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An estimate of Sumudu transforms for Boehmians

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Abstract
The space of Boehmians is constructed using an algebraic approach that utilizes convolution and approximate identities or delta sequences. A proper subspace can be identified with the space of distributions. In this paper, we first construct a suitable Boehmian space on which the Sumudu transform can be defined and the function space S can be embedded. In addition to this, our definition extends the Sumudu transform to more general spaces and the definition remains consistent for S elements. We also discuss the operational properties of the Sumudu transform on Boehmians and finally end with certain theorems for continuity conditions of the extended Sumudu transform and its inverse with respect to δ- and Δ-convergence.
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1 Introduction

The Sumudu transform of one variable function $f(x)$ is introduced as a new integral transform by Watugala in [1] and is given by

$$Sf(t)(y) = \frac{1}{y}\int_{R_+} f(t)\exp\left(\frac{-t}{y}\right)dt, \quad y \in (-\tau_1, \tau_2)$$

over the set of functions

$$A = \left\{f(t) : \exists M, \tau_1, \tau_2 > 0, \left|f(t)\right| < Me^{\frac{t}{\tau_j}}, t \in (-1)^j \times (0, \infty)\right\},$$

where $f(t)$ is a function that can be expressed as a convergent series [2, 3]. The Sumudu transform was applied to solve the ordinary differential equations in control engineering problems; see [3].

The Sumudu transform of the convolution product of f and u is given by

$$S(f \star u)(y) = yf^s(y)u^s(y),$$

where f^s and u^s are the Sumudu transforms of f and u, respectively.

Some of the properties were established by Weerakoon in [4, 5]. In [6], further fundamental properties of this transform were also established by Asiru. Similarly, this transform was applied to a one-dimensional neutron transport equation in [7] by Kadem.

In [8], the Sumudu transform was extended to the distributions and some of their properties were also studied. Recently, this transform has been applied to solve the system of differential equations; see Kılıçman *et al.* in [9].

Note that a very interesting fact about Sumudu transform is that the original function and its Sumudu transform have the same Taylor coefficients except the factor n; see Zhang [10]. Similarly, the Sumudu transform sends combinations $C(m,n)$ into permutations $P(m,n)$, and hence it will be useful in the discrete systems.

The following are the general properties of the Sumudu transform which are auxiliary from the substitution method and the properties of integral operators.

(i) *If k_1 and k_2 are non-negative integers and S_1 and S_2 are the corresponding Sumudu transforms of f_1 and f_2, respectively, then*

$$S(k_1f_1 + k_2f_2)(y) = k_1S_1(y) + k_2S_2(y).$$

(ii) $Sf(kt)(y) = S(ky)$, $k \in R_+$.

(iii) $\lim_{t\to 0} f(t) = \lim_{u\to 0} S(y) = f(0)$, *where $S(y)$ is the Sumudu transform of f.*

More properties of the Sumudu transforms a long with a some of applications were given in [11] and [12].

2 Boehmian space

Boehmians were first constructed as a generalization of regular Mikusinski operators [13]. The minimal structure necessary for the construction of Boehmians consists of the following elements:

(i) a nonempty set $\mathbb{A}$;

(ii) a commutative semigroup $(\mathbb{B}, *)$;

(iii) an operation $\odot : \mathbb{A} \times \mathbb{B} \to \mathbb{A}$ such that for each $x \in \mathbb{A}$ and $s_1, s_2, \in \mathbb{B}$, $x \odot (s_1 * s_2) = (x \odot s_1) \odot s_2$;

(iv) a collection $\Delta \subset \mathbb{B}^N$ such that

(a) If $x, y \in \mathbb{A}$, $(s_n) \in \Delta$, $x \odot s_n = y \odot s_n$ for all n, then $x = y$;

(b) If $(s_n), (t_n) \in \Delta$, then $(s_n * t_n) \in \Delta$.

Elements of Δ are called delta sequences. Consider

$$\mathbb{Q} = \{(x_n, s_n) : x_n \in \mathbb{A}, (s_n) \in \Delta, x_n \odot s_m = x_m \odot s_n, \forall m, n \in \mathbf{N}\}.$$

Now if $(x_n, s_n), (y_n, t_n) \in \mathbb{Q}$, $x_n \odot t_m = y_m \odot s_n$, $\forall m, n \in \mathbf{N}$, then we say $(x_n, s_n) \sim (y_n, t_n)$. The relation $\sim$ is an equivalence relation in $\mathbb{Q}$. The space of equivalence classes in $\mathbb{Q}$ is denoted by β. Elements of β are called Boehmians.

We note that between $\mathbb{A}$ and β there is a canonical embedding expressed as $x \to \frac{x \odot s_n}{s_n}$. The operation $\odot$ can also be extended to $\beta \times \mathbb{A}$ by $\frac{x_n}{s_n} \odot t = \frac{x_n \odot t}{s_n}$. The relationship between the notion of convergence and the product $\odot$ is given by:

(i) If $f_n \to f$ as $n \to \infty$ in $\mathbb{A}$ and $\phi \in \mathbb{B}$ is any fixed element, then $f_n \odot \phi \to f \odot \phi$ in A (as $n \to \infty$);

(ii) If $f_n \to f$ as $n \to \infty$ in $\mathbb{A}$ and $(\delta_n) \in \Delta$, then $f_n \odot \delta_n \to f$ in $\mathbb{A}$ (as $n \to \infty$).

The operation $\odot$ can be extended to $\beta \times \mathbb{B}$ as follows: *If* $[\frac{f_n}{s_n}] \in \beta$ *and* $\phi \in \mathbb{B}$, *then* $[\frac{f_n}{s_n}] \odot \phi = [\frac{f_n \odot \phi}{s_n}]$. In β, there are two types of convergence as follows.

(1) *A sequence* (h_n) *in* β *is said to be* δ*-convergent to* h *in* β*, denoted by* $h_n \xrightarrow{\delta} h$*, if there exists* $(s_n) \in \Delta$ *such that* $(h_n \odot s_n), (h \odot s_n) \in \mathbb{A}$*,* $\forall k, n \in \mathbf{N}$*, and* $(h_n \odot s_k) \to (h \odot s_k)$ *as* $n \to \infty$ *in* $\mathbb{A}$ *for every* $k \in \mathbf{N}$.
(2) *A sequence* (h_n) *in* β *is said to be* Δ*-convergent to* h *in* β*, denoted by* $h_n \xrightarrow{\Delta} h$*, if there exists a* $(s_n) \in \Delta$ *such that* $(h_n - h) \odot s_n \in \mathbb{A}$*,* $\forall n \in \mathbf{N}$*, and* $(h_n - h) \odot s_n \to 0$ *as* $n \to \infty$ *in* $\mathbb{A}$.

For further discussion, see [14–16].

3 The Boehmian space $\mathbb{H}(\mathbb{Y})$

Denote by $\mathbb{S}_+(\mathbb{R})$ and $\mathbb{D}_+(\mathbb{R})$ the space of all rapidly decreasing functions over $\mathbb{R}_+$ ($\mathbb{R}_+ = (0, \infty)$) and the space of all test functions of compact support, respectively. In what follows, we obtain preliminary results required to construct the Boehmian space $\mathbb{H}(\mathbb{Y})$, where $\mathbb{Y} = (\mathbb{S}_+, \mathbb{D}_+, \Delta_+)$.

Lemma 3.1

(1) *If* $u_1, u_2 \in \mathbb{D}_+(\mathbb{R})$*, then* $u_1 \star u_2 \in \mathbb{D}_+(\mathbb{R})$.
(2) *If* $f_1, f_2 \in \mathbb{S}_+(\mathbb{R})$ *and* $u_1 \in \mathbb{D}_+(\mathbb{R})$*, then* $(f_1 + f_2) \star u_1 = f_1 \star u_1 + f_2 \star u_1$.
(3) $u_1 \star u_2 = u_2 \star u_1$, $\forall u_1, u_2 \in \mathbb{D}_+(\mathbb{R})$.
(4) *If* $f \in S_+$, $u_1, u_2 \in \mathbb{D}_+(\mathbb{R})$*, then* $(f \star u_1) \star u_2 = f \star (u_1 \star u_2)$.

Proofs are analogous to those of classical cases and details are omitted.

Definition 3.2 A sequence (s_n) of functions from $\mathbb{D}_+(\mathbb{R})$ is said to be in Δ_+ if and only if

$$\Delta_+^1 \quad \int_{\mathbf{R}_+} s_n(x)\,dx = 1;$$

$$\Delta_+^2 \quad \int_{\mathbf{R}_+} \left|s_n(x)\right|dx \le M, \quad M \text{ is a positive integer;}$$

$$\Delta_+^3 \quad \operatorname{supp} s_n(x) \subset (0, \varepsilon_n), \quad \varepsilon_n \to 0 \text{ as } n \to \infty.$$

This means that (s_n) shrinks to zero as $n \to \infty$. Each member of Δ_+ is called a delta sequence or an approximate identity or, sometimes, a summability kernel. Delta sequences, in general, appear in many branches of mathematics, but probably the most important applications are those in the theory of generalized functions. The basic use of delta sequences is the regularization of generalized functions, and further, they can be used to define the convolution product and the product of generalized functions.

Lemma 3.3 *If* $(s_n), (t_n) \in \Delta_+$*, then* $\operatorname{supp}(s_n \star t_n) \subset \operatorname{supp} s_n + \operatorname{supp} t_n$.

Lemma 3.4 *If* $u_1, u_2 \in \mathbb{D}_+(\mathbb{R})$*, then so is* $u_1 \star u_2$ *and*

$$\int_{\mathbb{R}_+} |u_1 \star u_2| \le \int_{\mathbb{R}_+} |u_1| \cdot \int_{\mathbb{R}_+} |u_2|.$$

Theorem 3.5 *Let* $f_1, f_2 \in \mathbb{S}_+(\mathbb{R})$ *and* $(s_n) \in \Delta_+$ *such that* $f_1 \star s_n = f_2 \star s_n$*,* $n = 1, 2, 3, \ldots$*, then* $f_1 = f_2$ *in* $\mathbb{S}_+(\mathbb{R})$.

Proof We show that $f_1 \star s_n = f_1$ in $\mathbb{S}_+(\mathbb{R})$. Let K be a compact set containing the supp s_n for every $n \in \mathbf{N}$. Using Δ_+^1, we write

$$\left|x^k D^m(f_1 \star s_n - f_1)(x)\right| \le \int_K \left|s_n(t)\right|\left|x^k D^m\big(f_1(x-t) - f_1(x)\big)\right| dt. \tag{3.1}$$

The mapping $t \to f_1^t$, where $f_1^t(x) = f_1(x-t)$, is uniformly continuous from $\mathbb{R}_+ \to \mathbb{R}_+$. From the hypothesis that supp $s_n \to 0$ as $n \to \infty$ (by Δ_+^3), we choose $r > 0$ such that supp $s_n \subseteq [0, r]$ for large n and $t < r$. This implies

$$\left|f_1(x-t) - f_1(x)\right| = \left|f_1^t - f_1\right| < \frac{\varepsilon_n}{M}. \tag{3.2}$$

Hence using Δ_+^2 and (3.2), (3.1) becomes

$$\left|x^k D^m(f_1 \star s_n - f_1)(x)\right| < \varepsilon_n \to 0 \quad \text{as } n \to \infty.$$

Thus $f_1 \star s_n \to f_1$ in $\mathbb{S}_+(\mathbb{R})$. Similarly, we show that $f_2 \star s_n \to f_2$. This completes the proof of the theorem. □

Theorem 3.6 *If* $\lim_{n\to\infty} f_n = f$ *in* $\mathbb{S}_+(\mathbb{R})$ *and* $u \in \mathbb{D}_+(\mathbb{R})$, *then*

$$\lim_{n\to\infty} f_n \star u = f \star u.$$

Proof In view of the hypothesis of the theorem, we write

$$\left|x^k D^m(f_n \star u - f \star u)(x)\right| = \left|x^k\big(D^m(f_n - f) \star u\big)(x)\right|. \tag{3.3}$$

The last equation follows from the fact that [17]

$$D^m f \star u = D^m f \star u = f \star D^m u.$$

Hence, for each $u \in \mathbb{D}_+(\mathbb{R})$, we have

$$\begin{aligned}\left|x^k D^m(f_n \star u - f \star u)(x)\right| &\le \int_K x^k \left|D^m(f_n - f)(x-t)\right|\left|u(t)\right| dt \\ &\le M\gamma_k(f_n - f) \quad \text{for some constant } M \\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

The proof of the theorem is completed. □

Theorem 3.7 *If* $\lim_{n\to\infty} f_n = f$ *in* $\mathbb{S}_+(\mathbb{R})$ *and* $(s_n) \in \Delta_+$, *then* $\lim_{n\to\infty} f_n \star s_n = f$.

Proof In view of the analysis employed for Theorem 3.5, we get

$$\lim_{n\to\infty} f_n \star s_n = f_n \to f \quad \text{as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} f_n \star s_n = f \quad \text{as } n \to \infty.$$

This completes the proof. The Boehmian space $\mathbb{H}(\mathbb{Y})$ is therefore constructed. □

The canonical embedding between $\mathbb{S}_+(\mathbb{R})$ and $\mathbb{H}(\mathbb{Y})$ is expressed as $x \to [\frac{x\star s_n}{s_n}]$. The extension of $\star$ to $\mathbb{H}(\mathbb{Y}) \times \mathbb{S}_+$ is given by $[\frac{x_n}{s_n}] \star t = [\frac{x_n \star t}{s_n}]$. Convergence in $\mathbb{H}(\mathbb{Y})$ is defined in a natural way:

δ-convergence: *A sequence (h_n) in $\mathbb{H}(\mathbb{Y})$ is said to be δ-convergent to h in $\mathbb{H}(\mathbb{Y})$, denoted by $h_n \xrightarrow{\delta} h$, if there exists a delta sequence (s_n) such that $(h_n \star s_n), (h \star s_n) \in \mathbb{S}_+(\mathbb{R})$, $\forall k, n \in \mathbf{N}$, and $(h_n \star s_k) \to (h \star s_k)$ as $n \to \infty$ in $\mathbb{S}_+(\mathbb{R})$ for every $k \in \mathbf{N}$.*

Δ_+-convergence: *A sequence (h_n) in $\mathbb{H}(\mathbb{Y})$ is said to be Δ_+-convergent to h in $\mathbb{H}(\mathbb{Y})$, denoted by $h_n \xrightarrow{\Delta} h$, if there exists a $(s_n) \in \Delta_+$ such that $(h_n - h) \star s_n \in \mathbb{S}_+(\mathbb{R})$, $\forall n \in N$, and $(h_n - h) \star s_n \to 0$ as $n \to \infty$ in $\mathbb{S}_+(\mathbb{R})$.*

Theorem 3.8 *The mapping $f \to [\frac{f\star s_n}{s_n}]$ is a continuous embedding of $\mathbb{S}_+(\mathbb{R})$ into $\mathbb{H}(\mathbb{Y})$.*

Proof The mapping is one-to-one. For detailed proof, let $[\frac{f_1\star s_n}{s_n}] = [\frac{f_2\star t_n}{t_n}]$, then $(f_1 \star s_n) \star t_m = (f_2 \star t_m) \star s_n$. Then since $(s_n), (t_n) \in \Delta_+$, $f_1 \star (s_m \star t_n) = f_2 \star (t_n \star s_m) = f_2 \star (s_m \star t_n)$. Using Theorem 3.5, we get $f_1 = f_2$. To show the mapping is continuous, let $f_n \to 0$ as $n \to \infty$ in $\mathbb{S}_+(\mathbb{R})$. Then we show that

$$\left[\frac{f_n \star s_m}{s_m}\right] \xrightarrow{\delta} 0 \quad \text{as } n \to \infty.$$

From Theorem 3.5, $[\frac{f_n\star s_m}{s_m}] \star s_m = f_n \star s_m \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

Theorem 3.9 *Let $f \in \mathbb{S}_+(\mathbb{R})$ and $u \in \mathbb{D}_+(\mathbb{R})$, then*

$$S(f \star u)(y) = yf^s(y)u^s(y).$$

4 The Boehmian space $\mathbb{H}(\mathbb{Y}^s)$

We describe another Boehmian space as follows. Let $\mathbb{S}_+(\mathbb{R})$ be the space of rapidly decreasing functions [17]. Define

$$\mathbb{D}_+^s(\mathbb{R}) = \{u^s : \text{ for all } u \in \mathbb{D}_+(\mathbb{R})\}, \tag{4.1}$$

where u^s denotes the Sumudu transform of u. We also define $f \bullet u^s$ by

$$(f \bullet u^s)(y) = yf(y)u^s(y). \tag{4.2}$$

Lemma 4.1 *Let $f \in \mathbb{S}_+(\mathbb{R})$ and $u^s \in \mathbb{D}_+^s(\mathbb{R})$, then $f \bullet u^s \in \mathbb{S}_+(\mathbb{R})$.*

Proof If $f \in \mathbb{S}_+(\mathbb{R})$ and $u^s \in \mathbb{D}_+^s(\mathbb{R})$, then using the topology of $\mathbb{S}_+(\mathbb{R})$ and Leibnitz' theorem, we get

$$\begin{aligned} \left|x^k D_x^m\left(f \bullet u^s\right)(x)\right| &\le \left|x^k \sum_{j=1}^m D^{m-j}\left(xf(x)\right) D^j u^s(x)\right| \\ &\le \sum_{j=1}^m \left|x^k D^{m-j}\left(xf(x)\right)\right|\left|D^j u^s(x)\right| \\ &= \sum_{j=1}^m \left|x^k D^{m-j} f_1(x)\right|\left|\int_K u(t) D_x^j \frac{e^{-\frac{t}{x}}}{x}\, dt\right|, \end{aligned}$$

where $f_1(x) = xf(x) \in \mathbb{S}_+(\mathbb{R})$ and K is a compact subset containing the supp $u(t)$. Hence

$$\left|x^k D_x^m\left(f \bullet u^s\right)(x)\right| \le M \gamma_{k,m-j}(f_1) < \infty$$

for some positive constant M. This completes the proof of the lemma. □

Lemma 4.2 *The mapping*

$$\begin{aligned} &\mathbb{S}_+ \times \mathbb{D}_+^s \to \mathbb{S}_+, \\ &\left(f, u^s\right) \to f \bullet u^s \end{aligned}$$

satisfies the following properties:

(1) *If* $u_1^s, u_2^s \in \mathbb{D}_+^s(\mathbb{R})$, *then* $u_1^s \bullet u_2^s \in \mathbb{D}_+^s(\mathbb{R})$.
(2) *If* $f_1, f_2 \in \mathbb{S}_+(\mathbb{R})$, $u^s \in \mathbb{D}_+^s(\mathbb{R})$, *then* $(f_1 + f_2) \bullet u^s = f_1 \bullet u^s + f_2 \bullet u^s$.
(3) *For* $u_1^s, u_2^s \in \mathbb{D}_+^s(\mathbb{R})$, $u_1^s \bullet u_2^s = u_2^s \bullet u_1^s$.
(4) *For* $f \in \mathbb{S}_+(\mathbb{R})$, $u_1^s, u_2^s \in \mathbb{D}_+^s(\mathbb{R})$, *then* $(f \bullet u_1^s) \bullet u_2^s = f \bullet (u_1^s \bullet u_2^s)$.

Proof The proof of the above lemma is straightforward. Detailed proof is as follows.

Proof of (1). Let $u_1, u_2 \in \mathbb{D}_+(\mathbb{R})$, then $u_1 \star u_2 \in \mathbb{D}_+(\mathbb{R})$. Hence $(u_1 \star u_2)^s \in \mathbb{D}_+^s(\mathbb{R})$ by (4.1). Theorem 3.9 implies $u_1^s \bullet u_2^s \in \mathbb{D}_+^s(\mathbb{R})$.

Proof of (2) is obvious.

Proof of (3). We have

$$\begin{aligned} \left(u_1^s \bullet u_2^s\right)(x) &= x u_1^s(x) u_2^s(x) \\ &= x u_2^s(x) u_1^s(x) \\ &= \left(u_2^s \bullet u_1^s\right)(x). \end{aligned}$$

Hence $u_1^s \bullet u_2^s = u_2^s \bullet u_1^s$.

Proof of (4). Let $f \in \mathbb{S}_+(\mathbb{R})$, $u_1^s, u_2^s \in \mathbb{D}_+^s(\mathbb{R})$, then

$$\begin{aligned} \left(\left(f \bullet u_1^s\right) \bullet u_2^s\right)(x) &= x\left(f \bullet u_1^s\right)(x) u_2^s \\ &= xxf(x) u_1^s(x) u_2^s(x) \\ &= xf(x) x u_1^s(x) u_2^s(x) \end{aligned}$$

$$\begin{aligned} &= xf(x)\bigl(u_1^s \bullet u_2^s\bigr)(x) \\ &= f \bullet \bigl(u_1^s \bullet u_2^s\bigr)(x), \end{aligned}$$

that is,

$$\bigl(f \bullet u_1^s\bigr) \bullet u_2^s = f \bullet \bigl(u_1^s \bullet u_2^s\bigr).$$

This completes the proof of the theorem. □

Denote by Δ_+^s the set of all Sumudu transforms of delta sequences from Δ_+. That is,

$$\Delta_+^s = \bigl\{\bigl(s_n^s\bigr) : (s_n) \in \Delta_+, \forall n \in \mathbf{N}\bigr\}. \tag{4.3}$$

Lemma 4.3 *Let* $f_1, f_2 \in \mathbb{S}_+(\mathbb{R})$, $(s_n^s) \in \Delta_+^s$ *be such that* $f_1 \bullet s_n^s = f_2 \bullet s_n^s$, $\forall n$, *then* $f_1 = f_2$ *in* $\mathbb{S}_+(\mathbb{R})$.

Proof Let $f_1, f_2 \in \mathbb{S}_+(\mathbb{R})$ and $(s_n^s) \in \Delta_+^s$. Since $f_1 \bullet s_n^s = f_2 \bullet s_n^s$, (4.2) implies $xf_1(x)s_n^s(x) = xf_2(x)s_n^s(x)$. Hence $f_1(x) = f_2(x)$ for all x. The proof is completed. □

Lemma 4.4 *For each* $(s_n), (t_n) \in \Delta_+$, $(s_n^s \bullet t_n^s) \in \Delta_+^s$.

Proof Since $(s_n), (t_n) \in \Delta_+$, $s_n \star t_n \in \Delta_+$ for all n. Hence, from Theorem 3.9, we get $S(s_n \star t_n)(x) = xs_n^s(x)t_n^s(x) = s_n^s \bullet t_n^s \in \Delta_+^s$ for every n. This completes the proof of the lemma. □

By aid of Lemma 4.3. and Lemma 4.4, Δ_+^s can be regarded as a delta sequence.

Lemma 4.5 *Let* $f_n \to f$ *in* $\mathbb{S}_+(\mathbb{R})$, $u^s \in \mathbb{D}_+^s(\mathbb{R})$, *then* $f_n \bullet u^s \to f \bullet u^s$ *in* $\mathbb{S}_+(\mathbb{R})$.

Proof It is clear that u^s is bounded in $\mathbb{D}_+^s(\mathbb{R})$. Further,

$$\begin{aligned} \bigl(f_n \bullet u^s\bigr)(x) &\to xf(x)u^s(x) \\ &\to \bigl(f_n \bullet u^s\bigr)(x). \end{aligned}$$

Hence $(f_n \bullet u^s) \to f \bullet u^s$. □

Lemma 4.6 *Let* $f_n \to f$ *in* $\mathbb{S}_+(\mathbb{R})$, $(s_n^s) \in \Delta_+^s$, *then* $f_n \bullet s_n^s \to f$ *in* $\mathbb{S}_+(\mathbb{R})$.

Proof Let $(s_n) \in \Delta_+$, then $s_n^s(x) \to \frac{1}{x}$ uniformly on compact subsets of $\mathbb{R}_+$. Hence

$$\begin{aligned} \bigl|x^k D_x^m\bigl(f_n \bullet s_n^s - f\bigr)(x)\bigr| &= \bigl|x^k D_x^m\bigl(xf_n(x)s_n^s(x) - f(x)\bigr)\bigr| \\ &\to \bigl|x^k D_x^m(f_n - f)(x)\bigr| \end{aligned}$$

as $n \to \infty$. Thus $|x^k D_x^m(f_n \bullet s_n^s - f)(x)| \to 0$ as $n \to \infty$. This yields $f_n \bullet s_n^s \to f$ in the topology of $\mathbb{S}_+(\mathbb{R})$. The proof is therefore completed. The space $\mathbb{H}(\mathbb{Y}^s)$ can be regarded as a Boehmian space, where $\mathbb{Y}^s = (\mathbb{S}_+, \mathbb{D}_+^s, \Delta_+^s)$. □

Lemma 4.7 *The mapping*

$$f \to \left[\frac{f \bullet s_n^s}{s_n^s}\right] \tag{4.4}$$

is a continuous embedding of $\mathbb{S}_+(\mathbb{R})$ *into* $\mathbb{H}(\mathbb{Y}^s)$.

Proof For $f \in \mathbb{S}_+(\mathbb{R})$, $s_n^s \in \Delta_+^s$, $\frac{f \bullet s_n^s}{s_n^s}$ is a quotient of sequences in the sense that $(f \bullet s_n^s) \bullet s_m^s = f \bullet (s_m^s \bullet s_n^s)$. We show that the map (4.4) is one-to-one. Let $[\frac{f_1 \bullet s_n^s}{s_n^s}] = [\frac{f_2 \bullet t_n^s}{t_n^s}]$, then $(f_1 \bullet s_n^s) \bullet t_m^s = (f_2 \bullet t_m^s) \bullet s_n^s$, $m, n \in \mathbf{N}$. Using Lemma 4.2 and Lemma 4.3, we conclude $f_1 = f_2$. □

To establish the continuity of (4.4), let $f_n \to 0$ as $n \to \infty$ in $\mathbb{S}_+(\mathbb{R})$. Then $f_n \bullet s_n^s \to 0$ as $n \to \infty$ by Lemma 4.6, and hence

$$\left[\frac{f_n \bullet s_n^s}{s_n^s}\right] \to 0$$

as $n \to \infty$ in $\mathbb{H}(\mathbb{Y}^s)$. This completes the proof of the lemma.

5 The Sumudu transform of Boehmians

Let $\beta = [\frac{f_n}{s_n}] \in \mathbb{H}(\mathbb{Y})$, then we define the Sumudu transform of β by the relation

$$\beta_1^s = \left[\frac{f_n^s}{s_n^s}\right] \quad \text{in } \mathbb{H}(\mathbb{Y}^s). \tag{5.1}$$

Theorem 5.1 $\beta_1^s : \mathbb{H}(\mathbb{Y}) \to \mathbb{H}(\mathbb{Y}^s)$ *is well defined.*

Proof Let $\beta_1 = \beta_2 \in \mathbb{H}(\mathbb{Y})$, where $\beta_1 = [\frac{f_n}{s_n}]$, $\beta_2 = [\frac{g_n}{t_n}]$. Then the concept of quotients yields $f_n \star t_m = g_m \star s_n$. Employing Theorem 3.9, we get $xf_n^s(x)t_m^s(x) = xg_m^s(x)s_n^s(x)$, *i.e.*, $f_n^s \bullet t_m^s = g_m^s \bullet s_n^s$. Equivalently, $\frac{f_n^s}{s_n^s} \sim \frac{g_n^s}{t_n^s}$. Thus $\beta_1^s = \beta_2^s$. This completes the proof of the theorem. □

Theorem 5.2 $\beta^s : \mathbb{H}(\mathbb{Y}) \to \mathbb{H}(\mathbb{Y}^s)$ *is continuous with respect to* δ*-convergence.*

Proof Let $\beta_n \to 0$ in $\mathbb{H}(\mathbb{Y})$, then by [14], $\beta_n = [\frac{f_{n,k}}{s_k}]$ and $f_{n,k} \to 0$ as $n \to \infty$ in $\mathbb{S}_+(\mathbb{R})$. Applying the Sumudu transform to both sides yields $f_{n,k}^s \to 0$ as $n \to \infty$. Hence

$$\beta_n^s = \left[\frac{f_{n,k}^s}{s_k^s}\right] \to 0$$

as $n \to \infty$ in $\mathbb{H}(\mathbb{Y}^s)$. This proves the theorem. □

Theorem 5.3 $\beta^s : \mathbb{H}(\mathbb{Y}) \to \mathbb{H}(\mathbb{Y}^s)$ *is a one-to-one mapping.*

Proof Assume $\beta_1^s = [\frac{f_n^s}{s_n^s}] = [\frac{g_n^s}{t_n^s}] = \beta_2^s$, then $f_n^s \bullet t_m^s = g_m^s \bullet s_n^s$. Hence

$$(f_n \star t_m)^s = (g_m \star s_n)^s.$$

Since the Sumudu transform is one-to-one, we get $f_n \star t_m = g_m \star s_n$. Thus

$$\frac{f_n}{s_n} \sim \frac{g_n}{t_n}.$$

Hence

$$\left[\frac{f_n}{s_n}\right] = \beta_1 = \left[\frac{g_n}{t_n}\right] = \beta_2.$$

This completes the proof of the theorem. □

Theorem 5.4 *Let $\beta_1, \beta_2 \in \mathbb{H}(\mathbb{Y})$, then*

(1) $(\beta_1 + \beta_2)^s = \beta_1^s + \beta_2^s$;
(2) $(k\beta)^s = k\beta^s$, $\lambda \in \mathbb{C}$.

Proof is immediate from the definitions.

Theorem 5.5 $\beta^s : \mathbb{H}(\mathbb{Y}) \to \mathbb{H}(\mathbb{Y}^s)$ *is continuous with respect to Δ_+-convergence.*

Proof Let $\beta_n \xrightarrow{\Delta} \beta$ in $\mathbb{H}(\mathbb{Y})$ as $n \to \infty$. Then there exist $f_n \in \mathbb{S}_+(\mathbb{R})$ and $(s_n) \in \Delta_+$ such that $(\beta_n - \beta) \star s_n = [\frac{f_n \star s_k}{s_k}]$ and $f_n \to 0$ as $n \to \infty$. Employing Eq. (5.1), we get

$$S\big((\beta_n - \beta) \star s_n\big) = \left[\frac{S(f_n \star s_k)}{s_k^s}\right].$$

Hence, we have $S((\beta_n - \beta) \star s_n) = [\frac{y f_n^s s_k^s}{s_k^s}] \to 0$ as $n \to \infty$ in $\mathbb{H}(\mathbb{Y}^s)$. Therefore

$$\begin{aligned} S\big((\beta_n - \beta) \star s_n\big) &= y\big(\beta_n^s - \beta^s\big)s_n^s \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence, $\beta_n^s \xrightarrow{\Delta} \beta^s$ as $n \to \infty$. □

Theorem 5.6 $\beta^s : \mathbb{H}(\mathbb{Y}) \to \mathbb{H}(\mathbb{Y}^s)$ *is onto.*

Proof Let $[\frac{f_n^s}{s_n^s}] \in \mathbb{H}(\mathbb{Y}^s)$ be arbitrary, then $f_n^s \bullet s_m^s = f_m^s \bullet s_n^s$ for every $m, n \in \mathbf{N}$. Then $f_n \star s_m = f_m \star s_n$. That is, $\frac{f_n}{s_n}$ is the corresponding quotient of sequences of $\frac{f_n^s}{s_n^s}$. Thus $[\frac{f_n}{s_n}] \in \mathbb{H}(\mathbb{Y})$ is such that $S[\frac{f_n}{s_n}] = [\frac{f_n^s}{s_n^s}]$ in $\mathbb{H}(\mathbb{Y}^s)$. This completes the proof of the lemma.

Let $\beta^s = [\frac{f_n^s}{s_n^s}] \in \mathbb{H}(\mathbb{Y}^s)$, then we define the inverse Sumudu transform of β^s by

$$\beta^{s^{-1}} = \left[\frac{f_n}{s_n}\right]$$

in the space $\mathbb{H}(\mathbb{Y})$. □

Theorem 5.7 *Let $[\frac{f_n^s}{s_n^s}] \in \mathbb{H}(\mathbb{Y}^s)$ and $u \in \mathbb{D}_+(\mathbb{R})$, $u^s \in \mathbb{D}_+^s(\mathbb{R})$*

$$\beta\left(\left[\frac{f_n}{s_n}\right] \star u\right) = \left[\frac{f_n^s}{s_n^s}\right] \bullet u \quad \text{and} \quad \beta^{s^{-1}}\left(\left[\frac{f_n^s}{s_n^s}\right] \bullet u^s\right) = \left[\frac{f_n}{s_n}\right] \star u.$$

Proof is immediate from the definitions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both of the authors contributed equally to the manuscript and read and approved the final draft.

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