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On growth of meromorphic solutions for linear difference equations with meromorphic coefficients

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Abstract
In this paper, we consider the value distribution of meromorphic solutions for linear difference equations with meromorphic coefficients.
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1 Introduction and preliminaries

Recently, several papers (including [1–7]) have been published regarding value distribution of meromorphic solutions of linear difference equations. We recall the following results. Chiang and Feng proved the following theorem.

Theorem A ([2]) *Let $P_0(z), \dots, P_n(z)$ be polynomials such that there exists an integer l, $0 \le l \le n$, such that*

$$\deg(P_l) > \max_{0\le j\le n, j\ne l} \{\deg(P_j)\} \tag{1.1}$$

holds. Suppose $f(z)$ is a meromorphic solution of the difference equation

$$P_n(z)f(z+n) + \cdots + P_1(z)f(z+1) + P_0(z)f(z) = 0. \tag{1.2}$$

Then we have $\sigma(f) \ge 1$.

In this paper, we use the basic notions of Nevanlinna's theory (see [8, 9]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ to denote the exponent of convergence of zeros of $f(z)$.

Chen [1] weakened the condition (1.1) of Theorem A and proved the following results.

Theorem B ([1]) *Let $P_n(z), \dots, P_0(z)$ be polynomials such that $P_nP_0 \not\equiv 0$ and*

$$\deg(P_n + \cdots + P_0) = \max\{\deg P_j : j = 0, \dots, n\} \ge 1. \tag{1.3}$$

Then every finite order meromorphic solution $f(z)$ ($\not\equiv 0$) of equation (1.2) satisfies $\sigma(f) \ge 1$, and $f(z)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often and $\lambda(f - a) = \sigma(f)$.

Theorem C ([1]) *Let $F(z)$, $P_n(z),\ldots,P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$ and* (1.3). *Then every finite order transcendental meromorphic solution $f(z)$ of the equation*

$$P_n(z)f(z+n)+\cdots+P_1(z)f(z+1)+P_0(z)f(z)=F(z) \tag{1.4}$$

satisfies $\sigma(f)\geq 1$ and $\lambda(f)=\sigma(f)$.

Theorem D ([1]) *Let $F(z)$, $P_n(z),\ldots,P_0(z)$ be polynomials such that $FP_nP_0 \not\equiv 0$. Suppose that $f(z)$ is a meromorphic solution with infinitely many poles of* (1.2) (*or* (1.4)). *Then $\sigma(f)\geq 1$.*

For the linear difference equation with transcendental coefficients

$$A_n(z)f(z+n)+\cdots+A_1(z)f(z+1)+A_0(z)f(z)=0, \tag{1.5}$$

Chiang and Feng proved the following result.

Theorem E ([2]) *Let $A_0(z),\ldots,A_n(z)$ be entire functions such that there exists an integer l, $0\leq l\leq n$, such that*

$$\sigma(A_l)>\max\bigl\{\sigma(A_j):0\leq j\leq n, j\neq l\bigr\}. \tag{1.6}$$

If $f(z)$ is a meromorphic solution of (1.5), *then we have $\sigma(f)\geq\sigma(A_l)+1$.*

Laine and Yang proved the following theorem.

Theorem F ([5]) *Let $A_0,\ldots,A_n$ be entire functions of finite order so that among those having the maximal order $\sigma := \max\{\sigma(A_k):0\leq k\leq n\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution of*

$$A_n(z)f(z+C_n)+\cdots+A_1(z)f(z+C_1)+A_0(z)f(z)=0, \tag{1.7}$$

we have $\sigma(f)\geq\sigma+1$.

Remark 1.1 If $A_0,\ldots,A_n$ are meromorphic functions satisfying (1.6), then Theorem E does not hold. For example, the equation

$$y(z+1)-\left(e^i+\frac{e^i-1}{e^{iz}-1}\right)y(z)=0$$

has a solution $y(z)=e^{iz}-1$, which $\sigma(y)=1<\sigma(A_0)+1$.

This example shows that for the linear difference equation with meromorphic coefficients, the condition (1.6) cannot guarantee that every transcendental meromorphic solution $f(z)$ of (1.7) satisfies $\sigma(f)\geq\sigma(A_l)+1$.

Thus, a natural question to ask is what conditions will guarantee every transcendental meromorphic solution $f(z)$ of (1.7) with meromorphic coefficients satisfies $\sigma(f)\geq\sigma(A_l)+1$.

In this note, we consider this question and prove the following results.

Theorem 1.1 *Let c_1, c_2 ($\neq c_1$), a be nonzero constants, $h_1(z)$ be a nonzero meromorphic function with $\sigma(h_1) < 1$, $B(z)$ be a nonzero meromorphic function.*

If $B(z)$ satisfies any one of the following three conditions:

(i) $\sigma(B) > 1$ *and* $\delta(\infty, B) > 0$;

(ii) $\sigma(B) < 1$;

(iii) $B(z) = h_0(z)e^{bz}$ *where b is a nonzero constant, $h_0(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(h_0) < 1$,*

then every meromorphic solution f ($\not\equiv 0$) of the difference equation

$$f(z+c_2) + h_1(z)e^{az}f(z+c_1) + B(z)f(z) = 0 \tag{1.8}$$

satisfies $\sigma(f) \geq \max\{\sigma(B), 1\} + 1$.

Further, if $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with

$$\sigma(\varphi) < \max\{\sigma(B), 1\} + 1,$$

then

$$\lambda(f - \varphi) = \sigma(f) \geq \max\{\sigma(B), 1\} + 1.$$

Corollary *Under conditions of Theorem* 1.1, *every finite order solution $f(z)$ ($\not\equiv 0$) of* (1.8) *has infinitely many fixed points, satisfies $\tau(f) = \sigma(f)$, and for any nonzero constant c,*

$$\lambda\bigl(f(z) - c\bigr) = \sigma(f) \geq \max\{\sigma(B), 1\} + 1.$$

Example 1.1 The equation

$$f(z+2) - \frac{1}{2}e^{2z+3}f(z+1) - \frac{1}{2}e^{4z+4}f(z) = 0$$

satisfies conditions of Theorem 1.1 and has a solution $f(z) = e^{z^2}$ satisfying $\lambda(f) = 0$ and $\tau(f) = \sigma(f) = 2$. This example shows that under conditions of Theorem 1.1, a meromorphic solution of (1.8) may have no zero.

Theorem 1.2 *Let $h_1(z)$, c_1, c_2, a, $B(z)$ satisfy conditions of Theorem* 1.1, *and let $F(z)$ ($\not\equiv 0$) be a meromorphic function with $\sigma(F) < \max\{\sigma(B), 1\} + 1$. Then all meromorphic solutions with finite order of the equation*

$$f(z+c_2) + h_1(z)e^{az}f(z+c_1) + B(z)f(z) = F(z) \tag{1.9}$$

satisfy

$$\lambda(f) = \sigma(f) \geq \max\{\sigma(B), 1\} + 1$$

with at most one possible exceptional solution with $\sigma(f) < \max\{\sigma(B), 1\} + 1$.

Remark 1.2 Under conditions of Theorem 1.1, equation (1.8) has no rational solution. But equation (1.9) in Theorem 1.2 may have a rational solution. For example, the equation

$$f(z+2) + e^z f(z+1) - e^z f(z) = z + 2 - e^z$$

satisfies conditions of Theorem 1.2 and has a solution $f(z) = z$. This shows that in Theorem 1.2, there exists one possible exceptional solution with $\sigma(f) < \max\{\sigma(B), 1\} + 1$.

2 Proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 ([2, 10]) *Given two distinct complex constants η_1, η_2, let f be a meromorphic function of finite order σ. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O\left(r^{\sigma-1+\varepsilon}\right).$$

Lemma 2.2 (see [11]) *Suppose that $P(z) = (\alpha + i\beta)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)$ ($\not\equiv 0$) is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then, for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero such that for any $\theta \in [0, 2\pi) \backslash (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have that*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\left\{(1-\varepsilon)\delta(P,\theta)r^n\right\} < \left|g\left(re^{i\theta}\right)\right| < \exp\left\{(1+\varepsilon)\delta(P,\theta)r^n\right\}; \tag{2.1}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\left\{(1+\varepsilon)\delta(P,\theta)r^n\right\} < \left|g\left(re^{i\theta}\right)\right| < \exp\left\{(1-\varepsilon)\delta(P,\theta)r^n\right\}, \tag{2.2}$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3 *Let c_1, c_2 ($\neq c_1$), a be nonzero constants, $A_j(z)$ ($j = 0, 1, 2$), $F(z)$ be nonzero meromorphic functions. Suppose that $f(z)$ is a finite order meromorphic solution of the equation*

$$A_2(z)f(z+c_2) + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z). \tag{2.3}$$

If $\sigma(f) > \max\{\sigma(F), \sigma(A_j)\ (j = 0, 1, 2)\}$, then $\lambda(f) = \sigma(f)$.

Proof Suppose that $\sigma(f) = \sigma$, $\max\{\sigma(F), \sigma(A_j)\ (j = 0, 1, 2)\} = \alpha$. Then $\sigma > \alpha$. Equation (2.3) can be rewritten as the form

$$\frac{1}{f(z)} = \frac{F(z)}{f(z)}\left(A_2(z)\frac{f(z+c_2)}{f(z)} + A_1(z)\frac{f(z+c_1)}{f(z)} + A_0(z)\right). \tag{2.4}$$

Thus, by (2.4), we deduce that

$$\begin{aligned} T(r,f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &= m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + O(1) \end{aligned}$$

$$\leq N\left(r,\frac{1}{f}\right)+m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{2}m(r,A_j)$$
$$+m\left(r,\frac{f(z+c_2)}{f(z)}\right)+m\left(r,\frac{f(z+c_1)}{f(z)}\right)+O(1). \tag{2.5}$$

For any given ε $(0<\varepsilon<\min\{\frac{1}{4},\frac{\sigma-\alpha}{4}\})$, and for sufficiently large r, we have that

$$m\left(r,\frac{1}{F}\right)\leq T(r,F)\leq r^{\alpha+\varepsilon},\qquad m(r,A_j)\leq r^{\alpha+\varepsilon}\quad (j=0,1,2). \tag{2.6}$$

By Lemma 2.1, we obtain

$$m\left(r,\frac{f(z+c_2)}{f(z)}\right)\leq Mr^{\sigma-1+\varepsilon}\quad\text{and}\quad m\left(r,\frac{f(z+c_1)}{f(z)}\right)\leq Mr^{\sigma-1+\varepsilon}, \tag{2.7}$$

where M (>0) is some constant.

By $\sigma(f)=\sigma$, there exists a sequence $\{r_n\}$ satisfying $r_1<r_2<\cdots$, $r_n\to\infty$ such that

$$\lim_{n\to\infty}\frac{\log T(r_n,f)}{\log r_n}=\sigma. \tag{2.8}$$

Thus, for sufficiently large r_n, we have that

$$T(r_n,f)\geq r_n^{\sigma-\varepsilon}. \tag{2.9}$$

Substituting (2.6)-(2.9) into (2.5), we obtain for sufficiently large r_n

$$r_n^{\sigma-\varepsilon}\leq T(r_n,f)\leq N\left(r_n,\frac{1}{f}\right)+4r_n^{\alpha+\varepsilon}+2Mr_n^{\sigma-1+\varepsilon}. \tag{2.10}$$

Since $\varepsilon<\min\{\frac{1}{4},\frac{\sigma-\alpha}{4}\}$ and ε is arbitrary, by (2.10), we obtain

$$\overline{\lim_{n\to\infty}}\frac{\log N(r_n,\frac{1}{f})}{\log r_n}=\sigma.$$

Hence, $\lambda(f)=\sigma(f)=\sigma$. □

Proof of Theorem 1.1 Suppose that $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of equation (1.8) with $\sigma(f)<\infty$.

(1) Suppose that $B(z)$ satisfies the condition (i): $\sigma(B)>1$ and $\delta(\infty,B)=\delta>0$. Thus, for sufficiently large r,

$$m(r,B)>\frac{\delta}{2}T(r,B). \tag{2.11}$$

Clearly, $\sigma(f)\geq\sigma(B)$ by (1.8). By Lemma 2.1, we see that for any given ε $(0<\varepsilon<\frac{\sigma(B)-1}{3})$,

$$m\left(r,\frac{f(z+c_j)}{f(z)}\right)=O\big(r^{\sigma(f)-1+\varepsilon}\big)\quad (j=1,2), \tag{2.12}$$

and

$$m\big(r,h_1(z)e^{az}\big) \le T\big(r,h_1(z)e^{az}\big) \le r^{1+\varepsilon}. \tag{2.13}$$

By (1.8), we have that

$$-B(z) = \frac{f(z+c_2)}{f(z)} + h_1(z)e^{az}\frac{f(z+c_1)}{f(z)}. \tag{2.14}$$

Substituting (2.11)-(2.13) into (2.14), we deduce that

$$\begin{aligned}\frac{\delta}{2}T(r,B) &\le m(r,B)\\ &\le m\big(r,h_1(z)e^{az}\big) + m\left(r,\frac{f(z+c_2)}{f(z)}\right) + m\left(r,\frac{f(z+c_1)}{f(z)}\right)\\ &\le r^{1+\varepsilon} + O\big(r^{\sigma(f)-1+\varepsilon}\big).\end{aligned} \tag{2.15}$$

By $\sigma(B) = \sigma$, there is a sequence r_j $(1 < r_1 < r_2 < \cdots, r_j \to \infty)$ satisfying

$$T(r_j,B) > r_j^{\sigma(B)-\varepsilon}. \tag{2.16}$$

Thus, by (2.15) and (2.16), we obtain

$$\frac{\delta}{2}r_j^{\sigma(B)-\varepsilon} \le r_j^{1+\varepsilon} + Mr_j^{\sigma(f)-1+\varepsilon}, \tag{2.17}$$

where M (> 0) is some constant. Combining (2.17) and $\varepsilon < \frac{\sigma(B)-1}{3}$, it follows that

$$\frac{\delta}{2}r_j^{\sigma(B)-\varepsilon}\big(1+o(1)\big) \le Mr_j^{\sigma(f)-1+\varepsilon}.$$

So that, it follows that $\sigma(f) \ge \sigma(B) + 1 = \max\{\sigma(B),1\} + 1$.

(2) Suppose that $B(z)$ satisfies the condition (ii): $\sigma(B) < 1$. Using the same method as in (1), we can obtain $\sigma(f) \ge \max\{\sigma(B),1\} + 1$.

(3) Suppose that $B(z)$ satisfies the condition (iii): $B(z) = h_0(z)e^{bz}$, where b is a nonzero constant, $h_0(z)$ $(\not\equiv 0)$ is a meromorphic function with $\sigma(h_0) < 1$.

Now we need to prove $\sigma(f) \ge 2$. Contrary to the assertion, suppose that $\sigma(f) = \alpha < 2$. We will deduce a contradiction. Set $z = re^{i\theta}$. Then

$$\begin{cases}\mathbf{Re}\{az\} = \delta(az,\theta)|a|r = |a|r\cos(\arg a + \theta),\\ \mathbf{Re}\{bz\} = \delta(bz,\theta)|b|r = |b|r\cos(\arg b + \theta).\end{cases} \tag{2.18}$$

In what follows, we divide this proof into three subcases: (a) $\arg a \ne \arg b$; (b) $\arg a = \arg b$ and $|a| \ne |b|$; (c) $a = b$.

Subcase (a). Since $\arg a \ne \arg b$ and (2.18), it is easy to see that there exists a ray $\arg z = \theta_0$ such that

$$\begin{cases}\mathbf{Re}\{az\} = \delta(az,\theta_0)|a|r = |a|r\cos(\arg a + \theta_0) < 0,\\ \mathbf{Re}\{bz\} = \delta(bz,\theta_0)|b|r = |b|r\cos(\arg b + \theta_0) > 0.\end{cases} \tag{2.19}$$

By (1.8) and (2.19), we see that $f(z)$ cannot be a rational function. By Lemma 2.1, (2.12) holds. By Lemma 2.2 and (2.19), it is easy to see that for any given ε_1 $(0 < \varepsilon_1 < \min\{\frac{1}{2}, \frac{2-\alpha}{2}\})$ and for sufficiently large r,

$$\left|h_0(re^{i\theta_0})e^{bre^{i\theta_0}}\right| \geq \exp\{(1-\varepsilon_1)|b|\delta(bz,\theta_0)r\}, \tag{2.20}$$

and

$$\left|h_1(re^{i\theta_0})e^{are^{i\theta_0}}\right| \leq \exp\{(1-\varepsilon_1)|a|\delta(az,\theta_0)r\} < 1. \tag{2.21}$$

Thus, by (1.8), (2.12), (2.20) and (2.21), we deduce that

$$\begin{aligned}\exp\{(1-\varepsilon_1)|b|\delta(bz,\theta_0)r\} &\leq \left|h_0(re^{i\theta_0})e^{bre^{i\theta_0}}\right| \\ &\leq \left|\frac{f(re^{i\theta_0}+c_2)}{f(re^{i\theta_0})}\right| + \left|h_1(re^{i\theta_0})e^{are^{i\theta_0}}\right|\left|\frac{f(re^{i\theta_0}+c_1)}{f(re^{i\theta_0})}\right| \\ &\leq 2\exp\{r^{\sigma(f)-1+\varepsilon_1}\}. \end{aligned} \tag{2.22}$$

By $\delta(bz,\theta_0) = \cos(\arg b + \theta_0) > 0$, $\sigma(f) = \alpha < 2$ and $\varepsilon_1 < \frac{2-\alpha}{2}$, it is easy to see that (2.22) is a contradiction. Hence, $\sigma(f) \geq 2$.

Subcase (b). By $\arg a = \arg b$ and $|a| \neq |b|$, we see that $f(z)$ cannot be a rational function. By Lemma 2.1, (2.12) holds. By $\arg a = \arg b$ and (2.18), we take $\theta_1 = -\arg a$, then $\delta(az,\theta_1) = \delta(bz,\theta_1) = 1$ and

$$\mathbf{Re}\{are^{i\theta_1}\} = |a|r \quad \text{and} \quad \mathbf{Re}\{bre^{i\theta_1}\} = |b|r. \tag{2.23}$$

Now suppose that $|b| > |a|$. By Lemma 2.2, for any given ε_2 $(0 < \varepsilon_2 < \min\{2-\alpha, \frac{|b|-|a|}{2(|b|+|a|)}\})$,

$$\left|h_0(re^{i\theta_1})e^{bre^{i\theta_1}}\right| \geq \exp\{(1-\varepsilon_2)|b|r\}, \tag{2.24}$$

and

$$\left|h_1(re^{i\theta_1})e^{are^{i\theta_1}}\right| \leq \exp\{(1+\varepsilon_2)|a|r\}. \tag{2.25}$$

Thus, by (1.8), (2.12), (2.24) and (2.25), we deduce that

$$\begin{aligned}\exp\{(1-\varepsilon_2)|b|r\} &\leq \left|h_0(re^{i\theta_1})e^{bre^{i\theta_1}}\right| \\ &\leq \left|\frac{f(re^{i\theta_1}+c_2)}{f(re^{i\theta_1})}\right| + \left|h_1(re^{i\theta_1})e^{are^{i\theta_1}}\right|\left|\frac{f(re^{i\theta_1}+c_1)}{f(re^{i\theta_1})}\right| \\ &\leq \exp\{r^{\sigma(f)-1+\varepsilon_2}\} + \exp\{(1+\varepsilon_2)|a|r\}\exp\{r^{\sigma(f)-1+\varepsilon_2}\}. \end{aligned} \tag{2.26}$$

Since $\varepsilon_2 < 2-\alpha$, we have that $\sigma(f) - 1 + \varepsilon_2 = \alpha - 1 + \varepsilon_2 < 1$. Combining this and (2.26), we obtain

$$\exp\{(1-\varepsilon_2)|b|r\} < \exp\{(1+\varepsilon_2)|a|r(1+o(1))\}(1+o(1)). \tag{2.27}$$

By $\varepsilon_2 < \frac{|b|-|a|}{2(|b|+|a|)}$, we see that (2.27) is a contradiction.

Now suppose that $|b| < |a|$. Using the same method as above, we can also deduce a contradiction.

Hence, $\sigma(f) \geq 2$ in Subcase (b).

Subcase (c). We first affirm that $f(z)$ cannot be a nonzero rational function. In fact, if $f(z)$ is a rational function, then $e^{az}[h_1(z)f(z+c_1) + h_0(z)f(z)] = -f(z+c_2)$ is a rational function. So that $h_1(z)f(z+c_1) + h_0(z)f(z) \equiv 0$, that is, $f(z+c_2) \equiv 0$, a contradiction.

By Lemma 2.1, (2.12) holds. By $a = b$, equation (1.8) can be rewritten as

$$e^{-az}f(z+c_2) + h_1(z)f(z+c_1) + h_0(z)f(z) = 0. \tag{2.28}$$

Using the same method as in the proof of (1), we can obtain $\sigma(f) \geq 2$.

(4) Suppose that $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(\varphi) < \max\{\sigma(B), 1\} + 1$. Set $g(z) = f(z) - \varphi(z)$. Substituting $f(z) = g(z) + \varphi(z)$ into (1.8), we obtain

$$\begin{aligned} &g(z+c_2) + h_1(z)e^{az}g(z+c_1) + B(z)g(z) \\ &\quad = -\bigl[\varphi(z+c_2) + h_1(z)e^{az}\varphi(z+c_1) + B(z)\varphi(z)\bigr]. \end{aligned} \tag{2.29}$$

If $\varphi(z+c_2) + h_1(z)e^{az}\varphi(z+c_1) + B(z)\varphi(z) \equiv 0$, then $\varphi(z)$ is a nonzero meromorphic solution of (1.8). Thus, by the proof above, we have that $\sigma(\varphi) \geq \max\{\sigma(B), 1\} + 1$. This contradicts our condition that $\sigma(\varphi) < \max\{\sigma(B), 1\} + 1$. Hence, $\varphi(z+2) + h_1(z)e^{az}\varphi(z+1) + B(z)\varphi(z) \not\equiv 0$, and

$$\sigma\bigl(\varphi(z+c_2) + h_1(z)e^{az}\varphi(z+c_1) + B(z)\varphi(z)\bigr) < \max\bigl\{\sigma(B), 1\bigr\} + 1 \leq \sigma(f) = \sigma(g).$$

Applying this and Lemma 2.3 to (2.29), we deduce that

$$\lambda(f - \varphi) = \lambda(g) = \sigma(g) \geq \max\bigl\{\sigma(B), 1\bigr\} + 1.$$

Thus, Theorem 1.1 is proved. □

3 Proof of Theorem 1.2

Suppose that f_0 is a meromorphic solution of (1.9) with

$$\sigma(f_0) < \max\bigl\{\sigma(B), 1\bigr\} + 1.$$

If $f^*(z)$ ($\not\equiv f_0(z)$) is another meromorphic solution of (1.9) satisfying $\sigma(f^*) < \max\{\sigma(B), 1\} + 1$, then

$$\sigma\bigl(f^* - f_0\bigr) < \max\bigl\{\sigma(B), 1\bigr\} + 1.$$

But $f^* - f_0$ is a solution of the corresponding homogeneous equation (1.8) of (1.9). By Theorem 1.1, we have $\sigma(f^* - f_0) \geq \max\{\sigma(B), 1\} + 1$, a contradiction. Hence equation (1.9) possesses at most one exceptional solution f_0 with $\sigma(f_0) < \max\{\sigma(B), 1\} + 1$.

Now suppose that f is a meromorphic solution of (1.9) with

$$\max\bigl\{\sigma(B), 1\bigr\} + 1 \leq \sigma(f) < \infty.$$

Since $\sigma(f) > \max\{\sigma(B), \sigma(F), \sigma(h(z)e^{az})\}$, applying Lemma 2.3 to (1.9), we obtain

$$\lambda(f) = \sigma(f).$$

Thus, Theorem 1.2 is proved.

Competing interests
The author declares that they have no competing interests.

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