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On the fourth power mean of the general k th Kloosterman sums

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Abstract

Let $q > 2$ be an integer, and let χ be a Dirichlet character modulo q . For integers m, n and k , the general k th Kloosterman sum $S(m, n, k, \chi; q)$ is defined by

$S(m, n, k, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma^k + n\bar{a}^k}{q}\right)$, where $\sum'$ denotes the summation over all a with $(a, q) = 1$, $e(y) = e^{2\pi i y}$, and $\bar{a}$ is the inverse of a modulo q such that $a\bar{a} \equiv 1 \pmod{q}$ and $1 \leq \bar{a} \leq q$. In this paper we further study the fourth power mean $\sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, k, \chi; q)|^4$, and we give some identities.

MSC: 11F20

Keywords: general k th Kloosterman sum; fourth power mean; identity

1 Introduction

Let $q > 2$ be an integer, and let χ be a Dirichlet character modulo q . For arbitrary integers m and n , the general Kloosterman sum $S(m, n, \chi; q)$ is defined by

$$S(m, n, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma + n\bar{a}}{q}\right),$$

where $\sum'$ denotes the summation over all a with $(a, q) = 1$, $e(y) = e^{2\pi i y}$, and $\bar{a}$ is the inverse of a modulo q such that $a\bar{a} \equiv 1 \pmod{q}$ and $1 \leq \bar{a} \leq q$. For $q = p$ a prime, Chowla [1] and Malyshev [2] proved an upper bound:

$$|S(m, n, \chi; p)| \ll (m, n, p)^{1/2} p^{1/2+\epsilon},$$

where (m, n, p) is the great common divisor of m, n and p .

For general integer $q > 2$, we do not know how large $|S(m, n, \chi; q)|$ is. However, $|S(m, n, \chi; q)|$ enjoys good value distribution properties. For fixed integer n with $(n, q) = 1$, Zhang [3] showed the identity

$$\sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, \chi; q)|^4 = \phi^2(q) q^2 d(q) \prod_{p^\alpha \parallel q} \left(1 - \frac{2}{\alpha + 1} \cdot \frac{p^{\alpha-1} - 1}{p^\alpha(p-1)} + \frac{\alpha - 4p^{\alpha-1}}{(\alpha + 1)p^\alpha}\right),$$

where $d(q)$ is the divisor function, $\phi(q)$ is the Euler function, and $\prod_{p^\alpha \parallel q}$ denotes the product over all prime divisors p of q with $p^\alpha \mid q$ and $p^{\alpha+1} \nmid q$.

For integers m, n and k , the general k th Kloosterman sum $S(m, n, k, \chi; q)$ is defined by

$$S(m, n, k, \chi; q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma^k + n\bar{a}^k}{q}\right).$$

Suppose that $(nk, q) = 1$. Liu and Zhang [4] proved the identity:

$$\begin{aligned} & \sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, k, \chi; q)|^4 \\ &= \phi^2(q) q^2 \prod_{p^\alpha \parallel q} (k, p-1)^2 \left(\alpha - 1 + \frac{2(p-1)}{(k, p-1)p} + \frac{\alpha}{p^\alpha} - \frac{2(p^\alpha - 1)}{p^\alpha(p-1)} \right). \end{aligned}$$

In this paper we further consider the situation $(k, q) > 1$. Our result is a generalization of [4].

Theorem 1.1 *Let p be an odd prime, n be any integer. Let α and k be positive integers with $d = (k, p-1)$ and $(k, p^{\alpha-1}) = p^\delta$. Then we have*

$$\begin{aligned} & \sum_{\chi \bmod p^\alpha} \sum_{m=1}^{p^\alpha} |S(m, n, k, \chi; p^\alpha)|^4 \\ &= \begin{cases} 2d\phi^3(p^\alpha)p^{\alpha+\delta}\left(1 - \frac{d}{2\phi(p)}\right) + d^2\phi^2(p^\alpha)p^{2\alpha+\delta-1}(p^{\alpha-\delta-1} - 1), & \frac{\alpha-2}{2} \leq \delta \leq \alpha-1; \\ 2d\phi^3(p^\alpha)p^{\alpha+\delta}\left(1 - \frac{d}{2\phi(p)}\right) + d^2\phi^2(p^\alpha)p^{2\alpha+\delta-1}(p^{\delta+1} - 1) \\ \quad + d^2\phi^2(p^\alpha)p^{2\alpha+2\delta-1}(p-1)(\alpha-2\delta-2), & 0 \leq \delta \leq \frac{\alpha-3}{2}. \end{cases} \end{aligned}$$

From Theorem 1.1 we immediately get the following corollary.

Corollary 1.1 *Let p be an odd prime, n be any integer. Let α and k be positive integers. Assume that one of the following conditions holds:*

- (1) $\alpha = 1$;
- (2) $\alpha = 2$ and $(k, p^{\alpha-1}) > 1$;
- (3) $\alpha \geq 2$ and $(k, p^{\alpha-1}) = p^{\alpha-1}$.

Then we have

$$\sum_{\chi \bmod p^\alpha} \sum_{m=1}^{p^\alpha} |S(m, n, k, \chi; p^\alpha)|^4 = 2\phi^3(p^\alpha)p^\alpha(k, \phi(p^\alpha))\left(1 - \frac{d}{2\phi(p)}\right).$$

Theorem 1.2 *Let $q > 2$ be an odd number, n be an integer with $(n, q) = 1$, and let k be a positive integer. Then we have*

$$\begin{aligned} & \sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, k, \chi; q)|^4 \\ &= \prod_{\substack{p^\alpha \parallel q \\ (k, p^{\alpha-1}) = p^\delta \\ \frac{\alpha-2}{2} \leq \delta \leq \alpha-1}} \left(2(k, p-1)\phi^3(p^\alpha)p^{\alpha+\delta} \left(1 - \frac{(k, p-1)}{2\phi(p)} \right) \right) \end{aligned}$$

$$\begin{aligned}
 & + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+\delta-1} (p^{\alpha-\delta-1} - 1) \\
 & \times \prod_{\substack{p^\alpha \parallel q \\ (k, p^{\alpha-1}) = p^\delta \\ 0 \leq \delta \leq \frac{\alpha-3}{2}}} \left(2(k, p-1) \phi^3(p^\alpha) p^{\alpha+\delta} \left(1 - \frac{(k, p-1)}{2\phi(p)} \right) \right. \\
 & + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+\delta-1} (p^{\delta+1} - 1) \\
 & \left. + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+2\delta-1} (p-1)(\alpha-2\delta-2) \right).
 \end{aligned}$$

From Theorem 1.2 we can get the following corollary.

Corollary 1.2 *Let $q > 2$ be an odd number, n be an integer with $(n, q) = 1$, and let k be a positive integer. Assume that one of the following conditions holds:*

- (1) q is square-free;
- (2) $q = p_1^2 \cdots p_r^2$ and $p_1 \cdots p_r \mid k$;
- (3) $q = \prod_{i=1}^r p_i^{\alpha_i}$ and $q \mid k \prod_{i=1}^r p_i$.

Then we have

$$\sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, k, \chi; q)|^4 = 2^{\omega(q)} \phi^3(q) q(k, \phi(q)) \prod_{p^\alpha \parallel q} \left(1 - \frac{(k, p-1)}{2\phi(p)} \right).$$

2 Some lemmas

To complete the proof of theorems, we need the following lemmas.

Lemma 2.1 *Let p be an odd prime, and let α and k be positive integers with $(k, p^{\alpha-1}) = p^\delta$. Write $d = (k, p-1)$. Then we have*

$$\sum_{\substack{a=1 \\ p^\alpha \mid (a^k-1)}}^{p^\alpha} \sum_{\substack{b=1 \\ (b^k-1)}}^{p^\alpha} 1 = \begin{cases} 2d\phi(p^\alpha) p^\delta \left(1 - \frac{d}{2\phi(p)} \right) + d^2 p^{\alpha+\delta-1} (p^{\alpha-\delta-1} - 1), & \frac{\alpha-2}{2} \leq \delta \leq \alpha-1; \\ 2d\phi(p^\alpha) p^\delta \left(1 - \frac{d}{2\phi(p)} \right) + d^2 p^{\alpha+\delta-1} (p^{\delta+1} - 1) \\ + d^2 p^{\alpha+2\delta-1} (p-1)(\alpha-2\delta-2), & 0 \leq \delta \leq \frac{\alpha-3}{2}. \end{cases}$$

Proof For $\alpha = 1$, we get

$$\sum_{\substack{a=1 \\ p^\alpha \mid (a^k-1)}}^{p^\alpha} \sum_{\substack{b=1 \\ (b^k-1)}}^{p^\alpha} 1 = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} 1 = 2d(p-1) - d^2 = 2d\phi(p) \left(1 - \frac{d}{2\phi(p)} \right). \quad (2.1)$$

Now we assume that $\alpha \geq 2$. It is not hard to show that

$$\begin{aligned}
 \sum_{\substack{a=1 \\ p^\alpha \mid (a^k-1)}}^{p^\alpha} \sum_{\substack{b=1 \\ (b^k-1)}}^{p^\alpha} 1 &= \sum_{\substack{\beta=0 \\ \beta+\gamma \geq \alpha}}^{\alpha} \sum_{\gamma=0}^{\alpha} \sum_{\substack{a=1 \\ p^\beta \parallel a^{k-1}}}^{p^\alpha} \sum_{\substack{b=1 \\ p^\gamma \parallel b^{k-1}}}^{p^\alpha} 1 \\
 &= \sum_{\beta=0}^{\alpha} \sum_{\substack{a=1 \\ p^\beta \parallel a^{k-1}}}^{p^\alpha} \sum_{\gamma=\alpha-\beta}^{\alpha} \sum_{\substack{b=1 \\ p^\gamma \parallel b^{k-1}}}^{p^\alpha} 1 = \sum_{\beta=0}^{\alpha} \sum_{a=1}^{p^\alpha} \sum_{\substack{b=1 \\ p^{\alpha-\beta} \parallel b^{k-1}}}^{p^\alpha} 1
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 + \sum_{\beta=1}^{\alpha-1} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 + \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 \\
 &= (\phi(p^\alpha) - p^{\alpha-1}(k, \phi(p))) \cdot (k, \phi(p^\alpha)) + (k, \phi(p^\alpha)) \phi(p^\alpha) \\
 &\quad + \sum_{\beta=1}^{\alpha-1} (p^{\alpha-\beta}(k, \phi(p^\beta)) - p^{\alpha-\beta-1}(k, \phi(p^{\beta+1}))) \cdot p^\beta (k, \phi(p^{\alpha-\beta})) \\
 &= p^{\alpha-1}(2\phi(p) - (k, \phi(p))) \cdot (k, p^{\alpha-1}) \cdot (k, \phi(p)) \\
 &\quad + p^{\alpha-1}(k, \phi(p))^2 \sum_{\beta=1}^{\alpha-1} (p(k, p^{\beta-1}) - (k, p^\beta)) \cdot (k, p^{\alpha-\beta-1}) \\
 &= 2d\phi(p^\alpha)(k, p^{\alpha-1}) \left(1 - \frac{d}{2\phi(p)}\right) \\
 &\quad + d^2 p^{\alpha-1} \sum_{\beta=1}^{\alpha-1} (p(k, p^{\beta-1}) - (k, p^\beta)) \cdot (k, p^{\alpha-\beta-1}). \tag{2.2}
 \end{aligned}$$

If $(k, p^{\alpha-1}) = 1$, by (2.2) we have

$$\sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 = 2d\phi(p^\alpha) \left(1 - \frac{d}{2\phi(p)}\right) + d^2 \phi(p^\alpha)(\alpha - 1). \tag{2.3}$$

If $(k, p^{\alpha-1}) = p^{\alpha-1}$, then

$$\sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 = 2d\phi(p^\alpha) p^{\alpha-1} \left(1 - \frac{d}{2\phi(p)}\right). \tag{2.4}$$

On the other hand, for $\alpha = 2$ and $(k, p^{\alpha-1}) > 1$, we have

$$\sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 = 2d\phi(p^\alpha)(k, p^{\alpha-1}) \left(1 - \frac{d}{2\phi(p)}\right). \tag{2.5}$$

Next we suppose that $\alpha \geq 3$ and $(k, p^{\alpha-1}) = p^\delta$ with $1 \leq \delta \leq \alpha - 2$. Then

$$\begin{aligned}
 &\sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 \\
 &= 2d\phi(p^\alpha)(k, p^{\alpha-1}) \left(1 - \frac{d}{2\phi(p)}\right) + d^2 p^{\alpha-1} \sum_{\beta=1}^{\alpha-1} (p(k, p^{\beta-1}) - (k, p^\beta)) \cdot (k, p^{\alpha-\beta-1}) \\
 &= 2d\phi(p^\alpha) p^\delta \left(1 - \frac{d}{2\phi(p)}\right) + d^2 \phi(p^\alpha) p^\delta \sum_{\beta=\delta+1}^{\alpha-1} (k, p^{\alpha-\beta-1}).
 \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{\beta=\delta+1}^{\alpha-1} (k, p^{\alpha-\beta-1}) &= \sum_{u=0}^{\alpha-\delta-2} (k, p^u) \\ &= \begin{cases} \sum_{u=0}^{\alpha-\delta-2} p^u, & \alpha - \delta - 2 \leq \delta, \\ \sum_{u=0}^{\delta} p^u + \sum_{u=\delta+1}^{\alpha-\delta-2} p^\delta, & \alpha - \delta - 2 \geq \delta + 1 \end{cases} \\ &= \begin{cases} \frac{p^{\alpha-\delta-1}-1}{p-1}, & \delta \geq \frac{\alpha-2}{2}, \\ \frac{p^{\delta+1}-1}{p-1} + (\alpha - 2\delta - 2)p^\delta, & \delta \leq \frac{\alpha-3}{2}. \end{cases} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 \\ p^\alpha | (a^k-1)(b^k-1) \\ = \begin{cases} 2d\phi(p^\alpha)p^\delta(1 - \frac{d}{2\phi(p)}) + d^2p^{\alpha+\delta-1}(p^{\alpha-\delta-1} - 1), & \frac{\alpha-2}{2} \leq \delta \leq \alpha - 2; \\ 2d\phi(p^\alpha)p^\delta(1 - \frac{d}{2\phi(p)}) \\ + d^2p^{\alpha+\delta-1}(p^{\delta+1} - 1) + d^2p^{\alpha+2\delta-1}(p-1)(\alpha - 2\delta - 2), & 1 \leq \delta \leq \frac{\alpha-3}{2}. \end{cases} \end{aligned} \quad (2.6)$$

Now combining (2.1), (2.3)-(2.6) we immediately get

$$\begin{aligned} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1 \\ p^\alpha | (a^k-1)(b^k-1) \\ = \begin{cases} 2d\phi(p^\alpha)p^\delta(1 - \frac{d}{2\phi(p)}) + d^2p^{\alpha+\delta-1}(p^{\alpha-\delta-1} - 1), & \frac{\alpha-2}{2} \leq \delta \leq \alpha - 1; \\ 2d\phi(p^\alpha)p^\delta(1 - \frac{d}{2\phi(p)}) \\ + d^2p^{\alpha+\delta-1}(p^{\delta+1} - 1) + d^2p^{\alpha+2\delta-1}(p-1)(\alpha - 2\delta - 2), & 0 \leq \delta \leq \frac{\alpha-3}{2}, \end{cases} \end{aligned}$$

for $\alpha \geq 1$. □

Lemma 2.2 Let n, k, k_1, k_2 be integers with $(n, k_1k_2) = (k_1, k_2) = 1$. Then for any character $\chi \bmod k_1k_2$, there exist integers n_1 and n_2 with $(n_1, k_1) = (n_2, k_2) = 1$ such that

$$n \equiv n_1k_2^2 + n_2k_1^2 \pmod{k_1k_2},$$

and for these integers we have

$$|S(m, n, k, \chi; k_1k_2)| = |S(m\bar{k}_2, n_1k_2, k, \chi_1; k_1)| \cdot |S(m\bar{k}_1, n_2k_1, k, \chi_2; k_2)|,$$

where $\chi = \chi_1\chi_2$ with $\chi_1 \bmod k_1$ and $\chi_2 \bmod k_2$.

Proof This is Lemma 2.2 of [4]. □

3 Proof of the theorems

First we prove Theorem 1.1. Let p be an odd prime, and let α and k be positive integers. Assume that n is any integer. We have

$$\begin{aligned} |S(m, n, k, \chi; p^\alpha)|^2 &= \sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{ma^k + n\bar{a}^k}{p^\alpha}\right) \sum_{b=1}^{p^\alpha} \bar{\chi}(b) e\left(-\frac{mb^k + n\bar{b}^k}{p^\alpha}\right) \\ &= \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \chi(a) e\left(\frac{mb^k(a^k - 1) + n\bar{b}^k(\bar{a}^k - 1)}{p^\alpha}\right). \end{aligned}$$

Then from the orthogonality relation for characters and the trigonometric identity we get

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha} \sum_{m=1}^{p^\alpha} |S(m, n, k, \chi; p^\alpha)|^4 \\ &= \sum_{\chi \bmod p^\alpha} \sum_{m=1}^{p^\alpha} \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \chi(a) e\left(\frac{mb^k(a^k - 1) + n\bar{b}^k(\bar{a}^k - 1)}{p^\alpha}\right) \\ &\quad \times \sum_{c=1}^{p^\alpha} \sum_{d=1}^{p^\alpha} \bar{\chi}(c) e\left(-\frac{md^k(c^k - 1) + n\bar{d}^k(\bar{c}^k - 1)}{p^\alpha}\right) \\ &= \phi(p^\alpha) \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \sum_{d=1}^{p^\alpha} \sum_{m=1}^{p^\alpha} e\left(\frac{m(a^k - 1)(b^k - d^k) + n(\bar{a}^k - 1)(\bar{b}^k - \bar{d}^k)}{p^\alpha}\right) \\ &= \phi(p^\alpha) \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \sum_{d=1}^{p^\alpha} \sum_{m=1}^{p^\alpha} e\left(\frac{md^k(a^k - 1)(b^k - 1) + n\bar{d}^k(\bar{a}^k - 1)(\bar{b}^k - 1)}{p^\alpha}\right) \\ &= \phi(p^\alpha) \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \sum_{d=1}^{p^\alpha} \sum_{m=1}^{p^\alpha} e\left(\frac{md^k(a^k - 1)(b^k - 1) + n\bar{d}^k\bar{a}^k\bar{b}^k(a^k - 1)(b^k - 1)}{p^\alpha}\right) \\ &= \phi^2(p^\alpha) p^\alpha \sum_{a=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} 1. \end{aligned}$$

Write $d = (k, p - 1)$ and $(k, p^{\alpha-1}) = p^\delta$. By Lemma 2.1 we immediately have

$$\begin{aligned} &\sum_{\chi \bmod p^\alpha} \sum_{m=1}^{p^\alpha} |S(m, n, k, \chi; p^\alpha)|^4 \\ &= \begin{cases} 2d\phi^3(p^\alpha)p^{\alpha+\delta}(1 - \frac{d}{2\phi(p)}) + d^2\phi^2(p^\alpha)p^{2\alpha+\delta-1}(p^{\alpha-\delta-1} - 1), & \frac{\alpha-2}{2} \leq \delta \leq \alpha - 1; \\ 2d\phi^3(p^\alpha)p^{\alpha+\delta}(1 - \frac{d}{2\phi(p)}) + d^2\phi^2(p^\alpha)p^{2\alpha+\delta-1}(p^{\delta+1} - 1) \\ \quad + d^2\phi^2(p^\alpha)p^{2\alpha+2\delta-1}(p - 1)(\alpha - 2\delta - 2), & 0 \leq \delta \leq \frac{\alpha-3}{2}. \end{cases} \end{aligned}$$

This completes the proof of Theorem 1.1.

Now we prove Theorem 1.2. Let $q > 2$ be an odd number, n be an integer with $(n, q) = 1$, and let k be a positive integer. Write

$$q = \prod_{i=1}^r p_i^{\alpha_i} \quad \text{and} \quad m = \sum_{i=1}^r \frac{m_i q}{p_i^{\alpha_i}}.$$

By Lemma 2.2 and Theorem 1.1 we have

$$\begin{aligned}
 & \sum_{\chi \bmod q} \sum_{m=1}^q |S(m, n, k, \chi; q)|^4 \\
 &= \prod_{i=1}^r \left(\sum_{\chi_i \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}} \left| S \left(m_i \frac{q}{p_i^{\alpha_i}}, n_i \frac{q}{p_i^{\alpha_i}}, k, \chi_i; p_i^{\alpha_i} \right) \right|^4 \right) \\
 &= \prod_{i=1}^r \left(\sum_{\chi_i \bmod p_i^{\alpha_i}} \sum_{m_i=1}^{p_i^{\alpha_i}} \left| S \left(m_i, n_i \frac{q}{p_i^{\alpha_i}}, k, \chi_i; p_i^{\alpha_i} \right) \right|^4 \right) \\
 &= \prod_{\substack{p^\alpha \parallel q \\ (k, p^{\alpha-1}) = p^\delta \\ \frac{\alpha-2}{2} \leq \delta \leq \alpha-1}} \left(2(k, p-1) \phi^3(p^\alpha) p^{\alpha+\delta} \left(1 - \frac{(k, p-1)}{2\phi(p)} \right) \right. \\
 &\quad \left. + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+\delta-1} (p^{\alpha-\delta-1} - 1) \right) \\
 &\quad \times \prod_{\substack{p^\alpha \parallel q \\ (k, p^{\alpha-1}) = p^\delta \\ 0 \leq \delta \leq \frac{\alpha-3}{2}}} \left(2(k, p-1) \phi^3(p^\alpha) p^{\alpha+\delta} \left(1 - \frac{(k, p-1)}{2\phi(p)} \right) \right. \\
 &\quad \left. + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+\delta-1} (p^{\delta+1} - 1) \right. \\
 &\quad \left. + (k, p-1)^2 \phi^2(p^\alpha) p^{2\alpha+2\delta-1} (p-1)(\alpha-2\delta-2) \right).
 \end{aligned}$$

This proves Theorem 1.2.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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