next, we apply the fractional calculus operator ${}_zO_\beta^\alpha$ on both sides of (4.3) with $w = z$ after operation. We thus have for the l.h.s.:

$$\begin{aligned} {}_zO_\beta^\alpha F_1[a;b_1,b_2;c;z,w-z]|_{w=z} &= \sum_{m,n\geq 0}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}\, {}_zO_\beta^\alpha z^m(w-z)^n\Big|_{w=z} \\ &= \sum_{m,n\geq 0}\frac{(a)_{m+n}(b_1)_m(\alpha)_m(b_2)_n(\beta-\alpha)_n}{(c)_{m+n}(\beta)_{m+n}}\frac{z^m}{m!}\frac{z^n}{n!}. \end{aligned} \tag{4.4}$$

We obtain for the r.h.s.:

$$\begin{aligned}
&{}_zO^{\alpha}_{\beta}(1-z)^{-b_1}(1-w+z)^{-b_2}F_1\left[c-a;b_1,b_2;c;\frac{z}{z-1},\frac{w-z}{w-z-1}\right]\Big|_{w=z}\\
&\quad=\sum_{m,n\geq 0}\frac{(c-a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}(-1)^{m+n}{}_zO^{\alpha}_{\beta}z^m(1-z)^{-b_1-m}(w-z)^n(1-w+z)^{-b_2-n}\Big|_{w=z}\\
&\quad=\sum_{m,n,k,j\geq 0}\frac{(c-a)_{m+n}(b_1)_m(b_2)_n(b_1+m)_k(b_2+n)_j}{(c)_{m+n}m!n!k!j!}(-1)^{m+n}{}_zO^{\alpha}_{\beta}z^{m+k}(w-z)^{n+j}\Big|_{w=z}\\
&\quad=\sum_{m,n,k,j\geq 0}\frac{(c-a)_{m+n}(b_1)_m(b_2)_n(b_1+m)_k(b_2+n)_j(\alpha)_{m+k}(\beta-\alpha)_{n+j}}{(c)_{m+n}(\beta)_{m+n+k+j}}\frac{(-z)^m}{m!}\frac{(-z)^n}{n!}\frac{z^k}{k!}\frac{z^j}{j!}.
\end{aligned} \tag{4.5}$$

This completes the proof. □

Let us give a special case of Theorem 4.1 in which we recover a result given recently by Wei *et al.* [14, Theorem 1].

Corollary 4.2 *Let b_1 and b_2 be two nonpositive integers or α be a nonpositive integer and let $c,\beta\neq 0,-1,-2,\ldots$. Then the following summation formula*:

$$\begin{aligned}
&F^{1:2;2}_{2:0;0}\left[\begin{matrix}a: & b_1,e+b_2; & b_2,d+b_1-e; \\ c,d+b_1+b_2: & -; & -;\end{matrix}\ 1,1\right]\\
&\quad=\frac{\Gamma(e)\Gamma(d-e)\Gamma(d+b_1+b_2)}{\Gamma(d)\Gamma(e+b_2)\Gamma(d+b_1-e)}\sum_{m,n\geq 0}\frac{(c-a)_{m+n}(b_1)_m(b_2)_n(e)_{m-n}}{(c)_{m+n}(1+e-d)_{m-n}m!n!}
\end{aligned} \tag{4.6}$$

holds true.

Proof Setting $z=1$, $\alpha=e+b_2$ and $\beta=d+b_1+b_2$ in Theorem 4.1 and using twice the Gauss summation formula [1]

$${}_2F_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\ 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad\bigl(\mathrm{Re}(c-a-b)>0\bigr) \tag{4.7}$$

gives the result. □

Theorem 4.3 *Let β, c and $1+a+b-c\neq 0,-1,-2,\ldots$, and let $\mathrm{Re}(\beta-\alpha)>0$. Then the following transformation*:

$$\begin{aligned}
&F^{1:2;2}_{1:1;1}\left[\begin{matrix}\alpha: & a,b; & a,b; \\ \beta: & c; & 1+a+b-c;\end{matrix}\ z,z\right]\\
&\quad=\sum_{m,n,k\geq 0}\frac{(\alpha)_{m+n+k}(a)_{m+n}(b)_{m+n}(-m-n)_k}{(\beta)_{m+n+k}(c)_m(1+a+b-c)_n}\frac{z^{m+n+k}}{m!n!k!}
\end{aligned} \tag{4.8}$$

holds true.

Proof Beginning with the following transformation formula [30, Eq. (8)] with $x = y = z$:

$$F_4\bigl[a;b;c,1+a+b-c;z(1-z),z(1-z)\bigr] = {}_2F_1\left[\begin{matrix} a,b; \\ c; \end{matrix}\, z\right] {}_2F_1\left[\begin{matrix} a,b; \\ 1+a+b-c; \end{matrix}\, z\right] \tag{4.9}$$

and applying the operator ${}_zO_\beta^\alpha$ on both sides of (4.9), we get for the l.h.s.

$$\begin{aligned} &{}_zO_\beta^\alpha \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(1+a+b-c)_n} \frac{z^{m+n}}{m!} \frac{(1-z)^{m+n}}{n!} \\ &\quad = {}_zO_\beta^\alpha \sum_{m,n,k\geq 0} \frac{(a)_{m+n}(b)_{m+n}(-m-n)_k}{(c)_m(1+a+b-c)_n} \frac{z^{m+n+k}}{m!n!k!} \\ &\quad = \sum_{m,n,k\geq 0} \frac{(a)_{m+n}(b)_{m+n}(-m-n)_k(\alpha)_{m+n+k}}{(c)_m(1+a+b-c)_n(\alpha)_{m+n+k}} \frac{z^{m+n+k}}{m!n!k!} \end{aligned} \tag{4.10}$$

and for the r.h.s.

$$\begin{aligned} &{}_zO_\beta^\alpha \sum_{m,n\geq 0} \frac{(a)_m(b)_m(a)_n(b)_n}{(c)_m(1+a+b-c)_n} \frac{z^{m+n}}{m!n!} \\ &\quad = \sum_{m,n\geq 0} \frac{(a)_m(b)_m(a)_n(b)_n}{(c)_m(1+a+b-c)_n} \frac{(\alpha)_{m+n}}{(\beta)_{m+n}} \frac{z^{m+n}}{m!n!}. \end{aligned} \tag{4.11}$$

Rewriting (4.11) into the form of (1.2) leads to the desired result. □

Corollary 4.4 *Let* β, c *and* $1+a+b-c \neq 0,-1,-2,\dots$. *Then the following formula*:

$$\begin{aligned} &F_{1:1;1}^{1:2;2}\left[\begin{matrix} \alpha: & a,b; & a,b; \\ \beta: & c; & 1+a+b-c; \end{matrix}\, 1,1\right] \\ &\quad = F_{2:1;1}^{4:0;0}\left[\begin{matrix} \alpha,\beta-\alpha,a,b: & -; & -; \\ \frac{\beta}{2},\frac{\beta+1}{2}: & c; & 1+a+b-c; \end{matrix}\, \frac{1}{4},\frac{1}{4}\right] \end{aligned} \tag{4.12}$$

holds true.

Proof Putting $z = 1$ in Theorem 4.3, using the Gauss summation formula (4.7) and making elementary simplifications yields the result. □

This special case of Theorem 4.3 corresponds to a result also given by Wei *et al.* [14, Eq. (2.4)].

Corollary 4.5 *Let* $\frac{1+\beta}{2}$, β, c *and* $1+a+b-c \neq 0,-1,-2,\dots$. *Then the following formula*:

$$\begin{aligned} &F_{1:1;1}^{1:2;2}\left[\begin{matrix} 1: & a,b; & a,b; \\ \beta: & c; & 1+a+b-c; \end{matrix}\, \frac{1}{2},\frac{1}{2}\right] \\ &\quad = \frac{\Gamma(\beta)2^{1-\beta}\sqrt{\pi}}{\Gamma(\frac{\beta}{2})\Gamma(\frac{1+\beta}{2})} F_{1:1;1}^{3:0;0}\left[\begin{matrix} a,b,1: & -; & -; \\ \frac{1+\beta}{2}: & c; & 1+a+b-c; \end{matrix}\, \frac{1}{4},\frac{1}{4}\right] \end{aligned} \tag{4.13}$$

holds true.

Proof Letting $z = \frac{1}{2}$ and $\alpha = 1$ in Theorem 4.3 gives

$$\begin{aligned} &F_{1:1;1}^{1:2;2}\left[\begin{matrix} 1: & a,b; & a,b; \\ \beta: & c; & 1+a+b-c; \end{matrix} \frac{1}{2},\frac{1}{2}\right] \\ &\quad = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_{m+n}(1)_{m+n}}{(c)_m(1+a+b-c)_n(\beta)_{m+n}} \frac{(\frac{1}{2})^{m+n}}{m!n!} {}_2F_1\left[\begin{matrix} 1+m+n,-m-n; \\ \beta+m+n; \end{matrix} \frac{1}{2}\right]. \end{aligned} \tag{4.14}$$

With the help of the well-known Bailey summation theorem [31]:

$${}_2F_1\left[\begin{matrix} a,1-a; \\ c; \end{matrix} \frac{1}{2}\right] = \frac{2^{1-c}\Gamma(c)\sqrt{\pi}}{\Gamma(\frac{a+c}{2})\Gamma(\frac{c-a+1}{2})} \tag{4.15}$$

the result follows easily after simple calculations. □

Theorem 4.6 *Let* β, c, λ *and* $1+a+b-c \neq 0,-1,-2,\ldots$. *Then the following transformation*:

$$\begin{aligned} &F_{0:2;2}^{0:3;3}\left[\begin{matrix} -: & \alpha,a,b; & \gamma,a,b; \\ -: & \beta,c; & \lambda,1+a+b-c; \end{matrix} x,y\right] \\ &\quad = \sum_{m,n\geq 0} \frac{(a)_{m+n}(b)_{m+n}(\alpha)_m(\gamma)_n x^m y^n {}_2F_1[{}^{-n,\alpha+m;}_{\beta+m;} x]{}_2F_1[{}^{-m,\gamma+n;}_{\lambda+n;} y]}{(c)_m(\beta)_m(1+a+b-c)_n(\lambda)_n m!n!} \end{aligned} \tag{4.16}$$

holds true.

Proof Considering the transformation formula [30, Eq. (8)]

$$F_4\bigl[a;b;c,1+a+b-c;x(1-y),y(1-x)\bigr] = {}_2F_1\left[\begin{matrix} a,b; \\ c; \end{matrix} x\right]{}_2F_1\left[\begin{matrix} a,b; \\ 1+a+b-c; \end{matrix} y\right] \tag{4.17}$$

and applying successively the operator ${}_xO_\beta^\alpha$ and the operator ${}_yO_\lambda^\gamma$ on both sides of (4.17) gives the result. □

Setting $x = y = 1$ in Theorem 4.6 and using twice the Gauss summation formula (4.7) leads to a result given by Wei *et al.* [14, p.5], that is,

Corollary 4.7 *Let* β, c, λ *and* $1+a+b-c \neq 0,-1,-2,\ldots$, $\mathrm{Re}(\beta-\alpha) > 0$ *and* $\mathrm{Re}(\lambda-\gamma) > 0$. *Then the following transformation*:

$$\begin{aligned} &F_{0:2;2}^{0:3;3}\left[\begin{matrix} -: & \alpha,a,b; & \gamma,a,b; \\ -: & \beta,c; & \lambda,1+a+b-c; \end{matrix} 1,1\right] \\ &\quad = F_{2:1;1}^{2:2;2}\left[\begin{matrix} a,b: & \alpha,\lambda-\gamma; & \beta-\alpha,\gamma; \\ \beta,\lambda: & c; & 1+a+b-c; \end{matrix} 1,1\right] \end{aligned} \tag{4.18}$$

holds true.

Theorem 4.8 *The following transformation*:

$$F^{(3)}\left[\begin{matrix} a,\alpha_1+\alpha_2 & :: & 1-\beta_1-\beta_2;-;-: & \alpha_1;\alpha_2;-; \\ b & :: & \alpha_1+\alpha_2;-;-: & 1-\beta_1;1-\beta_2;-; \end{matrix} -x,-x,x\right]$$
$$=\sum_{m,n\geq 0}\frac{(a)_{m+n}(\alpha_1)_m(\alpha_2)_n(\beta_1)_{n-m}(\beta_2)_{m-n}}{(b)_{m+n}}\frac{(-x)^{m+n}}{m!n!} \tag{4.19}$$

holds true.

Proof We start from the following transformation formula between the Appell function F_2 and the Horn function G_2 [32, Eq. (5.6)] with $x \mapsto -x$ and $y \mapsto -x$:

$$G_2(\alpha_1,\alpha_2;\beta_1,\beta_2;-x,-x)$$
$$=(1-x)^{-\alpha_1-\alpha_2}F_2\left[1-\beta_1-\beta_2;\alpha_1,\alpha_2;1-\beta_1,1-\beta_2;\frac{-x}{1-x},\frac{-x}{1-x}\right]. \tag{4.20}$$

Applying the operator ${}_xO_b^a$ on both sides of (4.20) in a similar way as in the proofs of the previous theorems gives the result. □

If we set $x=1$ in Theorem 4.8, we obtain the following corollary which has been given by Wei *et al.* [14, p.8].

Corollary 4.9 *Let e be a nonpositive integer. Then the following transformation*:

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix} e,1-\beta_1-\beta_2: & \alpha_1; & \alpha_2; \\ 1+e-d: & 1-\beta_1; & 1-\beta_2; \end{matrix} 1,1\right]$$
$$=\frac{\Gamma(d)\Gamma(\alpha_1+\alpha_2+d-e)}{\Gamma(d-e)\Gamma(\alpha_1+\alpha_2+d)}\sum_{m,n\geq 0}\frac{(e)_{m+n}(\alpha_1)_m(\alpha_2)_n(\beta_1)_{n-m}(\beta_2)_{m-n}}{(\alpha_1+\alpha_2+d)_{m+n}m!n!} \tag{4.21}$$

holds true.

Proof Making the following substitutions: $a=e$, $b=\alpha_1+\alpha_2+d$, (4.20) can be written in the form

$$\sum_{m,n\geq 0}\frac{(1-\beta_1-\beta_2)_{m+n}(e)_{m+n}(\alpha_1)_m(\alpha_2)_n(-1)^{m+n}}{(\alpha_1+\alpha_2+d)_{m+n}(1-\beta_1)_m(1-\beta_2)_n m!n!}{}_2F_1\left[\begin{matrix}\alpha_1+\alpha_2+m+n,e+m+n; \\ \alpha_1+\alpha_2+d+m+n;\end{matrix} 1\right]$$
$$=\sum_{m,n\geq 0}\frac{(e)_{m+n}(\alpha_1)_m(\alpha_2)_n(\beta_1)_{n-m}(\beta_2)_{m-n}}{(\alpha_1+\alpha_2+d)_{m+n}}\frac{(-1)^{m+n}}{m!n!}. \tag{4.22}$$

Summing the hypergeometric function ${}_2F_1$ in the left member of (4.22) with the help of the Gauss summation formula (4.7) gives the result. □

Note that this result has been given recently by Wei *et al.* [14, p.8].

Let us complete this paper by giving one last transformation.

Theorem 4.10 *The following transformation*:

$$F^{(3)}\left[\begin{matrix} a & :: & \beta_1;-;\alpha: & -;-;-; \\ b & :: & -;-;-: & \gamma_1;-;-; \end{matrix} x,x,x\right]$$
$$= \sum_{m,n\geq 0} \frac{(a)_{m+n}(\alpha+\beta_1+n)_{2m}(\alpha)_m(\beta_1)_m(\alpha+\beta_1)_n}{(b)_{m+n}(\alpha+\beta_1)_{2m}(\gamma_1)_m} \frac{x^{m+n}}{m!n!} \tag{4.23}$$

holds true.

Proof From the following identity between the triple hypergeometric function H_A and the hypergeometric function ${}_2F_1$ [10, p.103, Eq. (5.3)] with $y = z = x$:

$$H_A(\alpha,\beta_1,\beta_2;\gamma_1,\beta_2;x,x,x) = (1-x)^{-\alpha-\beta_1} {}_2F_1\left[\begin{matrix} \alpha,\beta_1; \\ \gamma_1; \end{matrix} \frac{x}{(1-x)^2}\right], \tag{4.24}$$

if we apply the operator ${}_xO_\beta^\alpha$ on both sides (4.24), the result follows easily after simple calculations. □

Corollary 4.11 *Let* α *and* β_1 *be two nonpositive integers or* a *be a nonpositive integer. Then the following transformation*:

$$F^{(3)}\left[\begin{matrix} a & :: & \beta_1;-;\alpha: & -;-;-; \\ b & :: & -;-;-: & \gamma_1;-;-; \end{matrix} 1,1,1\right]$$
$$= \frac{\Gamma(b)\Gamma(b-a-\alpha-\beta_1)}{\Gamma(b-a)\Gamma(b-\alpha-\beta_1)} {}_4F_3\left[\begin{matrix} a,\alpha,\beta_1,1-b+\alpha+\beta_1; \\ \gamma_1, \frac{1-b+a+\alpha+\beta_1}{2}, \frac{2-b+a+\alpha+\beta_1}{2}; \end{matrix} \frac{-1}{4}\right] \tag{4.25}$$

holds true.

Proof Putting $x = 1$ in Theorem 4.10, we have, after simple manipulations,

$$F^{(3)}\left[\begin{matrix} a & :: & \beta_1;-;\alpha: & -;-;-; \\ b & :: & -;-;-: & \gamma_1;-;-; \end{matrix} 1,1,1\right]$$
$$= \sum_{m\geq 0} \frac{(a)_m(\alpha)_m(\beta_1)_m}{(b)_m(\gamma_1)_m m!} {}_2F_1\left[\begin{matrix} a+m,\alpha+\beta_1+2m; \\ b+m; \end{matrix} 1\right]. \tag{4.26}$$

Using the Gauss summation theorem (4.7), the result follows easily. □

The previous corollary has been given by Wei *et al.* [14, p.11].

It is important to mention here that the fractional calculus operator ${}_zO_\beta^\alpha$ used in this paper can provide many very general transformation formulas involving hypergeometric functions of several variables. Tremblay [15] obtained many new transformation formulas with the help of this fractional calculus operator. A paper dealing with these new relations is in preparation.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper together. Both authors read and approved the final manuscript.

Acknowledgements

The authors wish to thank the referees for valuable suggestions and comments.

Received: 14 January 2014 Accepted: 10 April 2014 Published: 06 May 2014

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10.1186/1687-1847-2014-126
Cite this article as: Gaboury and Tremblay: **Remark on certain transformations for multiple hypergeometric functions.** *Advances in Difference Equations* 2014, 2014:126