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Homoclinic solutions for a class of neutral Duffing differential systems

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Abstract

By using an extension of Mawhin's continuation theorem and some analysis methods, the existence of a set with $2kT$-periodic for a n-dimensional neutral Duffing differential systems, $(u(t) - Cu(t-\tau))'' + \beta(t)x'(t) + g(u(t-\gamma(t))) = p(t)$, is studied. Some new results on the existence of homoclinic solutions is obtained as a limit of a certain subsequence of the above set. Meanwhile, $C = [c_{ij}]_{n\times n}$ is a constant symmetrical matrix and $\beta(t)$ is allowed to change sign.

Keywords: homoclinic solution; continuation theorem; periodic solution

1 Introduction

The aim of this paper is to consider a kind of neutral Duffing differential systems as follows:

$$\big(u(t) - Cu(t-\tau)\big)'' + \beta(t)x'(t) + g\big(u\big(t-\gamma(t)\big)\big) = p(t), \tag{1.1}$$

where $\beta \in C^1(R,R)$ with $\beta(t+T) \equiv \beta(t)$, $g \in C(R^n,R^n)$, $p \in C(R,R^n)$, and $\gamma(t)$ is a continuous T-periodic function with $\gamma(t) \ge 0$; $T > 0$ and τ are given constants; $C = [c_{ij}]_{n\times n}$ is a constant symmetrical matrix and $\beta(t)$ is allowed to change sign.

As is well known, a solution $u(t)$ of Eq. (1.1) is called homoclinic (to O) if $u(t) \to 0$ and $u'(t) \to 0$ as $|t| \to +\infty$. In addition, if $u \not\equiv 0$, then u is called a nontrivial homoclinic solution.

Under the condition of $C = O$, system (1.1) transforms into a classic second-order Duffing equation

$$u''(t) + \beta(t)x'(t) + g\big(t, u\big(t-\gamma(t)\big)\big) = p(t), \tag{1.2}$$

which has been studied by Li *et al.* [1] and some new results on the existence and uniqueness of periodic solutions for (1.2) are obtained. Very recently, by using Mawhin's continuation theorem, Du [2] studied the following neutral differential equations:

$$\big(u(t) - Cu(t-\tau)\big)'' + \frac{d}{dt}\nabla F\big(u(t)\big) + \nabla G\big(u(t)\big) = e(t), \tag{1.3}$$

where $F \in C^2(R^n,R)$; $G \in C^1(R^n,R)$; $e \in C(R,R^n)$; $C = \operatorname{diag}(c_1, c_2, \dots, c_n)$, c_i $(i = 1, 2, \dots, n)$ and τ are given constants, obtaining the existence of homoclinic solutions for (1.3).

In this paper, like in the work of Rabinowitz in [3], Izydorek and Janczewska in [4] and Tan and Xiao in [5], the existence of a homoclinic solution for (1.1) is obtained as a limit of a certain sequence of $2kT$-periodic solutions for the following equation:

$$\bigl(u(t) - Cu(t-\tau)\bigr)'' + \beta(t)u'(t) + g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = p_k(t), \tag{1.4}$$

where $k \in N$, $p_k : R \to R^n$ is a $2kT$-periodic function such that

$$p_k(t) = \begin{cases} p(t), & t \in [-kT, kT - \varepsilon_0), \\ p(kT - \varepsilon_0) + \frac{p(-kT)-p(kT-\varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0), & t \in [kT - \varepsilon_0, kT], \end{cases} \tag{1.5}$$

$\varepsilon_0 \in (0, T)$ is a constant independent of k. However, the approaches to show $u'(t) \to 0$ as $|t| \to +\infty$ are different from the corresponding ones used in the past and the existence of $2kT$-periodic solutions to Eq. (1.4) is obtained by using an extension of Mawhin's continuation theorem, which is quite different from the approach of [3–5]. Furthermore, $C = [c_{ij}]_{n\times n}$ is a constant symmetrical matrix and $\beta(t)$ is allowed to change sign, different from the corresponding ones of [2].

2 Preliminary

Throughout this paper, $\langle\cdot,\cdot\rangle : R^n \times R^n \to R$ denotes the standard inner product, and $|\cdot|$ denotes the absolute value and the Euclidean norm on R^n. For each $k \in N$, let $C_{2kT} = \{x | x \in C(R, R^n), x(t + 2kT) \equiv x(t)\}$, $C^1_{2kT} = \{x | x \in C^1(R, R^n), x(t + 2kT) \equiv x(t)\}$ and $|x|_0 = \max_{t\in[0,2kT]} |x(t)|$. If the norms of C_{2kT} and C^1_{2kT} are defined by $\|\cdot\|_{C_{2kT}} = |\cdot|_0$ and $\|\cdot\|_{C^1_{2kT}} = \max\{|x|_0, |x'|_0\}$, respectively, then C_{2kT} and C^1_{2kT} are all Banach spaces. Furthermore, for $\varphi \in C_{2kT}$, $\|\varphi\|_r = (\int_{-kT}^{kT} |\varphi(t)|^r\,dt)^{\frac{1}{r}}$, $r > 1$.

Define the linear operator

$$A : C_T \to C_T, \quad [Ax](t) = x(t) - Cx(t-\tau).$$

Lemma 2.1 [6] *Suppose that Ω is an open bounded set in X such that the following conditions are satisfied*:

[A$_1$] *For each $\lambda \in (0,1)$, the equation*

$$\bigl(u(t) - Cu(t-\tau)\bigr)'' + \lambda\beta(t)u'(t) + \lambda g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = \lambda p_k(t)$$

has no solution on $\partial\Omega$.

[A$_2$] *The equation*

$$\Delta(a) := \frac{1}{2kT}\int_{-kT}^{kT}\bigl[g(a) - p_k(t)\bigr]\,dt = 0$$

has no solution on $\partial\Omega \cap R^n$.

[A$_3$] *The Brouwer degree*

$$d_B\bigl\{\Delta, \Omega \cap R^n, 0\bigr\} \neq 0.$$

Equation (1.4) *has a $2kT$-periodic solution in $\bar{\Omega}$.*

Lemma 2.2 [7] *If set* $P_T = \{x|x \in C(R,R), x(t+T) \equiv x(t)\}$ *and* $A_0 : P_T \to P_T$, $[A_0x](t) = x(t) - cx(t)$, *where* $c \in R$ *is a constant with* $|c| \neq 1$, *then operator* A_0 *has continuous inverse* A_0^{-1} *on* P_T, *satisfying*

$$\left[A_0^{-1}f\right](t) = \begin{cases} \sum_{j\geq 0} c^j f(t - j\tau), & |c| < 1, \forall f \in P_T, \\ -\sum_{j\geq 1} c^{-j} f(t + j\tau), & |c| > 1, \forall f \in P_T. \end{cases}$$

Lemma 2.3 [5] *If* $u : R \to R^n$ *is continuously differentiable on* R, $a > 0$, $\mu > 1$, *and* $p > 1$ *are constants, then for every* $t \in R$, *the following inequality holds*:

$$\left|u(t)\right| \leq (2a)^{-\frac{1}{\mu}} \left(\int_{t-a}^{t+a} \left|u(s)\right|^{\mu} ds \right)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} \left|u'(s)\right|^{p} ds \right)^{\frac{1}{p}}.$$

This lemma is a special case of Lemma 2.2 *in* [5].

Lemma 2.4 [6] *Suppose that* $c_1, c_2, \ldots, c_n$ *are eigenvalues of matrix* C. *If* $|c_i| \neq 1$ $(i = 1, 2, \ldots, n)$, *then* A *has a continuous bounded inverse with the following relationships*:

(1) $\|A^{-1}f\| \leq (\sum_{i=1}^{n} \frac{1}{|1-|c_i||})\|f\|$, $\forall f \in C_T$,

(2) $\int_0^T |(A^{-1}f)(t)|^p\,dt \leq \alpha \int_0^T |f(t)|^p\,dt$, $\forall f \in C_T$, $p \geq 1$, *where*

$$\alpha = \begin{cases} \max(\frac{1}{(1-|c_i|)^2}), & p = 2, \\ (\sum_{i=1}^{n} \frac{1}{(1-|c_i|)^{\frac{2p}{2-p}}})^{\frac{2-p}{2}}, & p \in [1,2), \\ (\sum_{i=1}^{n} \frac{1}{1-|c_i|^q})^{\frac{p}{q}}, & p \in [2,+\infty), \end{cases}$$

q is a constant with $\frac{1}{p} + \frac{1}{q} = 1$.

(3) $(Ax)' = Ax'$, $\forall x \in C_T^1$.

Lemma 2.5 [7] *Let* $s \in C(R,R)$ *with* $s(t+\omega) \equiv s(t)$ *and* $s(t) \in [0,\omega]$, $\forall t \in R$. *Suppose* $p \in (1,+\infty)$, $|s|_0 = \max_{t\in[0,\omega]} s(t)$ *and* $u \in C^1(R,R)$ *with* $u(t+\omega) \equiv u(t)$. *Then*

$$\int_0^{\omega} \left|u(t) - u\left(t - s(t)\right)\right|^p dt \leq |s|_0^p \int_0^{\omega} \left|u'(t)\right|^p dt.$$

Throughout this paper, we suppose in addition that $c_m = \max\{|c_i|\}$, $i = 1, 2, \ldots, n$, where $c_1, c_2, \ldots, c_n$ are eigenvalues of matrix C with $|c_i| \neq 1$ and let $\beta_L' = \min|\beta'(t)|$, $\beta_M = \max|\beta(t)|$, $\forall t \in [0,T]$.

For convenience, we list the following assumptions which will be used to study the existence of homoclinic solutions to Eq. (1.1) in Section 3.

[H$_1$] There are constants $L > 0$ and $m > 0$ such that

$$\left|g(x_1) - g(x_2)\right| \leq L|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in R^n,$$

and

$$\left\langle (E - C)x, g(x)\right\rangle \leq -m|x|^2, \quad \text{for all } x \in R^n,$$

[H_2] $p \in C(R, R^n)$ is a bounded function with $p(t) \not\equiv O = (0, 0, \dots, 0)^\top$ and

$$B := \left(\int_R \left|p(t)\right|^2 dt\right)^{\frac{1}{2}} + \sup_{t\in R}\left|p(t)\right| < +\infty.$$

Remark 2.1 [8] From (1.5), we see that $|p_k(t)| \leq \sup_{t\in R} |p(t)|$. So if assumption [$H_2$] holds, for each $k \in \mathbf{N}$, $(\int_{-kT}^{kT} |p_k(t)|^2 dt)^{\frac{1}{2}} < B$.

3 Main results

In order to investigate the existence of $2kT$-periodic solutions to system (1.4), we need to study some properties of all possible $2kT$-periodic solutions to the following system:

$$\big(x(t) - Cx(t-\tau)\big)'' + \lambda\beta(t)x'(t) + \lambda g\big(x\big(t-\gamma(t)\big)\big) = \lambda p_k(t), \quad \lambda \in (0,1]. \tag{3.1}$$

For each $k \in \mathbf{N}$, let $\Sigma \subset C^1_{2kT}$ represent the set of all the $2kT$-periodic solutions to system (3.1).

Theorem 3.1 *Suppose assumptions* [H_1]-[H_2] *hold*, $\beta'_L > -2m$, *and*

$$\frac{\alpha[c_m^{\frac{1}{2}}L(|\gamma|_0 + |\tau|) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M]^2}{(\frac{1}{2}\beta'_L + m)} < 1,$$

then for each $k \in \mathbf{N}$, *if* $u \in \Sigma$, *then there are positive constants* A_0, A_1, ρ_0, *and* ρ_1 *which are independent of k and* λ, *such that*

$$\|u\|_2 \leq A_0, \qquad \left\|u'\right\|_2 \leq A_1, \qquad |u|_0 \leq \rho_0, \qquad \left|u'\right|_0 \leq \rho_1.$$

Proof For each $k \in \mathbf{N}$, if $u \in \Sigma$, then u must satisfy

$$\big(u(t) - Cu(t-\tau)\big)'' + \lambda\beta(t)u'(t) + \lambda g\big(u\big(t-\gamma(t)\big)\big) = \lambda p_k(t), \quad \lambda \in (0,1]. \tag{3.2}$$

Multiplying both sides of Eq. (3.2) by $[Au](t)$ and integrating on the interval $[-kT, kT]$, we have

$$\begin{aligned} -\left\|Au'\right\|_2^2 &+ \lambda\int_{-kT}^{kT}\big\langle [Au](t), \beta(t)u'(t)\big\rangle dt + \lambda\int_{-kT}^{kT}\big\langle [Au](t), g\big(u\big(t-\gamma(t)\big)\big)\big\rangle dt \\ &= \lambda\int_{-kT}^{kT}\big\langle [Au](t), p_k(t)\big\rangle dt. \end{aligned} \tag{3.3}$$

Clearly, $\int_{-kT}^{kT}\langle u(t), \beta(t)u'(t)\rangle dt = -\frac{1}{2}\int_{-kT}^{kT}\beta'(t)u^2(t)\, dt$, then we have

$$\begin{aligned} \lambda&\int_{-kT}^{kT}\big\langle [Au](t), p_k(t)\big\rangle dt \\ &= -\left\|Au'\right\|_2^2 - \lambda\frac{1}{2}\int_{-kT}^{kT}\beta'(t)u^2(t)\,dt + \lambda\int_{-kT}^{kT}\big\langle Cu'(t-\tau), \beta(t)u'(t)\big\rangle dt \\ &\quad + \lambda\int_{-kT}^{kT}\big\langle u(t), g\big(u\big(t-\gamma(t)\big)\big) - g\big(u(t)\big)\big\rangle dt + \lambda\int_{-kT}^{kT}\big\langle u(t), g\big(u(t)\big)\big\rangle dt \end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_{-kT}^{kT} \bigl\langle Cu(t-\tau), g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t-\tau)\bigr)\bigr\rangle \, dt \\
&- \lambda \int_{-kT}^{kT} \bigl\langle Cu(t-\tau), g\bigl(u(t-\tau)\bigr)\bigr\rangle \, dt
\end{aligned} \tag{3.4}
$$

and from (3.4) and $[\mathrm{H}_1]$ that

$$
\begin{aligned}
&\|Au'\|_2^2 + \lambda\left(\frac{1}{2}\beta_L' + m\right)\|u\|_2^2 \\
&\quad \le \lambda \int_{-kT}^{kT} \bigl|\bigl\langle Cu(t-\tau), \beta(t)u'(t)\bigr\rangle\bigr| \, dt \\
&\qquad + \lambda \int_{-kT}^{kT} \bigl|\bigl\langle u(t), g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t)\bigr)\bigr\rangle\bigr| \, dt \\
&\qquad + \lambda \int_{-kT}^{kT} \bigl|\bigl\langle Cu(t-\tau), g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t-\tau)\bigr)\bigr\rangle\bigr| \, dt \\
&\qquad + \lambda \int_{-kT}^{kT} \bigl|\bigl\langle Au(t), p_k(t)\bigr\rangle\bigr| \, dt.
\end{aligned} \tag{3.5}
$$

By using $[\mathrm{H}_1]$ and Lemma 2.5, we get

$$
\begin{aligned}
&\int_{-kT}^{kT} \bigl|\bigl\langle u(t), g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t)\bigr)\bigr\rangle\bigr| \, dt \\
&\quad \le \left(\int_{-kT}^{kT} \bigl|u(t)\bigr|^2 \, dt\right)^{\frac{1}{2}} \left(\int_{-kT}^{kT} \bigl|g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t)\bigr)\bigr|^2 \, dt\right)^{\frac{1}{2}} \\
&\quad \le L|\gamma|_0 \|u\|_2 \|u'\|_2.
\end{aligned} \tag{3.6}
$$

In a similar way as in the proof of (3.6), we have

$$
\int_{-kT}^{kT} \bigl|\bigl\langle Cu(t-\tau), g\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) - g\bigl(u(t-\tau)\bigr)\bigr\rangle\bigr| \, dt \le c_m^{\frac{1}{2}} L\bigl(|\gamma|_0 + |\tau|\bigr)\|u\|_2 \|u'\|_2. \tag{3.7}
$$

By using $[\mathrm{H}_2]$, we get

$$
\begin{aligned}
\int_{-kT}^{kT} \bigl|\bigl\langle [Au](t), p_k(t)\bigr\rangle\bigr| \, dt &\le \|e_k\|_2 \|u\|_2 + c_m^{\frac{1}{2}} \|p_k\|_2 \|u\|_2 \\
&\le B\bigl(1 + c_m^{\frac{1}{2}}\bigr)\|u\|_2
\end{aligned} \tag{3.8}
$$

and

$$
\int_{-kT}^{kT} \bigl|\bigl\langle Cu(t-\tau), \beta(t)u'(t)\bigr\rangle\bigr| \, dt \le c_m^{\frac{1}{2}} \beta_M \|u\|_2 \|u'\|_2. \tag{3.9}
$$

By applying (3.6)-(3.9), we see that

$$
\begin{aligned}
\|Au'\|_2^2 + \lambda\left(\frac{1}{2}\beta_L' + m\right)\|u\|_2^2 &\le \lambda\bigl[c_m^{\frac{1}{2}} L\bigl(|\gamma|_0 + |\tau|\bigr) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M\bigr]\|u\|_2 \|u'\|_2 \\
&\quad + \lambda B\bigl(1 + c_m^{\frac{1}{2}}\bigr)\|u\|_2.
\end{aligned} \tag{3.10}
$$

Thus, from (3.10)

$$\left(\frac{1}{2}\beta_L' + m\right)\|u\|_2^2 \le \left[c_m^{\frac{1}{2}}L\left(|\gamma|_0 + |\tau|\right) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M\right]\|u\|_2\left\|u'\right\|_2 + B\left(1 + c_m^{\frac{1}{2}}\right)\|u\|_2. \tag{3.11}$$

By using Lemma 2.4, we have $\|u'\|_2 = \|A^{-1}Au'\|_2 \le \alpha^{\frac{1}{2}}\|Au'\|_2$, and from (3.10)-(3.11)

$$\begin{aligned}\left\|Au'\right\|_2^2 \le{}& \frac{\alpha[c_m^{\frac{1}{2}}L(|\gamma|_0 + |\tau|) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M]^2}{(\frac{1}{2}\beta_L' + m)}\left\|Au'\right\|_2^2 \\ &+ \frac{2\alpha^{1/2}B(1 + c_m^{\frac{1}{2}}[c_m^{\frac{1}{2}}L(|\gamma|_0 + |\tau|) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M]}{(\frac{1}{2}\beta_L' + m)}\left\|Au'\right\|_2 \\ &+ \frac{B^2(1 + c_m^{\frac{1}{2}})^2}{(\frac{1}{2}\beta_L' + m)}.\end{aligned} \tag{3.12}$$

Since

$$\frac{\alpha[c_m^{\frac{1}{2}}L(|\gamma|_0 + |\tau|) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M]^2}{(\frac{1}{2}\beta_L' + m)} < 1,$$

there is a constant $M > 0$ such that

$$\left\|Au'\right\|_2 \le M, \tag{3.13}$$

$$\left\|u'\right\|_2 \le \alpha^{\frac{1}{2}}\left\|Au'\right\|_2 \le \alpha^{\frac{1}{2}}M := A_1, \tag{3.14}$$

and by (3.11)

$$\|u\|_2 \le \frac{[c_m^{\frac{1}{2}}L(|\gamma|_0 + |\tau|) + L|\gamma|_0 + c_m^{\frac{1}{2}}\beta_M]A_1 + B(1 + c_m^{\frac{1}{2}})}{(\frac{1}{2}\beta_L' + m)} := A_0. \tag{3.15}$$

Obviously, A_0 and A_1 are constants independent of k and λ. Thus by using Lemma 2.2, for all $t \in [-kT, kT]$, we get

$$\begin{aligned}\left|u(t)\right| &\le (2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u(s)\right|^2\,ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u'(s)\right|^2\,ds\right)^{\frac{1}{2}} \\ &\le (2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}\left|u(s)\right|^2\,ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-kT}^{t+kT}\left|u'(s)\right|^2\,ds\right)^{\frac{1}{2}} \\ &= (2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}\left|u(s)\right|^2\,ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{-kT}^{kT}\left|u'(s)\right|^2\,ds\right)^{\frac{1}{2}}.\end{aligned}$$

From (3.14) and (3.15), we obtain

$$|u|_0 \le (2T)^{-\frac{1}{2}}\|u\|_2 + T(2T)^{-\frac{1}{2}}\left\|u'\right\|_2 \le (2T)^{-\frac{1}{2}}A_0 + T(2T)^{-\frac{1}{2}}A_1 := \rho_0, \tag{3.16}$$

where ρ_0 is a constant independent of k and λ.

For $i = -k, -k+1, \ldots, k-1$, from the continuity of $[Au'](t)$, one can find that there is a $t_i \in [iT, (i+1)T]$ such that

$$\left|[Au'](t_i)\right| = \left|\frac{1}{T}\int_{iT}^{(i+1)T}[Au'](s)\,ds\right| = \left|\frac{[Au]((i+1)T) - [Au](iT)}{T}\right| \le \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0,$$

and it follows from (3.14) that for $t \in [iT, (i+1)T]$, $i = -k, -k+1, \ldots, k-1$,

$$\begin{aligned}
\left|[Au'](t)\right| &= \left|\int_{t_i}^{t}[Au]''(s)\,ds + [Au'](t_i)\right| \\
&\le \int_{t_i}^{t}\left|[Au]''(s)\right|ds + \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0 \\
&\le \int_{iT}^{(i+1)T}\left|[Au]''(s)\right|ds + \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0 \\
&\le \int_{iT}^{(i+1)T}\left|\beta(s)u'(s)\right|ds + \int_{iT}^{(i+1)T}\left|g\left(u\left(s - \gamma(s)\right)\right)\right|ds \\
&\quad + \int_{iT}^{(i+1)T}\left|p_k(s)\right|ds + \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0 \\
&\le \beta_M T^{\frac{1}{2}}\left(\int_{-kT}^{kT}\left|u'(s)\right|^2 ds\right)^{\frac{1}{2}} + Tg_M + TB + \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0 \\
&\le \beta_M T^{\frac{1}{2}}A_1 + Tg_M + TB + \frac{2}{T}\left(1 + c_m^{\frac{1}{2}}\right)\rho_0 := \rho,
\end{aligned}$$

i.e.,

$$\left|Au'\right|_0 \le \rho, \tag{3.17}$$

where $g_M = \max_{|u|_0 \le \rho_0}|g(u(t - \tau(t)))|$.

By Lemma 2.4 and (3.17), we get

$$\left|u'\right|_0 = \left|A^{-1}Au'\right|_0 \le \left(\sum_{i=1}^{n}\frac{1}{|1 - |c_i||}\right)\left|Au'\right|_0 \le \left(\sum_{i=1}^{n}\frac{1}{|1 - |c_i||}\right)\rho := \rho_1.$$

Clearly, ρ_1 is a constant independent of k and λ. Hence the conclusion of Theorem 3.1 holds. □

Theorem 3.2 *Assume that the conditions of Theorem* 3.1 *are satisfied. Then for each* $k \in N$, *Eq.* (3.2) *has at least one* $2kT$*-periodic solution* $u_k(t)$ *such that*

$$\|u_k\|_2 \le A_0, \qquad \left\|u_k'\right\|_2 \le A_1, \qquad |u_k|_0 \le \rho_0, \qquad \left|u_k'\right|_0 \le \rho_1,$$

where A_0, A_1, ρ_0, *and* ρ_1 *are constants defined by Theorem* 3.1.

Proof In order to use Lemma 2.1, for each $k \in N$, we consider the following equation:

$$\left(u(t) - Cu(t - \tau)\right)'' + \lambda\beta(t)u'(t) + \lambda g\left(u\left(t - \gamma(t)\right)\right) = \lambda p_k(t), \quad \lambda \in (0,1). \tag{3.18}$$

Let $\Omega_1 \subset C_{2kT}^1$ represent the set of all the $2kT$-periodic of system (3.18), since $(0,1) \subset (0,1]$, then $\Omega_1 \subset \Sigma$, where Σ is defined by Theorem 3.1. If $u \in \Omega_1$, by using Theorem 3.1, we have

$$|u|_0 \le \rho_0, \qquad |u'|_0 \le \rho_1.$$

Let $\Omega_2 = \{x : x \in \operatorname{Ker} L, QNx = 0\}$, where
$L : D(L) \subset C_{2kT} \to C_{2kT}$, $Lu = (Au)''$,
$N : C_{2kT} \to C^1_{2kT}$, $Nu = -\beta(t)u'(t) - g(u(t-\gamma(t))) + p_k(t)$,
$Q : C_{2kT} \to C_{2kT}/\operatorname{Im} L$, $Qy = \frac{1}{2kT}\int_{-kT}^{kT} y(s)\,ds$.
If $x \in \Omega_2$, then $x = a \in R^n$ (constant vector) and by [H$_1$], we see that

$$2kTm|a|^2 \le \int_{-kT}^{kT} \bigl|\langle (E-C)a, p_k(t)\rangle\bigr|\,dt \le B|a|(1+c_m)(2kT)^{\frac{1}{2}},$$

i.e.,

$$|a| \le m^{-1}BT^{\frac{-1}{2}}(1+c_m) := B_0.$$

Now, if we set $\Omega = \{x : x \in C^1_{2kT}, |x|_0 < \rho_0 + B_0, |x'|_0 < \rho_1 + 1\}$, then $\Omega \supset \Omega_1 \cup \Omega_2$. So condition [A$_1$] and condition [A$_2$] of Lemma 2.1 are satisfied. What remains is verifying condition [A$_3$] of Lemma 2.1. In order to do this, let

$$H(x,\mu) : (\Omega \cap R^n) \times [0,1] \longrightarrow R^n : H(x,\mu) = -\mu x + (1-\mu)\Delta(x),$$

where $\Delta(x) = \frac{1}{2kT}\int_{-kT}^{kT}[g(x) - p_k(t)]\,dt$ is determined by Lemma 2.1. From assumption [H$_1$], we have

$$H(x,\mu) \ne 0, \quad \forall (x,\mu) \in \bigl[\partial(\Omega \cap R^n)\bigr] \times [0,1].$$

Hence

$$\begin{aligned} \deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} &= \deg\bigl\{H(x,0), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &= \deg\bigl\{H(x,1), \Omega \cap \operatorname{Ker} L, 0\bigr\} \\ &\ne 0. \end{aligned}$$

So condition [A$_3$] of Lemma 2.1 is satisfied. Therefore, by using Lemma 2.1, we see that Eq. (1.2) has a $2kT$-periodic solution $u_k \in \bar{\Omega}$. Evidently, $u_k(t)$ is a $2kT$-periodic solution to Eq. (3.1) for the case of $\lambda = 1$, so $u_k \in \Sigma$. Thus, by using Theorem 3.1, we get

$$\|u_k\|_2 \le A_0, \qquad \|u_k'\|_2 \le A_1, \qquad |u_k|_0 \le \rho_0, \qquad |u_k'|_0 \le \rho_1. \tag{3.19}$$

□

Theorem 3.3 *Suppose that the conditions in Theorem* 3.1 *hold, then Eq.* (1.1) *has a nontrivial homoclinic solution.*

Proof From Theorem 3.2, we see that for each $k \in N$, there exists a $2kT$-periodic solution $u_k(t)$ to Eq. (1.2). So for every $k \in N$, $u_k(t)$ satisfies

$$\bigl(u_k(t) - Cu_k(t-\tau)\bigr)'' + \beta(t)u_k'(t) + g\bigl(u_k\bigl(t-\gamma(t)\bigr)\bigr) = p_k(t). \tag{3.20}$$

Let $y_k = (Au'_k)$ for $k > k_0$. By (3.17),

$$|y_k|_0 \leq \rho$$

and, by (3.20),

$$\left|y'_k\right|_0 \leq \beta_M \left|u'_k\right|_0 + g_M + \sup_{t\in R}\left|p(t)\right| := \rho_2.$$

Obviously, ρ_2 is a constant independent of k. Similar to the proof of Lemma 2.4 in [5], we see that there exists a $u_0 \in C^1(R,R^n)$ such that for each interval $[c,d] \subset R$, there is a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ with R, $u_{k_j}(t) \to u_0(t)$ and $u'_{k_j}(t) \to u'_0(t)$ uniformly on $[c,d]$.

For all $a,b \in R$ with $a < b$, there must be a positive integer j_0 such that for $j > j_0$, $[-k_jT, k_jT - \varepsilon_0] \supset [a - |\gamma|_0, b + |\gamma|_0]$. So for $t \in [a - |\gamma|_0, b + |\gamma|_0]$, from (1.5) and (3.20) we see that

$$\bigl(u_{k_j}(t) - Cu_{k_j}(t-\tau)\bigr)'' = -\beta(t)u'_{k_j}(t) - g\bigl(u_{k_j}\bigl(t-\gamma(t)\bigr)\bigr) + p(t). \tag{3.21}$$

By (3.21),

$$\begin{aligned} y'_k &= \bigl(Au'_{k_j}\bigr)' \\ &= -\beta(t)u'_{k_j}(t) - g\bigl(u_{k_j}\bigl(t-\gamma(t)\bigr)\bigr) + p(t) \\ &\to -\beta(t)u'_0(t) - g\bigl(u_0\bigl(t-\gamma(t)\bigr)\bigr) + p(t) \\ &:= \chi(t), \end{aligned}$$

uniformly on $[a,b]$.

By the fact that $y'_{k_j}(t)$ is a continuous differential on (a,b), for $j > j_0$, $y'_{k_j}(t) \to \chi(t)$ uniformly $[a,b]$. We have $\chi(t) = (u_0(t) - Cu_0(t-\tau))''$, $t \in R$, in view of $a,b \in R$ being arbitrary, that is, $u_0(t)$ is a solution to system (1.1).

Now, we will prove $u_0(t) \to 0$ and $u'_0(t) \to 0$ for $|t| \to +\infty$. We have

$$\begin{aligned} \int_{-\infty}^{+\infty} \bigl(\left|u_0(t)\right|^2 + \left|u'_0(t)\right|^2\bigr)\,dt &= \lim_{i\to+\infty} \int_{-iT}^{iT} \bigl(\left|u_0(t)\right|^2 + \left|u'_0(t)\right|^2\bigr)\,dt \\ &= \lim_{i\to+\infty}\lim_{j\to+\infty} \int_{-iT}^{iT} \bigl(\left|u_{k_j}(t)\right|^2 + \left|u'_{k_j}(t)\right|^2\bigr)\,dt. \end{aligned}$$

Clearly, for every $i \in N$ if $k_j > i$, by (3.14) and (3.15), we get

$$\int_{-iT}^{iT} \bigl(\left|u_{k_j}(t)\right|^2 + \left|u'_{k_j}(t)\right|^2\bigr)\,dt \leq \int_{-k_jT}^{k_jT} \bigl(\left|u_{k_j}(t)\right|^2 + \left|u'_{k_j}(t)\right|^2\bigr)\,dt \leq A_0^2 + A_1^2.$$

Let $i \to +\infty$ and $j \to +\infty$; we have

$$\int_{-\infty}^{+\infty} \bigl(\left|u_0(t)\right|^2 + \left|u'_0(t)\right|^2\bigr)\,dt \leq A_0^2 + A_1^2, \tag{3.22}$$

and then

$$\int_{|t|\geq r} \left(\left|u_0(t)\right|^2 + \left|u_0'(t)\right|^2\right) dt \to 0, \tag{3.23}$$

as $r \to +\infty$.

From (3.13), in a similar way we get

$$\int_{-\infty}^{+\infty} \left|u_0'(t) - Cu_0'(t-\tau)\right|^2 dt \leq M^2. \tag{3.24}$$

So, by using Lemma 2.3,

$$\begin{aligned}\left|u_0(t)\right| &\leq (2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u_0(s)\right|^2 ds\right)^{\frac{1}{2}} + T(2T)^{-\frac{1}{2}}\left(\int_{t-T}^{t+T}\left|u_0'(s)\right|^2 ds\right)^{\frac{1}{2}} \\ &\leq \max\left\{(2T)^{-\frac{1}{2}}, T(2T)^{-\frac{1}{2}}\right\}\int_{t-T}^{t+T}\left(\left|u_0(t)\right|^2 + \left|u_0'(t)\right|^2\right) dt \to 0, \quad |t| \to +\infty.\end{aligned}$$

Finally, in order to obtain

$$\left|u_0'(t)\right| \to 0, \quad |t| \to +\infty,$$

we show that

$$\left|\left[\tilde{A}u'\right]_0(t)\right| := \left|u_0'(t) - Cu_0'(t-\tau)\right| \to 0, \quad |t| \to +\infty. \tag{3.25}$$

From (3.16), we have $|u|_0 \leq \rho_0$ and by (1.1), we get

$$\begin{aligned}\left|\left(\left[\tilde{A}u_0'\right](t)\right)'\right| &\leq \left|\beta(t)u_0(t)\right| + \left|g\left(u_0\left(t-\gamma(t)\right)\right)\right| + \sup_{t\in R}\left|p(t)\right| \\ &\leq \beta_M\rho_0 + \sup_{|u|\leq\rho_0}\left|g(u)\right| + \sup_{t\in R}\left|p(t)\right| := \tilde{M}, \quad \text{for } t \in R.\end{aligned}$$

If (3.25) does not hold, then there exist $\varepsilon_0 \in (0, \frac{1}{2})$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < |t_3| < \cdots < |t_k| + 1 < |t_{k+1}|, \quad k = 1, 2, \ldots,$$

and

$$\left|\left[\tilde{A}u_0'\right](t_k)\right| \geq 2\varepsilon_0, \quad k = 1, 2, \ldots.$$

From this, we have, for $t \in [t_k, t_k + \varepsilon_0/(1+\tilde{M})]$,

$$\left|\left[\tilde{A}u_0'\right](t)\right| = \left|\left[\tilde{A}u_0'\right](t_k) + \int_{t_k}^{t}\left(\left[\tilde{A}u_0'\right](s)\right)' ds\right| \geq \left|\left[\tilde{A}u_0'\right](t_k)\right| - \int_{t_k}^{t}\left|\left(\left[\tilde{A}u_0'\right](s)\right)'\right| ds \geq \varepsilon_0.$$

It follows that

$$\int_{-\infty}^{+\infty}\left|\left[\tilde{A}u_0'\right](t_k)\right|^2 dt \geq \sum_{k=1}^{\infty}\int_{t_k}^{t_k+\varepsilon_0/(1+\tilde{M})}\left|\left[\tilde{A}u_0'\right](t_k)\right|^2 dt = \infty,$$

which contradicts (3.24), so (3.25) holds.

Since C is symmetrical, it is easy to see that there is an orthogonal matrix T such that $TCT^\top = E_c = \operatorname{diag}(c_1, c_2, \ldots, c_n)$.

Let $y'_{k_j}(t) = Tu'_{k_j}(t) = (y'^{(1)}_{k_j}(t), y'^{(2)}_{k_j}(t), \ldots, y'^{(n)}_{k_j}(t)) = T(u'^{(1)}_{k_j}(t), u'^{(2)}_{k_j}(t), \ldots, u'^{(n)}_{k_j}(t))^\top$, then we get $y'_0(t) = (y'^{(1)}_0(t), y'^{(2)}_0(t), \ldots, y'^{(n)}_0(t)) = Tu'_0(t) = T(u'^{(1)}_0(t), u'^{(2)}_0(t), \ldots, u'^{(n)}_0(t))^\top$ as $j \to \infty$. By (3.25), we have

$$\left|y'_0(t) - E_c y'_0(t-\tau)\right| \to 0, \quad |t| \to +\infty. \tag{3.26}$$

By using (3.19), we see that $|Au'_k| < (1 + c_m^{\frac{1}{2}})\rho_1 := \tilde{B}$, which implies

$$\left|TAu'_k\right| = \left|\langle TAu'_k, TAu'_k\rangle\right|^{\frac{1}{2}} < \tilde{B},$$

i.e.,

$$\left|y'_k(t) - E_c y'_k(t-\tau)\right| < \tilde{B}, \quad \forall t \in R. \tag{3.27}$$

For all $\varepsilon > 0$, there exists $N = [\log_{|c_i|}^{\frac{\varepsilon(1-|c_i|)}{2\tilde{B}}}] > 0$ such that $\sum_{h=N+1}^{\infty} |c_i|^h < \frac{\varepsilon}{2\tilde{B}}$ ($|c_i| < 1$), for $t > N$. Similarly, by (3.26), we see that there is a constant $G > 0$ such that $|y'_{0_i}(t) - c_i y'_{0_i}(t-\tau)| < \frac{\varepsilon}{2(N+1)}$, for $t > G$.

Then, by using Lemma 2.2 and (3.27), when $|c_i| < 1$, we get

$$\begin{aligned}
\left|y'^{(i)}_0(t)\right| &= \lim_{j\to+\infty} \left|\left[A_0^{-1} A_0 y'^{(i)}_{k_j}\right](t)\right| \\
&\le \left|\lim_{j\to\infty} \sum_{h\ge 0}^{N} c_i^h \left[A_0 y'^{(i)}_{k_j}\right](t-h\tau) + \sum_{h=N+1}^{\infty} c_i^h \left[A_0 y'^{(i)}_{k_j}\right](t-h\tau)\right| \\
&\le \left|\lim_{j\to\infty} \sum_{h\ge 0}^{N} c_i^h \left[A_0 y'^{(i)}_{k_j}\right](t-h\tau)\right| + \left|\lim_{j\to\infty} \sum_{h=N+1}^{\infty} c_i^h \left[A_0 y'^{(i)}_{k_j}\right](t-h\tau)\right| \\
&\le \lim_{j\to\infty} \sum_{h\ge 0}^{N} |c_i|^h \left|\left[A_0 y'^{(i)}_{k_j}\right](t-h\tau)\right| + \tilde{B} \sum_{h=N+1}^{\infty} |c_i|^h \\
&= \sum_{h\ge 0}^{N} |c_i|^h \left|\left(y'^{(i)}_0(t-h\tau) - c_i y'^{(i)}_0\left(t-(h+1)\tau\right)\right)\right| + \tilde{B} \sum_{h=N+1}^{\infty} |c_i|^h.
\end{aligned} \tag{3.28}$$

Now, by (3.27) and (3.28), we conclude that $\forall \varepsilon > 0$, there exists $\bar{N} = G + N$ such that for $t > \bar{N}$,

$$\begin{aligned}
\left|y'_{0_i}(t)\right| &\le \sum_{h\ge 0}^{N} |c_i|^h \left|\left(y'^{(i)}_0(t-h\tau) - c_i y'^{(i)}_0\left(t-(h+1)\tau\right)\right)\right| + \left|\tilde{B} \sum_{h=N+1}^{\infty} c_i^h\right| \\
&< (N+1)\frac{\varepsilon}{2(N+1)} + \tilde{B}\frac{\varepsilon}{2\tilde{B}} \\
&= \varepsilon.
\end{aligned}$$

Thus, we get $|y'^{(i)}_0(t)| \to 0$, as $|t| \to +\infty$.

In the similar way, when $|c_i| > 1$, we can proof $|y_0'^{(i)}(t)| \to 0$, as $|t| \to +\infty$.

Therefore, $|y_0'(t)| \to 0$, as $|t| \to +\infty$; *i.e.*,

$$T\Big(\lim_{|t|\to+\infty} u_0'^{(1)}(t), \lim_{|t|\to+\infty} u_0'^{(2)}(t), \dots, \lim_{|t|\to+\infty} u_0'^{(n)}(t)\Big)^\top = O,$$

we know T is an orthogonal matrix, then $u_0'^{(i)}(t) \to 0$ as $|t| \to +\infty$.

Thus, we have

$$\big|u_0'(t)\big| \to 0, \quad |t| \to +\infty.$$

Clearly, $u_0(t) \not\equiv 0$; otherwise, $p(t) = 0$, which contradicts the assumption $[\mathrm{H}_2]$.

As an application, we consider the following equation:

$$\big(u(t) - Cu(t - 0.01)\big)'' + \sin(t)x'(t) + g\big(u\big(t - \cos^2 t\big)\big) = p(t), \tag{3.29}$$

where $C = \begin{pmatrix} 26 & 3 \\ 3 & 17 \end{pmatrix}$, $u(t) = (u_1(t), u_1(t))^\top$, $g(x) = x = (x_1, x_2)^\top$ and $p(t) = (p_1(t), p_2(t))^\top = (\frac{1}{\sqrt{1+t^2}}, \frac{2}{\sqrt{1+t^2}})^\top$. Clearly, $\lambda_{1,2} = \frac{43\pm\sqrt{117}}{2} \neq \pm 1$. Also, $\langle (E - C)x, g(x)\rangle = -25x_1^2 - 6x_1x_2 - 16x_2^2 < -10(x_1^2 + x_2^2)$ and $g(x) = x$, which implies that assumption $[\mathrm{H}_1]$ is satisfied with $L = 2$, $m = 10$. $p(t) = (\frac{1}{\sqrt{1+t^2}}, \frac{2}{\sqrt{1+t^2}})^\top$ is a bounded function and $(\int_R |p(t)|^2\,dt)^{\frac{1}{2}} + \sup_{t\in R}|p(t)| = \sqrt{5}(1 + \frac{\sqrt{2}}{2}\pi)$, which implies that assumption $[\mathrm{H}_2]$ holds. Furthermore, we can choose $\alpha = \frac{4}{(\sqrt{117}-41)^2}$, $c_m = \frac{43+\sqrt{117}}{2}$, $|\gamma|_0 = 1$, $\beta_M = 1$ and $\beta_L' > -20$, then

$$\frac{\frac{1}{(\sqrt{117}-41)^2}[(\frac{43+\sqrt{117}}{2})^{\frac{1}{2}}2(1 + 0.01) + 2 + (\frac{43+\sqrt{117}}{2})^{\frac{1}{2}}]^2}{-\frac{1}{2} + 10} < 1.$$

By applying Theorem 3.3, we see that Eq. (3.29) has a nontrivial homoclinic solution. □

Competing interests
The authors declare that they have no competing interests.

Author's contributions
The author drafted the manuscript, read and approved the final manuscript.

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