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# Chaotification for linear delay difference equations

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## Abstract

This paper is concerned with chaotification of linear delay difference equations via the feedback control technique. The controlled system is first reformulated into a linear discrete dynamical system. Then, a chaotification theorem based on the snap-back repeller theory for maps is established. The controlled system is proved to be chaotic in the sense of both Devaney and Li-Yorke. An illustrative example is provided with computer simulations.

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**Keywords:** chaos; chaotification; delay difference equation; chaos in the sense of Li-Yorke; chaos in the sense of Devaney

## 1 Introduction

Chaotification (or called anticontrol of chaos) is a process that makes a nonchaotic system chaotic, or enhances a chaotic system to present stronger or different type of chaos. In recent years, it has been found that chaos can actually be useful under some circumstances, for example, in human brain analysis [1, 2], heartbeat regulation [3, 4], encryption [5], digital communications [6], *etc.* So, sometimes it is useful and even important to make a system chaotic or create new types of chaos. It has attracted increasing interest in research on chaotification of dynamical systems due to the great potentials of chaos in many non-traditional applications.

In the pursuit of chaotifying discrete dynamical systems, a simple yet mathematically rigorous chaotification method was first developed by Chen and Lai [7–9] from a feedback control approach. They showed that the Lyapunov exponents of a controlled system are positive [7], and the controlled system via the mod-operation is chaotic in the sense of Devaney when the original system is linear and is chaotic in a weaker sense of Wiggins when the original system is nonlinear [9]. Later, Wang and Chen [10] further showed that the Chen-Lai algorithm for chaotification also leads to chaos in the sense of Li-Yorke. This method plays an important role in studying chaotification problems of discrete dynamical systems. Recently, Shi and Chen [11] studied chaotification for some discrete dynamical systems governed by continuous maps and showed that the controlled systems are chaotic in the sense of both Devaney and Li-Yorke. It is noticed that all the above chaotification problems for discrete dynamical systems are formulated in finite-dimensional real spaces. More recently, Shi *et al.* [12] first studied chaotification for discrete dynamical systems in Banach spaces via the feedback control technique. They showed that some controlled systems in finite-dimensional real spaces studied by [9] and [13] are chaotic in the sense of

Devaney as well as in the sense of both Li-Yorke and Wiggins. Particularly, they extended and proved the Marotto theorem to general Banach spaces [14], established some chaotification theorems for discrete dynamical systems in (infinite-dimensional) Banach spaces [12], and showed the controlled systems are chaotic in the sense of both Devaney and Li-Yorke. The reader is referred to Chen and Shi [15] for a survey of chaotification of discrete dynamical systems and some references cited therein.

Time delay arises in many realistic systems with feedback in science and engineering. The delay difference equations have been studied by many researchers. Although there exist some general chaotification schemes for finite-dimensional or infinite-dimensional discrete dynamical systems as stated in the above, there are few results on the chaotification of linear delay difference equations. In this paper, we employ the Chen-Lai method, from the feedback control approach to study chaotification of linear delay difference equations and prove that the controlled system is chaotic in the sense of both Devaney and Li-Yorke by applying the snap-back repeller theory; see [14, 16, 17] for the theory.

The rest of the paper is organized as follows. In Section 2, the chaotification problem under investigation is described, and some concepts, one lemma, and reformulation of the controlled system are introduced. In Section 3, the chaotification problem is studied and a chaotification criterion is established. Finally, we provide an example to illustrate the theoretical result with computer simulations in Section 4.

## 2 Preliminaries

In this section, we describe the chaotification problem, give a reformulation of the linear delay difference equation, and introduce some fundamental concepts and a criterion of chaos, which will be used in the next section.

### 2.1 Description of chaotification problem

Consider the following linear delay difference equation:

$$x(n+1) = \sum_{j=1}^{k+1} a_j x(n+j-k-1), \quad n \geq 0, \tag{1}$$

where  $k \geq 1$  is a fixed integer and  $a_j, 1 \leq j \leq k+1$ , are constant parameters.

The object is to design a simple control input sequence  $\{v(n)\}$  such that the output of the controlled system

$$x(n+1) = \sum_{j=1}^{k+1} a_j x(n+j-k-1) + v(n), \quad n \geq 0, \tag{2}$$

is chaotic in the sense of both Devaney and Li-Yorke (see Definitions 1 and 2 below). The controller to be designed in this paper is in the form of

$$v(n) = \beta \sin \left( \sum_{j=1}^{k+1} \gamma_j x(n+j-k-1) \right), \tag{3}$$

where  $\beta$  and  $\gamma_j, 1 \leq j \leq k+1$ , are undetermined parameters.

## 2.2 Reformulation

In this subsection, we reformulate equations (1) and (2) into  $k + 1$ -dimensional discrete dynamical systems. By setting

$$u_j(n) := x(n + j - k - 1), \quad 1 \leq j \leq k + 1, n \geq 0,$$

equation (1) and the controlled system (2) with controller (3) can be written as the following discrete systems on  $\mathbf{R}^{k+1}$ :

$$u(n + 1) = Au(n), \quad n \geq 0, \tag{4}$$

$$u(n + 1) = Au(n) + bv(n) := F(u(n)), \quad n \geq 0, \tag{5}$$

respectively, where  $u = (u_1, u_2, \dots, u_{k+1})^T, b = (0, \dots, 0, 1)^T \in \mathbf{R}^{k+1}$ ,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{k+1} \end{pmatrix}_{(k+1) \times (k+1)},$$

$v(n) = \beta \sin(\sum_{j=1}^{k+1} \gamma_j u_j(n)) \in \mathbf{R}$ , and the map  $F : \mathbf{R}^{k+1} \rightarrow \mathbf{R}^{k+1}$  is given by

$$F \begin{pmatrix} u_1(n) \\ u_2(n) \\ \vdots \\ u_k(n) \\ u_{k+1}(n) \end{pmatrix} = \begin{pmatrix} u_2(n) \\ u_3(n) \\ \vdots \\ u_{k+1}(n) \\ \sum_{j=1}^{k+1} a_j u_j(n) + \beta \sin(\sum_{j=1}^{k+1} \gamma_j u_j(n)) \end{pmatrix}.$$

Systems (4) and (5) are said to be induced by equations (1) and (2), respectively. It is easy to see that the dynamical behaviors of equations (1) and (2) are the same as those of their induced systems (4) and (5), respectively.

## 2.3 Some concepts and a criterion of chaos

Since Li and Yorke [18] first introduced a precise mathematical definition of chaos, there appeared several different definitions of chaos, some are stronger and some are weaker, depending on the requirements in different problems; see [19–22] *etc.* For convenience, we list two definitions of chaos in the sense of Li-Yorke and Devaney, which are used in the paper.

**Definition 1** Let  $(X, d)$  be a metric space,  $f : X \rightarrow X$  be a map, and  $S$  be a set of  $X$  with at least two distinct points. Then  $S$  is called a scrambled set of  $f$  if for any two distinct points  $x, y \in S$ ,

$$\liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) > 0.$$

The map  $f$  is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set  $S$  of  $f$ .

**Definition 2** [19] Let  $(X, d)$  be a metric space. A map  $f : V \subset X \rightarrow V$  is said to be chaotic on  $V$  in the sense of Devaney if

- (i) the set of the periodic points of  $f$  is dense in  $V$ ;
- (ii)  $f$  is topologically transitive in  $V$ ;
- (iii)  $f$  has sensitive dependence on initial conditions in  $V$ .

In 1992, Banks *et al.* [23] proved that conditions (i) and (ii) together imply condition (iii) if  $f$  is continuous in  $V$ . So, condition (iii) is redundant in the above definition in this case. It has been proved in [24] that under some conditions, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.

Now, we introduce some relative concepts for system (2), which are motivated by [12, Definitions 5.1 and 5.2].

**Definition 3**

- (i) A point  $x \in \mathbf{R}^{k+1}$  is called an  $m$ -periodic point of system (2) if  $x \in \mathbf{R}^{k+1}$  is an  $m$ -periodic point of its induced system (5), that is,  $F^m(x) = x$ ,  $F^j(x) \neq x$ ,  $1 \leq j \leq m - 1$ . In the special case of  $m = 1$ ,  $x$  is called a fixed point or a steady state of system (2).
- (ii) The concepts of density of periodic points, topological transitivity, sensitive dependence on initial conditions, and the invariant set for system (2) are defined similarly to those for its induced system (5) in  $\mathbf{R}^{k+1}$ .
- (iii) System (2) is said to be chaotic in the sense of Devaney (or Li-Yorke) on  $V \subset \mathbf{R}^{k+1}$  if its induced system (5) is chaotic in the sense of Devaney (or Li-Yorke) on  $V \subset \mathbf{R}^{k+1}$ .

In this paper, we will use the criterion of chaos established by Shi *et al.* to chaotify the linear delay difference equation (1). For convenience, we state it as follows.

**Lemma 1** ([12, Theorem 2.1], [14, Theorem 4.4]) *Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$  be a map with a fixed point  $z \in \mathbf{R}^n$ . Assume that*

- (i)  *$f$  is continuously differentiable in a neighborhood of  $z$  and all the eigenvalues of  $Df(z)$  have absolute values larger than 1, which implies that there exist a positive constant  $r$  and a norm  $\| \cdot \|$  in  $\mathbf{R}^n$  such that  $f$  is expanding in  $\bar{B}_r(z)$  in  $\| \cdot \|$ , where  $\bar{B}_r(z)$  is the closed ball of radius  $r$  centered at  $z$  in  $(\mathbf{R}^n, \| \cdot \|)$ ;*
- (ii)  *$z$  is a snap-back repeller of  $f$  with  $f^m(x_0) = z$ ,  $x_0 \neq z$ , for some  $x_0 \in B_r(z)$  and some positive integer  $m$ , where  $B_r(z)$  is the open ball of radius  $r$  centered at  $z$  in  $(\mathbf{R}^n, \| \cdot \|)$ . Furthermore,  $f$  is continuously differentiable in some neighborhoods of  $x_0, x_1, \dots, x_{m-1}$ , respectively, and  $\det Df(x_j) \neq 0$  for  $0 \leq j \leq m - 1$ , where  $x_j = f(x_{j-1})$  for  $1 \leq j \leq m - 1$ .*

*Then for each neighborhood  $U$  of  $z$ , there exist a positive integer  $k > m$  and a Cantor set  $\Lambda \subset U$  such that  $f^k : \Lambda \rightarrow \Lambda$  is topologically conjugate to the symbolic dynamical system  $\sigma : \Sigma_2^+ \rightarrow \Sigma_2^+$ . Consequently, there exists a compact and perfect invariant set  $V \subset \mathbf{R}^n$ , containing the Cantor set  $\Lambda$ , such that  $f$  is chaotic on  $V$  in the sense of Devaney as well as in the sense of Li-Yorke, and has a dense orbit in  $V$ .*

**Remark 1** Lemma 1 extends and improves the Marroto theorem [17]. Under the conditions of Lemma 1,  $z$  is a regular and nondegenerate snap-back repeller. Therefore, Lemma 1

can be briefly stated as follows: ‘a regular and nondegenerate snap-back repeller in  $\mathbf{R}^n$  implies chaos in the sense of both Devaney and Li-Yorke’. We refer to [12, 14] for details.

### 3 Chaotification for linear delay difference equations

In this section, we will show that the controlled system (5) with a single-input state feedback controller (3), *i.e.*, the controlled system (2) with controller (3), is chaotic in the sense of both Devaney and Li-Yorke for some parameters  $\beta$  and  $\gamma_j$ ,  $1 \leq j \leq k$ .

Consider the fixed points of system (5). It is obvious that  $O := (0, \dots, 0)^T \in \mathbf{R}^{k+1}$  is always a fixed point, and other fixed points  $P := (x_0, \dots, x_0)^T \in \mathbf{R}^{k+1}$  satisfy the following equation:

$$\sin \left[ \left( \sum_{i=1}^{k+1} \gamma_i \right) x_0 \right] = \frac{1 - \sum_{i=1}^{k+1} a_i}{\beta} x_0. \tag{6}$$

In the following, we only show the fixed point  $O$  can be a regular and nondegenerate snap-back repeller of system (5) under some conditions.

**Theorem 1** *There exist some constants  $\beta > 0$  and  $\gamma_j$ ,  $1 \leq j \leq k$ , with*

$$\beta > \max \{ 1 + |a_1|, |a_2|, \dots, |a_{k+1}| \}, \quad |\gamma_1| > \sum_{j=2}^{k+1} |\gamma_j| + \frac{1 + \sum_{j=1}^{k+1} |a_j|}{\beta}, \tag{7}$$

*such that the controlled system (5), and consequently system (2), is chaotic in the sense of both Devaney and Li-Yorke.*

*Proof* We will use Lemma 1 to prove this theorem. So, it suffices to show that all the conditions in Lemma 1 are satisfied. As assumed in the statement of the theorem, let  $\beta$  and  $|\gamma_1|$  satisfy condition (7) throughout the proof.

First, we will show that  $O$  is an expanding fixed point of  $F$  in some norm in  $\mathbf{R}^{k+1}$ . In fact,  $F$  is continuously differentiable in  $\mathbf{R}^{k+1}$ , and the Jacobian matrix of  $F$  at  $O$  is

$$DF(O) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ \beta\gamma_1 + a_1 & \beta\gamma_2 + a_2 & \beta\gamma_3 + a_3 & \cdots & \beta\gamma_{k+1} + a_{k+1} \end{pmatrix}_{(k+1) \times (k+1)}.$$

Its eigenvalues are determined by

$$\lambda^{k+1} - (\beta\gamma_{k+1} + a_{k+1})\lambda^k - (\beta\gamma_k + a_k)\lambda^{k-1} - \cdots - (\beta\gamma_2 + a_2)\lambda - (\beta\gamma_1 + a_1) = 0. \tag{8}$$

It follows from (8) and the second relation of condition (7) that all the eigenvalues of  $DF(O)$  have absolute values larger than 1 in norm. Otherwise, suppose that there exists an eigen-

value  $\lambda_0$  of  $DF(O)$  with  $|\lambda_0| \leq 1$ , then we get the following inequality:

$$\begin{aligned} & 1 + \beta \sum_{j=2}^{k+1} |\gamma_j| + \sum_{j=2}^{k+1} |a_j| \\ & \geq |\lambda_0^{k+1} - (\beta\gamma_{k+1} + a_{k+1})\lambda_0^k - (\beta\gamma_k + a_k)\lambda_0^{k-1} - \dots - (\beta\gamma_2 + a_2)\lambda_0| \\ & = |\beta\gamma_1 + a_1| > \beta|\gamma_1| - |a_1| > 1 + \beta \sum_{j=2}^{k+1} |\gamma_j| + \sum_{j=2}^{k+1} |a_j|, \end{aligned}$$

which is a contradiction. Hence, it follows from the first condition of Lemma 1, there exist a positive constant  $r$  and a norm  $\|\cdot\|^*$  in  $\mathbf{R}^{k+1}$  such that  $O$  is an expanding fixed point of  $F$  in  $\bar{B}_r(O)$  in the norm  $\|\cdot\|^*$ , that is,

$$\|F(x) - F(y)\|^* \geq \mu \|x - y\|^*, \quad \forall x, y \in \bar{B}_r(O),$$

where  $\mu > 1$  is an expanding coefficient of  $F$  in  $\bar{B}_r(O)$  and  $\bar{B}_r(O)$  is the closed ball centered at  $O \in \mathbf{R}^{k+1}$  of radius  $r$  with respect to the norm  $\|\cdot\|^*$ .

From (6) and (7), we see that for sufficiently large  $\beta$  and  $|\gamma_1|$ , the absolute value of  $x_0$  can become very small. So, the distance of the fixed points  $O$  and  $P$  can be very small, and consequently, the  $r$  obtained above will be less than 1.

Next, we show  $O$  is a snap-back repeller of  $F$  in the norm  $\|\cdot\|^*$ . For fixed  $a_1$  and sufficiently large  $\beta$  and  $|\gamma_1|$ , the equation

$$\sin(\gamma_1 x) = \frac{1 - a_1 x}{\beta} \tag{9}$$

has a solution  $x^* \neq 0$  near 0, which implies that  $O_0 = (x^*, 0, \dots, 0)^T \in B_r(O)$  with  $O_0 \neq O$ . It follows from the first relation of (7) that  $|\frac{a_j}{\beta}| < 1, 1 \leq j \leq k + 1$ . Let

$$\gamma_j = 2m_j\pi + \arcsin\left(-\frac{a_j}{\beta}\right), \quad m_j \in \mathbf{Z}, 1 \leq j \leq k + 1.$$

Then

$$\sin \gamma_j = -\frac{a_j}{\beta}, \quad 1 \leq j \leq k + 1. \tag{10}$$

Set  $O_j = F(O_{j-1}), 1 \leq j \leq k + 1$ . We can easily get that  $O_j = (0, \dots, 0, \underbrace{1, 0, \dots, 0}_j)^T \notin B_r(O)$  for  $1 \leq j \leq k + 1$ , and

$$F^{k+2}(O_0) = O,$$

which implies that  $O$  is a snap-back repeller of  $F$ .

Finally, we show that

$$\det DF(O_j) \neq 0, \quad 0 \leq j \leq k + 1.$$

A direct calculation shows that for any  $u = (u_1, \dots, u_{k+1})^T \in \mathbf{R}^{k+1}$ ,

$$\det DF(u) = (-1)^k \left[ a_1 + \beta \gamma_1 \cos \left( \sum_{j=1}^{k+1} \gamma_j u_j \right) \right].$$

So, the following inequalities:

$$\det DF(O_0) = (-1)^k [a_1 + \beta \gamma_1 \cos(\gamma_1 x^*)] \neq 0,$$

$$\det DF(O_1) = (-1)^k (a_1 + \beta \gamma_1 \cos \gamma_{k+1}) \neq 0,$$

...

$$\det DF(O_{k+1}) = (-1)^k (a_1 + \beta \gamma_1 \cos \gamma_1) \neq 0$$

can be carried out from equations (9) and (10) by choosing sufficiently large  $\beta$ . Therefore, all the assumptions in Lemma 1 are satisfied and  $O$  is a regular and nondegenerate snap-back repeller of system (5). So, system (5), *i.e.*, equation (2), is chaotic in the sense of both Devaney and Li-Yorke. The proof is complete.  $\square$

**Remark 2** Zhang and Chen [25], Kwok and Tang [26] had independently studied the chaotification of system (4) with the following single-input controllers:

$$bv(n) = -b \sin \left( \sum_{j=1}^{k+1} \alpha_j u_j(n) \right) \quad \text{and} \quad bv(n) = -b \sum_{j=1}^{k+1} \alpha_j \tanh(\beta_j u_j(n)),$$

respectively, where  $b = (0, \dots, 0, 1)^T \in \mathbf{R}^{k+1}$  and  $\alpha_j, \beta_j, 1 \leq j \leq k + 1$ , are undetermined parameters. They showed that the controlled system is chaotic in the sense of Li-Yorke by using the Marotto theorem. However, there exists some problem in their proofs. We state as follows. By the definition of a snap-back repeller, if a point  $z$  is a snap-back repeller of a map  $g$ , then  $z$  is an expanding fixed point of  $g$  in  $\bar{B}_r(z)$  for some  $r > 0$ , and there exists a point  $x_0 \in B_r(z)$  with  $x_0 \neq z$  such that  $g^m(x_0) = z$  for some positive integer  $m$ . It is easy to show that there must exist a positive integer  $m_0 < m$  such that  $g^{m_0}(x_0) \notin B_r(z)$ . However, in the proofs of both [25] and [26], they all proved that the map  $g$  of the controlled system is expanding in a neighborhood  $B_r(O)$  of the origin  $O \in \mathbf{R}^{k+1}$ , and there exists a point  $x_0 \in B_r(O)$  such that  $g(x_0) \in B_r(O)$  and  $g^2(x_0) = O$ , and consequently  $O$  is a snap-back repeller of the map  $g$ . This is a contradiction with the definition of a snap-back repeller. In this paper, we use controller (3) to chaotify the linear delay system (1), which corresponds to the chaotification of linear system (4) with the following single-input controller:

$$bv(n) = b\beta \sin \left( \sum_{j=1}^{k+1} \alpha_j u_j(n) \right).$$

This controller is slightly different from that used in [25]. However, the problem is solved, and it is rigorously proved that system (4) with the above controller is chaotic in the sense of both Devaney and Li-Yorke by using the snap-back repeller theory.

#### 4 An example

In the last section, we present an example of chaotification for a linear delay difference equation with computer simulations.

**Example 1** The linear delay difference equation (1) is taken as the following:

$$x(n+1) = - \sum_{j=1}^{k+1} \frac{1}{2^j} x(n+j-k-1), \quad n \geq 0, \tag{11}$$

where  $k \geq 1$  is a fixed integer.

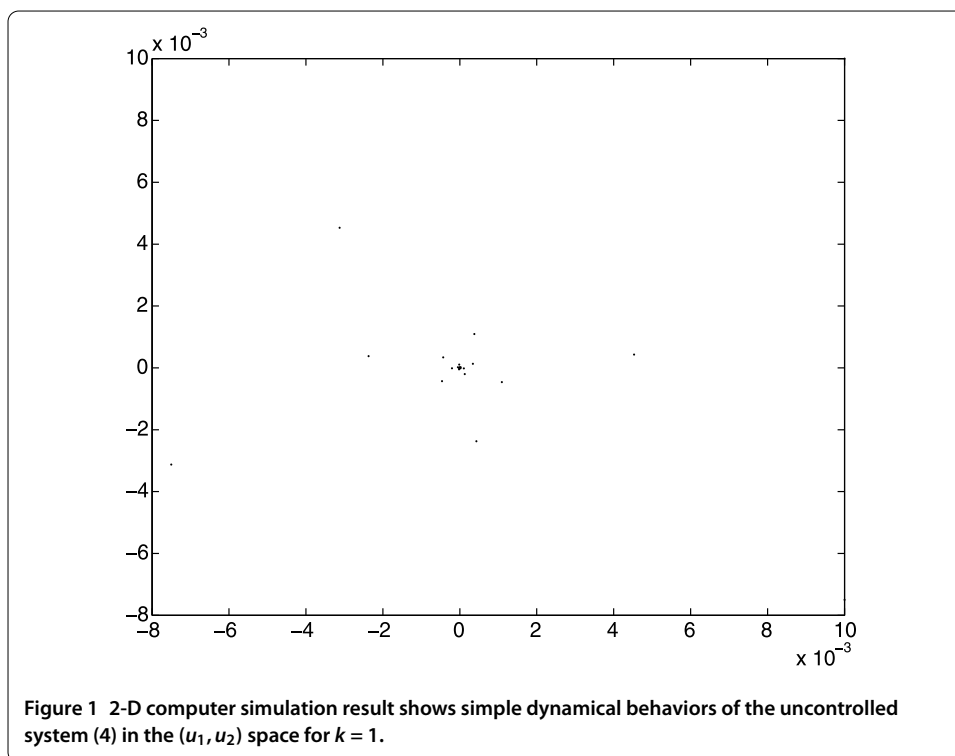
Obviously,  $O = (0, 0, \dots, 0)^T \in \mathbf{R}^{k+1}$  is a fixed point of equation (11). By Theorem 1, we can take controller (3) with  $\beta$  and  $\gamma_j$ ,  $1 \leq j \leq k+1$ , as specified in the proof, such that the output of the controlled system (5), *i.e.*, system (2), is chaotic in the sense of both Devaney and Li-Yorke. In order to help better visualize the theoretical result, we take  $k = 1, 2$  for computer simulations. For  $k = 1$ , we can take

$$\beta = 10, \quad \gamma_1 = 2\pi + \arcsin \frac{1}{20}, \quad \gamma_2 = \arcsin \frac{1}{40}.$$

For  $k = 2$ , we take

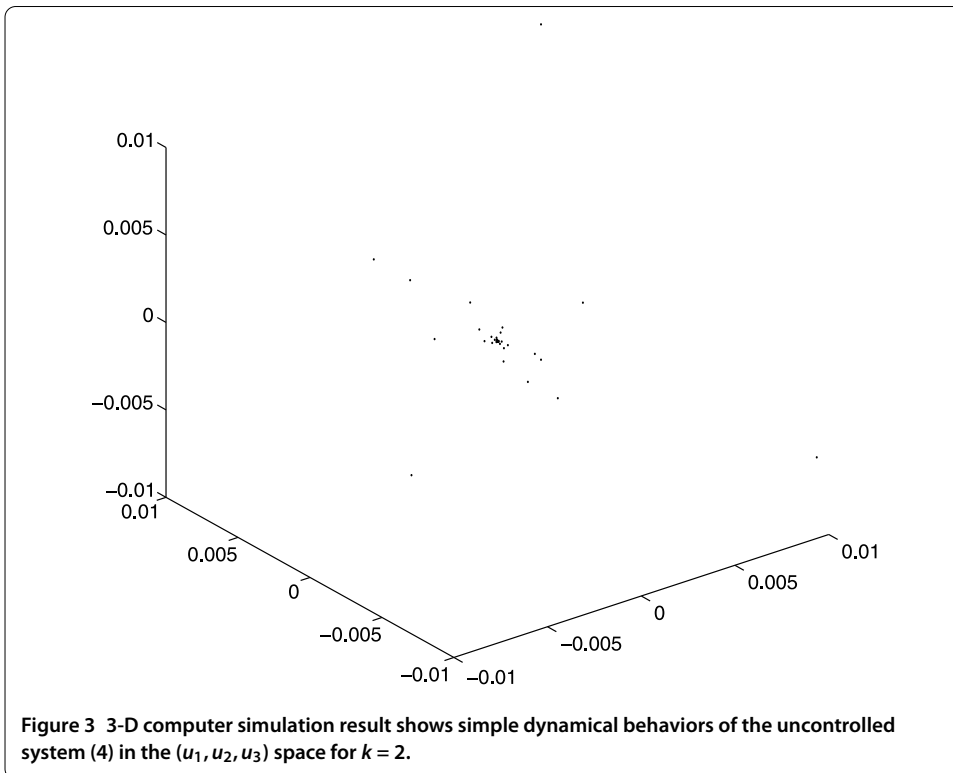
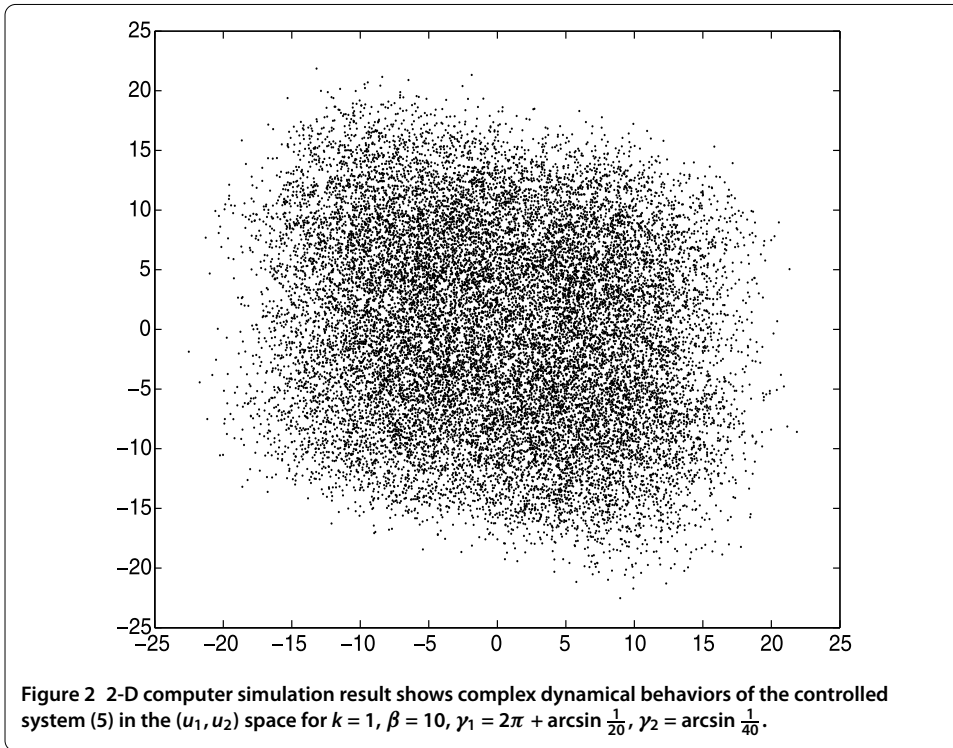
$$\beta = 10, \quad \gamma_1 = 2\pi + \arcsin \frac{1}{20}, \quad \gamma_2 = \arcsin \frac{1}{40}, \quad \gamma_3 = \arcsin \frac{1}{60}.$$

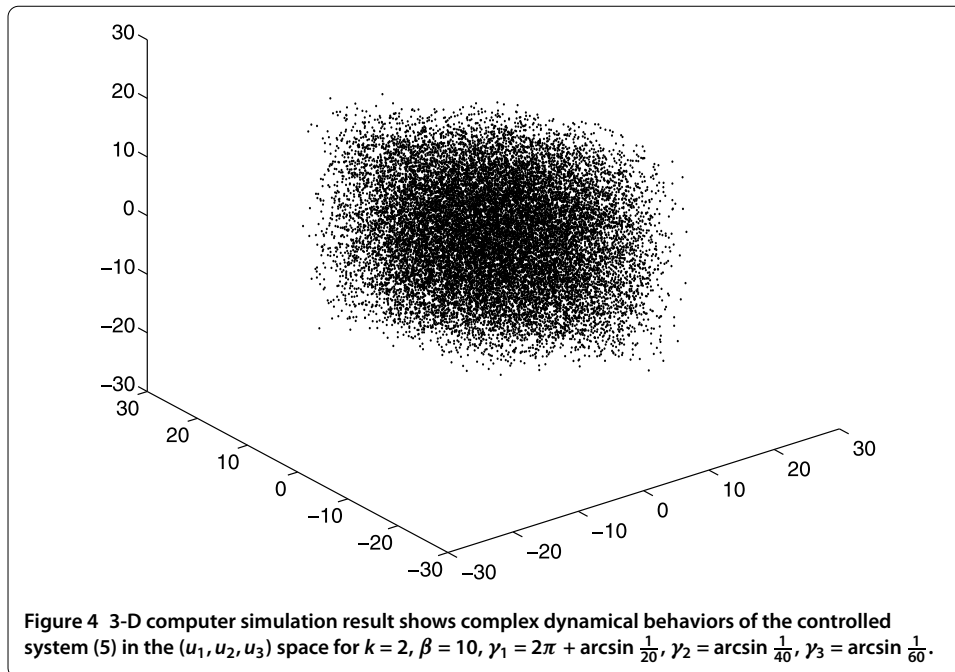
Both of them satisfy the condition (7). The simulated results show that the origin system (4), *i.e.*, system (11), has simple dynamical behaviors, and the controlled system (5), *i.e.*, system (2), has complex dynamical behaviors, see Figures 1-4.



**Figure 1** 2-D computer simulation result shows simple dynamical behaviors of the uncontrolled system (4) in the  $(u_1, u_2)$  space for  $k = 1$ .







#### Competing interests

The author declares that he has no competing interests.

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