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The poles and growth of solutions of systems of complex difference equations

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Abstract

In view of the Nevanlinna theory, we study the growth and poles of solutions of some classes of systems of complex difference equations and obtain some interesting results such as the lower bounds for Nevanlinna lower order, a counting function of poles and maximum modulus for solutions of such systems. They extend some results concerning functional equations to the systems of functional equations in the fields of complex equations.

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1 Introduction and main results

The purpose of this paper is to study some properties of the poles and growth of meromorphic solutions of the systems of complex difference equations. The fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions are used (see [1–3]). In this paper, a meromorphic function means being meromorphic in the whole complex plane $\mathbb{C}$; for a meromorphic function f, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t} < \infty$, and a meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a(z)) = S(r,f) = o(T(r,f))$.

In 1980, Shimomura [4] and Yanagihara [5] studied some existence of solutions of difference equations and obtained some theorems as follows.

Theorem 1.1 (see [4, Theorem 2.5]) *For any non-constant polynomial $P(y)$, the difference equation*

$$y(z+1) = P\bigl(y(z)\bigr)$$

has a non-trivial entire solution.

Theorem 1.2 (see [5, Corollary 6]) *For any non-constant rational function $R(y)$, the difference equation*

$$y(z+1) = R\bigl(y(z)\bigr)$$

has a non-trivial meromorphic solution in the complex plane.

It was proposed that the existence of sufficiently many meromorphic solutions of finite order would be a strong indicator of integrability of an equation (see [6–8]).

In 2000, Ablowitz, Halburd and Herbst [6] studied some classes of complex difference equations

$$f(z+1)+f(z-1)=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{1}$$

$$f(z+1)f(z-1)=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q}, \tag{2}$$

where the coefficients are meromorphic functions, and obtained the following results.

Theorem 1.3 (see [6]) *If difference equation* (1) (*or* (2)) *with polynomial coefficients* $a_i(z)$, $b_i(z)$ *admits a transcendental meromorphic solution of finite order, then* $d=\max\{p,q\}\le 2$.

In 2001, Heittokangas *et al.* [9] further investigated some complex difference equations which are similar to (1) and (2) and obtained the following results which are improvements of Theorems 1.1 and 1.2.

Theorem 1.4 (see [9, Proposition 8 and Proposition 9]) *Let* $c_1,\ldots,c_n\in\mathbb{C}\setminus\{0\}$. *If the equations*

$$\sum_{i=1}^{n} f(z+c_i)=R\bigl(z,f(z)\bigr),\qquad \prod_{i=1}^{n} f(z+c_i)=R\bigl(z,f(z)\bigr),$$

$$R\bigl(z,f(z)\bigr):=\frac{P(z,f(z))}{Q(z,f(z))}=\frac{a_0(z)+a_1(z)f+\cdots+a_s(z)f^s}{b_0(z)+b_1(z)f+\cdots+b_t(z)f^t}$$

with rational coefficients $a_i(z)$, $b_i(z)$ *admit a transcendental meromorphic solution of finite order, then* $d=\max\{s,t\}\le n$.

In the same paper, some results of the lower bound for the characteristic functions, poles and maximum modulus of transcendental meromorphic solutions of some complex difference equations are obtained as follows.

Theorem 1.5 (see [9, Theorem 10]) *Let* $c_1,\ldots,c_n\in C\setminus\{0\}$ *and let* $m\ge 2$. *Suppose* y *is a transcendental meromorphic solution of the difference equation*

$$\sum_{i=1}^{n} a_i(z)y(z+c_i)=\sum_{i=0}^{m} b_i(z)y(z)^i$$

with rational coefficients $a_i(z)$, $b_i(z)$. *Denote* $C:=\max\{|c_1|,\ldots,|c_n|\}$.

(1) *If* y *is entire or has finitely many poles, then there exist constants* $K>0$ *and* $r_0>0$ *such that*

$$\log M(r,y)\ge Km^{r/C}$$

holds for all $r\ge r_0$.

(2) *If y has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that*

$$n(r,y) \geq Km^{r/c}$$

holds for all $r \geq r_0$.

Theorem 1.6 (see [9, Theorem 11]) *Let $c_1, \dots, c_n \in \mathbb{C}\backslash\{0\}$ and suppose that y is a non-rational meromorphic solution of*

$$\sum_{i=1}^{n} d_i(z)y(z+c_i) = \frac{a_0(z) + a_1(z)f + \cdots + a_p(z)f^p}{b_0(z) + b_1(z)f + \cdots + b_q(z)f^q}, \tag{3}$$

where all coefficients in (3) *are of growth $o(T(r,y))$ without an exceptional set as $r \to \infty$, and d_i's are non-vanishing. If $d = \max\{p,q\} > n$, then for any ε $(0 < \varepsilon < (d-n)/(d+n))$, there exists an $r_0 > 0$ such that*

$$T(r,y) \geq K\left(\frac{d}{n}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\right)^{r/C}$$

for all $r \geq r_0$, where $C := \max\{|c_1|, \dots, |c_n|\}$ and $K > 0$ is a constant.

Recently, a number of papers have focused on difference equations, difference product and q-difference in the complex plane $\mathbb{C}$, and considerable attention has been paid to the growth of solutions of difference equations, value distribution and uniqueness of differences analogues of Nevanlinna's theory [4, 6–25].

In 2012, Gao [14–16] also investigated the growth and existence of meromorphic solutions of some systems of complex difference equations and obtained some existence theorems and estimates on the proximity function and the counting function of solutions of some systems of complex difference equations.

Inspired by the ideas of Refs. [14–16] and Ref. [9], we investigate the growth and poles of meromorphic solutions of some systems of complex difference equations and obtain the following results.

Theorem 1.7 *Suppose that (f_1,f_2) is a transcendental meromorphic solution of a system of difference equations of the form*

$$\begin{cases} \sum_{j=1}^{n} a_j^1(z)f_1(z+c_j) = \sum_{i=0}^{d_1} b_i^1(z)f_2(z)^i, \\ \sum_{j=1}^{n} a_j^2(z)f_2(z+c_j) = \sum_{i=0}^{d_2} b_i^2(z)f_1(z)^i, \end{cases} \tag{4}$$

where $d_1d_2 \geq 2$ and the coefficients $a_j^t(z)$, $b_i^t(z)$ $(t = 1,2)$ are rational functions. Denote $C := \max\{|c_1|, \dots, |c_n|\}$. If f_t $(t = 1,2)$ are entire or have finitely many poles, then there exist constants $K_t > 0$ $(t = 1,2)$ and $r_0 > 0$ such that for all $r \geq r_0$

$$\log M(r,f_t) \geq K_t(d_1d_2)^{r/(2C)}, \quad t = 1,2.$$

Theorem 1.8 *Suppose that* (f_1,f_2) *is a transcendental meromorphic solution of a system of difference equations of the form*

$$\begin{cases} \sum_{j=1}^{n_1} a_j^1(z)f_1(z+c_j) = R_2(z,f_2(z)) = \frac{P_2(z,f_2(z))}{Q_2(z,f_2(z))}, \\ \sum_{j=1}^{n_2} a_j^2(z)f_2(z+c_j) = R_1(z,f_1(z)) = \frac{P_1(z,f_1(z))}{Q_1(z,f_1(z))}, \end{cases} \tag{5}$$

where the coefficients $a_j^t(z)$, $t = 1,2$ *are rational functions, and* P_t, Q_t *are relatively prime polynomials in* f_t *over the field of rational functions satisfying* $p_t = \deg_{f_t} P_t$, $l_t = \deg_{f_t} Q_t$, $d_t = p_t - l_t \geq 2$, $t = 1,2$. *Denote* $C := \max\{|c_1|,\ldots,|c_n|\}$. *If* f_t *have infinitely many poles, then for sufficiently large* r,

$$n(r,f_t) \geq K_t(d_1d_2)^{r/(2C)}, \quad t = 1,2.$$

Remark 1.1 Since system (4) is a particular case of system (5), from the conclusions of Theorem 1.8, we can get the following result.

Under the assumptions of Theorem 1.7, if f_t $(t = 1,2)$ have infinitely many poles, then there exist constants $K_t > 0$ $(t = 1,2)$ and $r_0 > 0$ such that for all $r \geq r_0$,

$$n(r,f_t) \geq K_t(d_1d_2)^{r/(2C)}, \quad t = 1,2.$$

Theorem 1.9 *Suppose that* (f_1,f_2) *is a transcendental meromorphic solution of a system of complex difference equations of the form*

$$\begin{cases} \dfrac{\sum_{\lambda^1\in I_1} d_{\lambda^1}(z)f_2(z+c_1)^{i_{\lambda_1^1}}\cdots f_2(z+c_{n_1})^{i_{\lambda_{n_1}^1}}}{\sum_{\mu^1\in J_1} e_{\mu^1}(z)f_2(z+c_1)^{j_{\mu_1^1}}\cdots f_2(z+c_{n_1})^{j_{\mu_{n_1}^1}}} = \dfrac{\sum_{j=0}^{s_1} a_j^1(z)f_1(z)^j}{\sum_{j=0}^{l_1} b_j^1(z)f_1(z)^j}, \\ \dfrac{\sum_{\lambda^2\in I_2} d_{\lambda^2}(z)f_1(z+c_1)^{i_{\lambda_1^2}}\cdots f_1(z+c_{n_2})^{i_{\lambda_{n_2}^2}}}{\sum_{\mu^2\in J_2} e_{\mu^2}(z)f_1(z+c_1)^{j_{\mu_1^2}}\cdots f_1(z+c_{n_2})^{j_{\mu_{n_2}^2}}} = \dfrac{\sum_{j=0}^{s_2} a_j^2(z)f_2(z)^j}{\sum_{j=0}^{l_2} b_j^2(z)f_2(z)^j}, \end{cases} \tag{6}$$

where $I_t = \{i_{\lambda_1^t}, i_{\lambda_2^t},\ldots,i_{\lambda_{n_t}^t}\}$, $J_t = \{j_{\mu_1^t}, j_{\mu_2^t},\ldots,j_{\mu_{n_t}^t}\}$ *are finite index sets satisfying*

$$\max_{\lambda^t,\mu^t}\{i_{\lambda_1^t} + i_{\lambda_2^t} + \cdots + i_{\lambda_{n_t}^t}, j_{\mu_1^t} + j_{\mu_2^t} + \cdots + j_{\mu_{n_t}^t}\} = \sigma_t, \quad t = 1,2,$$

$d_t = \max\{s_t,l_t\} \geq 2$, $t = 1,2$, $n_1,n_2 \in \mathbb{N}_+$ *and all coefficients of* (6) *are of growth* $o(T(r,f_1))$, $o(T(r,f_2))$ *without an exceptional set. Denote* $C := \max\{|c_1|,\ldots,|c_n|\}$. *If* $d_1d_2 > 4n_1n_2\sigma_1\sigma_2$, *then for any* ε *satisfying*

$$0 < \varepsilon < \frac{\sqrt{d_1d_2} - \sqrt{4n_1n_2\sigma_1\sigma_2}}{\sqrt{d_1d_2} + \sqrt{4n_1n_2\sigma_1\sigma_2}},$$

there exist constants $r_0 > 0$ *and* $K_t > 0$ $(t = 1,2)$ *for all* $r > r_0$,

$$T(r,f_t) \geq K_t\left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right)^{r/(2C)}, \quad t = 1,2.$$

Theorem 1.10 *Suppose that all coefficients in* (6) *are of growth* $S(r,f_1), S(r,f_2)$ *and that all the other assumptions of Theorem* 1.9 *hold. Then* $\mu(f_t) = \infty$ $(t = 1,2)$.

2 Some lemmas

Lemma 2.1 (Valiron-Mohon'ko [21]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R\bigl(z,f(z)\bigr)=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T\bigl(r,R\bigl(z,f(z)\bigr)\bigr)=dT(r,f)+O\bigl(\Psi(r)\bigr),$$

where $d=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Lemma 2.2 (see [25]) *Let $f_1,f_2,\ldots,f_n$ be meromorphic functions. Then*

$$T\left(r,\sum_{\lambda\in I}f_1^{i_{\lambda_1}}f_2^{i_{\lambda_2}}\cdots f_n^{i_{\lambda_n}}\right)\le\sigma\sum_{i=1}^{n}T(r,f_i)+\log s,$$

where $I=\{i_{\lambda_1},i_{\lambda_2},\ldots,i_{\lambda_n}\}$ is an index set consisting of s elements, and $\sigma=\max_{\lambda\in I}\{i_{\lambda_1}+i_{\lambda_2}+\cdots+i_{\lambda_n}\}$.

Lemma 2.3 (see [21]) *Let $g:(0,+\infty)\to\mathbb{R}$, $h:(0,+\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\le h(r)$ outside of an exceptional set of finite logarithmic measure. Then for any $\alpha>1$, there exists $r_0>0$ such that $g(r)\le h(\alpha r)$ for all $r>r_0$.*

Lemma 2.4 (see [6, Lemma 1]) *Given $\epsilon>0$ and a meromorphic function y, the Nevanlinna characteristic function T satisfies*

$$T\bigl(r,y(z\pm1)\bigr)\le(1+\varepsilon)T\bigl(r+1,y(z)\bigr)+\kappa$$

for all $r\ge1/\epsilon$, for some constant κ.

3 The proof of Theorem 1.7

Since the coefficients $a_j^t(z)$, $b_i^t(z)$ $(t=1,2)$ are rational functions, we can rewrite (4) as

$$\begin{cases}\sum_{j=1}^{n}A_j^1(z)f_1(z+c_j)=\sum_{i=0}^{d_1}B_i^1(z)f_2(z)^i,\\ \sum_{j=1}^{n}A_j^2(z)f_2(z+c_j)=\sum_{i=0}^{d_2}B_i^2(z)f_1(z)^i,\end{cases}\tag{7}$$

where the coefficients $A_j^t(z)$, $B_i^t(z)$ $(t=1,2)$ are polynomials.

Next, two cases will be considered as follows.

Case 1. Since (f_1,f_2) is a transcendental solution of system (4) or (7) and f_t $(t=1,2)$ are entire, set $p_i^t=\deg A_j^t$ $(j=1,2,\ldots,n)$, $q_i^t=\deg B_i^t$ $(i=0,1,\ldots,d_i)$, $t=1,2$, taking $m_t=\max\{p_1^t,\ldots,p_n^t\}+1$, we have that

$$\begin{cases}M(r,\sum_{i=0}^{d_1}B_i^1(z)f_2(z)^i)=M(r,\sum_{j=1}^{n}A_j^1(z)f_1(z+c_j))\le nr^{m_1}M(r+C,f_1),\\ M(r,\sum_{i=0}^{d_2}B_i^2(z)f_1(z)^i)=M(r,\sum_{j=1}^{n}A_j^2(z)f_2(z+c_j))\le nr^{m_2}M(r+C,f_2),\end{cases}\tag{8}$$

when r is sufficiently large. Since f_t are transcendental entire functions and B_i^t ($i = 0,1,\ldots,d_t$; $t = 1,2$) are polynomials, we have $M(r, \sum_{i=0}^{d_1-1} B_i^1 f_2(z)^i) = o(M(r,f_2(z)^{d_1}))$ and $M(r, \sum_{i=0}^{d_2-1} B_i^2 f_1(z)^i) = o(M(r,f_1(z)^{d_2}))$. Thus, for sufficiently large r, we have

$$\begin{cases} M(r, \sum_{i=0}^{d_1} B_i^1(z) f_2(z)^i) \geq \frac{1}{2} M(r, B_{d_1}^1 f_2(z)^{d_1}), \\ M(r, \sum_{i=0}^{d_2} B_i^2(z) f_1(z)^i) \geq \frac{1}{2} M(r, B_{d_2}^2 f_1(z)^{d_2}). \end{cases} \tag{9}$$

From (8) and (9), we have

$$\begin{cases} \log M(r + C, f_1) \geq d_1 \log M(r, f_2) + g_1(r), \\ \log M(r + C, f_2) \geq d_2 \log M(r, f_1) + g_2(r), \end{cases} \tag{10}$$

where $|g_t(r)| < K_t \log r$, $t = 1,2$ for some constants $K_t > 0$ and sufficiently large r. From (10), for sufficiently large r, we have

$$\log M(r + 2C, f_1) \geq d_1 d_2 \log M(r, f_1) + g_1(r + C) + d_1 g_2(r). \tag{11}$$

Iterating (11), we have

$$\log M(r + 2kC, f_1) \geq (d_1 d_2)^k \log M(r, f_1) + E_k^1(r) + E_k^2(r) \quad (k \in \mathbb{N}), \tag{12}$$

where

$$\begin{aligned} \left|E_k^1(r)\right| &= \left|(d_1 d_2)^{k-1} g_1(r + C) + (d_1 d_2)^{k-2} g_1(r + 3C) + \cdots + g_1\bigl(r + (2k-1)C\bigr)\right| \\ &\leq K_1 (d_1 d_2)^{k-1} \sum_{j=1}^{k} \frac{\log[r + (2j-1)C]}{(d_1 d_2)^j} \\ &\leq K_1 (d_1 d_2)^{k-1} \sum_{j=1}^{\infty} \frac{\log[r + (2j-1)C]}{(d_1 d_2)^{j-1}}, \end{aligned}$$

and

$$\begin{aligned} \left|E_k^2(r)\right| &= \left|d_1 (d_1 d_2)^{k-1} g_2(r) + d_1 (d_1 d_2)^{k-2} g_2(r + 2C) + \cdots + d_1 g_2\bigl(r + 2(k-1)C\bigr)\right| \\ &\leq K_2 d_1 (d_1 d_2)^{k-1} \sum_{j=1}^{k} \frac{\log[r + 2(j-1)C]}{(d_1 d_2)^{j-1}} \\ &\leq K_2 d_1 (d_1 d_2)^{k-1} \sum_{j=1}^{\infty} \frac{\log[r + 2(j-1)C]}{(d_1 d_2)^{j-1}}. \end{aligned}$$

Since $\log[r + kC] \leq \log r \times \log kC$ for sufficiently large r and k, and since $d_1 d_2 \geq 2$, we know that the series $\sum_{j=1}^{\infty} \frac{\log[r+(2j-1)C]}{(d_1 d_2)^{j-1}}$ and $\sum_{j=1}^{\infty} \frac{\log[r+2(j-1)C]}{(d_1 d_2)^{j-1}}$ are convergent. Thus, for sufficiently large r, we have

$$\left|E_k^t(r)\right| \leq K_t' (d_1 d_2)^k \log r, \quad t = 1,2, \tag{13}$$

where $K_t' > 0$ $(t = 1, 2)$ are some constants. Since f_1 is a transcendental entire function, for sufficiently large r, we have

$$\log M(r, f_1) \geq 3K' \log r, \tag{14}$$

where $K' > \max\{K_1', K_2'\}$. Hence, from (12)-(14), there exists $r_0 \geq e$ such that for $r \geq r_0$, we have

$$\log M(r + 2kC, f_1) \geq K'(d_1 d_2)^k \log r. \tag{15}$$

Choosing $r \in [r_0, r_0 + C)$ and letting $k \to \infty$ for each choice of r, and for each sufficiently large $R := r + 2kC \geq R_0 := r_0 + C$, we have

$$R \in \bigl[r_0 + 2kC, r_0 + (2k+1)C\bigr), \quad \textit{i.e. } k > \frac{R - r_0 - C}{2C}. \tag{16}$$

From (15) and (16), we have

$$\log M(R, f_1) \geq \log M(r_0 + 2kC, f_1) \geq K'(d_1 d_2)^k \log r_0 \geq K''(d_1 d_2)^{R/(2C)},$$

where $K'' = K'(d_1 d_2)^{\frac{-r_0 - C}{2C}} \log r_0$.

By using the same argument as above, we can get that there exist constants $K > 0$ and $r_0 > 0$ such that for all $R \geq r_0$,

$$\log M(R, f_2) \geq K(d_1 d_2)^{R/(2C)}. \tag{17}$$

Case 2. Suppose that (f_1, f_2) is a solution of system (4) and f_t $(t = 1, 2)$ are meromorphic functions with finitely many poles. Then there exist polynomials $P_t(z)$ such that $g_t(z) = P_t(z) f_t(z)$ $(t = 1, 2)$ are entire functions. Substituting $f_t(z) = \frac{g_t(z)}{P_t(z)}$ into (7) and again multiplying away the denominators, we can get a system similar to (7). By using the same argument as above, we can obtain that for sufficiently large $r \geq r_1 \geq r_0$,

$$\log M(r, f_t) = \log M(r, g_t) + \log M\left(r, \frac{1}{P_t(z)}\right) \geq \bigl(K_t'' - \varepsilon\bigr)(d_1 d_2)^{r/(2C)} \geq K_t'''(d_1 d_2)^{r/(2C)},$$

where K_t''' (> 0) $(t = 1, 2)$ are some constants.

From Case 1 and Case 2, this completes the proof of Theorem 1.7.

4 The proof of Theorem 1.8

Suppose that (f_1, f_2) is a solution of system (5) and f_t $(t = 1, 2)$ are transcendental. Since the coefficients of $P_t(z, f_t(z))$, $Q_t(z, f_t(z))$ are rational functions, we can choose a sufficiently large constant R (> 0) such that the coefficients of $P_t(z, f_t(z))$, $Q_t(z, f_t(z))$ have no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since f_t $(t = 1, 2)$ have infinitely many poles, we can choose a pole z_0 of f_1 of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Then the right-hand side of the second equation in system (5) has a pole of multiplicity $d_1 \tau$ at z_0. Then there exists at least one index $j_1 \in \{1, 2, \ldots, n_2\}$ such that $z_0 + c_{j_1}$ is a pole of f_2 of multiplicity $\tau_1' \geq d_1 \tau$. Replacing z

by $z_0 + c_{j_1}$ in the first equation of (5), we have

$$\sum_{j=1}^{n_1} a_j^1(z_0 + c_{j_1})f_1(z_0 + c_{j_1} + c_j) = R_2\big(z_0 + c_{j_1}, f_2(z_0 + c_{j_1})\big). \tag{18}$$

We now have two possibilities as follows.

(i) If $z_0 + c_{j_1}$ is a pole or a zero of the coefficients of $R_2(z,f_2(z))$, then this process will be terminated and we can choose another pole z_0 of f_1 in the way we did above.

(ii) If $z_0 + c_{j_1}$ is neither a pole nor a zero of the coefficients of $R_2(z,f_2(z))$, thus the right-hand side of (18) has a pole of multiplicity $d_2\tau_1'$ at $z_0 + c_{j_1}$, then there exists at least one index $j_1' \in \{1,2,\dots,n_1\}$ such that $z_0 + c_{j_1} + c_{j_1'}$ is a pole of f_1 of multiplicity $\tau_1 \geq d_2\tau_1' \geq d_1d_2\tau$. Replacing z by $z_0 + c_{j_1} + c_{j_1'}$ in the second equation of (5), we have

$$\sum_{j=1}^{n_2} a_j^2(z_0 + c_{j_1} + c_{j_1'})f_2(z_0 + c_{j_1} + c_{j_1'} + c_j) = R_1\big(z_0 + c_{j_1} + c_{j_1'}, f_1(z_0 + c_{j_1} + c_{j_1'})\big).$$

We proceed to follow the step above. Since the coefficients of $R_t(z,f_t(z))$ have finitely many zeros and poles in $\{z \in \mathbb{C} : |z| > R\}$ and f_1 has infinitely many poles again, we may construct poles $\zeta_{2k} := z_0 + c_{j_1} + c_{j_1'} + c_{j_2} + c_{j_2'} + \cdots + c_{j_k} + c_{j_k'}$ ($j_i \in \{1,2,\dots,n_2\}$, $j_i' \in \{1,2,\dots,n_1\}$, $i = 1,2,\dots,k$) of f_1 of multiplicity τ_k satisfying $\tau^k \geq (d_1d_2)^k\tau$ as $k \to \infty$, $k \in \mathbb{N}$. Since $|\zeta_{2k}| \to \infty$ as $k \to \infty$, for sufficiently large k, say $k \geq k_0$ and any $\mathscr{R}_1 \in [|z_0|, |z_0| + C)$, we have

$$\begin{aligned}\tau(d_1d_2)^k &\leq \tau\big(1 + d_1d_2 + \cdots + (d_1d_2)^k\big) \leq n\big(|\zeta_{2k}|, f_1\big)\\ &= n\big(|z_0| + 2kC, f_1\big) \leq n\big(\mathscr{R}_1 + 2kC, f_1\big).\end{aligned} \tag{19}$$

If we can choose a pole z_1 of f_2 of multiplicity $\tau' \geq 1$ satisfying $|z_1| > R$, similar to the above discussion, we can get that for sufficiently large k and any $\mathscr{R}_2 \in [|z_1|, |z_1| + C)$,

$$\tau'(d_1d_2)^k \leq n\big(|z_1| + 2kC, f_2\big) \leq n(\mathscr{R}_2 + 2kC, f_2). \tag{20}$$

Thus, for each sufficiently large $\mathscr{R} := \mathscr{R}_1 + 2kC \geq r_0 := |z_0| + 2(k_0 + 1)C$, there exists a $k \in \mathbb{N}$ such that $\mathscr{R} \in [|z_0| + 2kC, |z_0| + (2k+1)C)$ (or $\mathscr{R} \in [|z_1| + 2kC, |z_1| + (2k+1)C)$), by using the same method as in the proof of Theorem 1.7, from (19) (or (20)), we have

$$n(\mathscr{R}, f_1) \geq \tau(d_1d_2)^k \geq \tau(d_1d_2)^{\frac{\mathscr{R} - |z_0| - C}{2C}} \geq K_1(d_1d_2)^{\mathscr{R}/(2C)}, \tag{21}$$

or

$$n(\mathscr{R}, f_2) \geq \tau'(d_1d_2)^k \geq \tau(d_1d_2)^{\frac{\mathscr{R} - |z_1| - C}{2C}} \geq K_2(d_1d_2)^{\mathscr{R}/(2C)}, \tag{22}$$

where $K_1 = \tau(d_1d_2)^{\frac{-|z_0|-C}{2C}}$ and $K_2 = \tau'(d_1d_2)^{\frac{-|z_1|-C}{2C}}$.

Thus, from (21) and (22), this completes the proof of Theorem 1.8.

5 The proof of Theorem 1.9

From the assumptions of Theorem 1.9 and f_t ($t = 1,2$) are transcendental, applying Lemma 2.1 and Lemma 2.2 for (6) and by Lemma 2.4, for any given ε ($0 < \varepsilon <$

$\frac{\sqrt{d_1d_2}-\sqrt{4n_1n_2\sigma_1\sigma_2}}{\sqrt{d_1d_2}+\sqrt{4n_1n_2\sigma_1\sigma_2}}$) and all $r \geq r_0 \geq \frac{1}{\varepsilon}$, we have

$$\begin{cases} d_1(1-\varepsilon)T(r,f_1) \leq d_1T(r,f_1) + o(T(r,f_1)) \leq 2\sigma_1 \sum_{j=1}^{n_1} T(r+C,f_2) + o(T(r,f_2)) \\ \qquad \leq 2n_1\sigma_1(1+\varepsilon)T(r+C,f_2), \\ d_2(1-\varepsilon)T(r,f_2) \leq d_2T(r,f_2) + o(T(r,f_2)) \leq 2\sigma_2 \sum_{j=1}^{n_2} T(r+C,f_1) + o(T(r,f_1)) \\ \qquad \leq 2n_2\sigma_2(1+\varepsilon)T(r+C,f_1). \end{cases} \tag{23}$$

From (23) and for all $r \geq r_0 \geq \frac{1}{\varepsilon}$, we have

$$d_1d_2(1-\varepsilon)^2T(r,f_1) \leq 4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2T(r+2C,f_1), \tag{24}$$

$$d_1d_2(1-\varepsilon)^2T(r,f_2) \leq 4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2T(r+2C,f_2). \tag{25}$$

Iterating (24) and (25), we can get

$$T(r+2kC,f_t) \geq \left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right)^k T(r,f_t), \quad t=1,2, \tag{26}$$

which holds for $r \geq r_0$ and $k \in \mathbb{N}$. Let $k \to \infty$, for any $r \in [r_0, r_0+C)$, set $R := r+2kC \geq r_0+C$, we have $k \geq \frac{R-r_0-C}{2C}$. Then from (26) we can get

$$\begin{aligned} T(R,f_t) &\geq \left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right)^{\frac{R-r_0-C}{2C}} T(r_0,f_t) \\ &\geq K_t\left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right)^{\frac{R}{2C}}, \end{aligned}$$

where

$$K_t = \left(\frac{d_1d_2}{4n_1n_2\sigma_1\sigma_2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2\right)^{\frac{-r_0-C}{2C}} T(r_0,f_t), \quad t=1,2.$$

Thus, this completes the proof of Theorem 1.9.

6 The Proof of Theorem 1.10

From the assumptions of Theorem 1.10, by using the same argument as in Theorem 1.9, for any given ε ($0 < \varepsilon < \frac{\sqrt{d_1d_2}-\sqrt{4n_1n_2\sigma_1\sigma_2}}{\sqrt{d_1d_2}+\sqrt{4n_1n_2\sigma_1\sigma_2}}$) and all $r \geq r_0 \geq \frac{1}{\varepsilon}$, we have

$$\begin{cases} d_1(1-\varepsilon)T(r,f_1) \leq d_1T(r,f_1) + S(r,f_1) \leq 2\sigma_1 \sum_{j=1}^{n_1} T(r+C,f_2) + S(r,f_2) \\ \qquad \leq 2n_1\sigma_1(1+\varepsilon)T(r+C,f_2), \\ d_2(1-\varepsilon)T(r,f_2) \leq d_2T(r,f_2) + S(r,f_2) \leq 2\sigma_2 \sum_{j=1}^{n_2} T(r+C,f_1) + S(r,f_1) \\ \qquad \leq 2n_2\sigma_2(1+\varepsilon)T(r+C,f_1), \end{cases}$$

outside of a possible exceptional set of finite logarithmic measure.

It follows that

$$d_1d_2(1-\varepsilon)^2T(r,f_t) \leq 4n_1n_2\sigma_1\sigma_2(1+\varepsilon)^2T(r+2C,f_t), \quad t=1,2, \tag{27}$$

outside of a possible exceptional set of finite logarithmic measure. From (27) and Lemma 2.3, it follows that for every $\alpha > 1$, there exists $r_0 > 0$ such that

$$T(\alpha^2 r + 2C, f_t) \geq \frac{d_1 d_2 (1-\varepsilon)^2}{4n_1 n_2 \sigma_1 \sigma_2 (1+\varepsilon)^2} T(r, f_t) =: \zeta T(r, f_t), \quad t = 1, 2, \tag{28}$$

and $T(r, f_t) > 1$ holds for all $r \geq r_0$, where $\zeta > 1$. Inductively, for any positive integer $k \in \mathbb{N}$ and $r \geq r_0$, from (28), we have

$$T\left(\alpha^{2k} r + \frac{\alpha^{2k}-1}{\alpha^2-1} 2C, f_t\right) \geq \zeta^k T(r, f_t), \quad t = 1, 2. \tag{29}$$

By using the same argument as in [9, Theorem 1.11], we can get that $\mu(f_t) = \infty$ $(t = 1, 2)$ easily.

Thus, this completes the proof of Theorem 1.10.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HW and HYX completed the main part of this article, HYX, HW and BXL corrected the main theorems. All authors read and approved the final manuscript.

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