Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Meromorphic functions sharing small functions with their linear difference polynomials

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Abstract

In this paper, we prove some results on the uniqueness of meromorphic functions sharing small functions CM with their linear difference polynomials. Examples are provided to show the existence of meromorphic functions satisfying the conditions of our results.

MSC: Primary 30D35; secondary 39A10; 39B32

Keywords: uniqueness; meromorphic functions; difference polynomials

1 Introduction and main results

In the following, we use the standard notations of Nevanlinna theory of meromorphic functions (see [1–3]). For any given nonconstant meromorphic function $f(z)$, we recall the hyper order of $f(z)$ defined as follows (see [3]):

$$\rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside of a set of r with a finite logarithmic measure. A meromorphic function $a(z)$ is said to be a small function of $f(z)$ if $T(r,a) = S(r,f)$. In what follows, we use $S(f)$ to denote the set of all small functions of $f(z)$.

For two meromorphic functions $f(z)$ and $g(z)$, and $a \in S(f) \cup S(g) \cup \{\infty\}$, we say that $f(z)$ and $g(z)$ share a CM when $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicity.

For a nonzero complex constant $c \in \mathbb{C}$, $f(z + c)$ is called a shift of $f(z)$. And a difference monomial of type $\prod_{i=1}^{m} f^{n_i}(z + c_i)$ is called a difference product of $f(z)$, where $c_1, \dots, c_m \in \mathbb{C}$ and $n_1, \dots, n_m \in \mathbb{N}$.

A difference polynomial of $f(z)$ is a finite sum of difference products of $f(z)$, with all coefficients being small functions of $f(z)$. In the following, we mainly consider a linear difference polynomial of $f(z)$ of the form

$$L(z,f) = \sum_{i=1}^{n} a_i(z) f(z + c_i),$$

where $c_1, \dots, c_n \in \mathbb{C}$, $a_1(z), \dots, a_n(z) \in S(f)$.

It is well known that the difference operators of $f(z)$ are defined as follows:

$$\Delta_c f(z) = f(z+c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c^{n-1}\big(\Delta_c f(z)\big), \quad n \in \mathbb{N}, n \geq 2.$$

In particular, $\Delta_c^n f(z) = \Delta^n f(z)$ for the case $c = 1$. We point out that a difference operator is just a special linear difference polynomial of $f(z)$ such that the sum of its coefficients equals 0.

The subject on the uniqueness of the entire function $f(z)$ sharing values with its derivative $f'(z)$ was initiated by Rubel and Yang [4]. For a nonconstant entire function $f(z)$, they proved that $f(z) \equiv f'(z)$ provided that $f(z)$ and $f'(z)$ share two distinct finite values CM.

Recently, a number of papers have focused on the Nevanlinna theory with respect to difference operators; see, *e.g.*, the papers [5, 6] by Chiang and Feng and [7, 8] by Halburd and Korhonen. Then, many authors started to investigate the uniqueness of meromorphic functions sharing values or small functions with their shifts (see, *e.g.*, [9–14]) or difference operators (see, *e.g.*, [9, 12]). The following Theorem A is indeed a corollary of Theorem 2.1 in [10] and Theorem 2 in [11].

Theorem A ([10, 11]) *Let $f(z)$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2 \in S(f)$ be two distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a_1, a_2, ∞ CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Theorem B below is Theorem 1.2 in [9], while Theorem C is Theorem 1.1 in [12].

Theorem B ([9]) *Let $f(z)$ be a transcendental meromorphic function such that its order of growth $\rho(f)$ is not an integer or infinite, and let $c \in \mathbb{C}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then $f(z+c) = 2f(z)$.*

Theorem C ([12]) *Let $f(z)$ be a nonconstant entire function of finite order, $c \in \mathbb{C}$, and n be a positive integer. Suppose that $f(z)$ and $\Delta_c^n f(z)$ share two distinct finite values a, b CM and one of the following cases is satisfied*:

(i) $ab = 0$;

(ii) $ab \neq 0$ *and* $\rho(f) \notin \mathbb{N}$.

Then $f(z) \equiv \Delta_c^n f(z)$.

Remark 1 The methods in [9] and [12] are quite different. Due to a result of Ozawa [15] (he proved that for any given $\rho \in [1,\infty)$, there exists a periodic entire function of order ρ), Chen and Yi [9] and Li and Gao [12] gave some examples to show the existence of functions satisfying the conditions of Theorem B and Theorem C respectively.

Considering Theorems A-C, due to some ideas of [9] and [12], we obtain the following result with a quite simple proof.

Theorem 1.1 *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f) < 1$, let $L(z,f)$ be a difference polynomial of $f(z)$, and let $a, b \in S(f)$ be two distinct meromorphic functions. Suppose that $f(z)$ and $L(z,f)$ share a, b, ∞ CM and one of the following cases holds*:

(i) $L(z,a) - a = L(z,b) - b \equiv 0$;

(ii) $L(z,a) - a \equiv 0$ *or* $L(z,b) - b \equiv 0$, *and* $N(r,f) < \lambda T(r,f)$ *for some* $\lambda \in (0,1)$;

(iii) $\rho(f) \notin \mathbb{N} \cup \{\infty\}$.
Then $f(z) \equiv L(z,f)$.

Example 1 We give two examples for Theorem 1.1.

(1) For the cases (i) and (ii): Let $f(z) = e^{z \log 3}$ and $L(z,f) = \Delta f(z) - f(z) = f(z+1) - 2f(z)$. Then $L(z,f) = f(z)$, and hence for $a = 0$ and any given $b \in S(f)$, $f(z)$ and $L(z,f)$ share a, b, ∞ CM.
(2) For the case (iii): Let $f(z) = g(z)e^{z \log 3}$ and $L(z,f) = \Delta f(z) - f(z) = f(z+1) - 2f(z)$, where $g(z)$ is a periodic entire function with period 1 such that $\rho(g) \in (1,\infty) \setminus \mathbb{N}$. Then $L(z,f) = f(z)$, and hence for any given $a, b \in S(f)$, $f(z)$ and $L(z,f)$ share a, b, ∞ CM.

For one CM shared value case, Li and Gao [12] proved the following results.

Theorem D ([12]) *Let $f(z)$ be a nonconstant entire function of finite order $\rho(f)$, $\eta \in \mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share one finite value a CM, and for a finite value $b \neq a$, $f(z) - b$ and $f(z+\eta) - b$ have* $\max\{1, [\rho(f)] - 1\}$ *distinct common zeros of multiplicity ≥ 2, then $f(z) \equiv f(z+\eta)$.*

Theorem E ([12]) *Let $f(z)$ be a nonconstant entire function of finite order $\rho(f)$, $\eta \in \mathbb{C}$, and n be a positive integer. If $f(z)$ and $\Delta_\eta^n f(z)$ share one finite value a CM, and for a finite value $b \neq a$, $f(z) - b$ and $\Delta_\eta^n f(z) - b$ have* $\max\{1, [\rho(f)]\}$ *distinct common zeros of multiplicity ≥ 2, then $f(z) \equiv \Delta_\eta^n f(z)$.*

To generalize Theorems D and E, we prove Theorem 1.2 below.

Theorem 1.2 *Let $f(z)$ be a meromorphic function of finite order $\rho(f)$, let $L(z,f)$ be a difference polynomial of $f(z)$, and let $a, b \in S(f)$ be two distinct meromorphic functions. Suppose that $f(z)$ and $L(z,f)$ share a, ∞ CM and $f(z) - b$ and $L(z,f) - b$ have $m =$* $\max\{1, [\rho(f)]\}$ *distinct common zeros of multiplicity ≥ 2, denoted by $z_1, z_2, \ldots, z_m$, such that $a(z_i) \neq b(z_i)$. Then $f(z) \equiv L(z,f)$.*

Example 2 Let $f(z) = g^2(z)e^{z \log 3}$ and $L(z,f) = \Delta f(z) - f(z) = f(z+1) - 2f(z)$, where $g(z)$ is a periodic entire function with period 1 such that $\rho(g) \in (1,\infty) \setminus \mathbb{N}$. Then $L(z,f) = f(z)$, and hence for any given $a \in S(f)$ and $b = 0$, $f(z)$ and $L(z,f)$ share a, ∞ CM.

Remark 2 Chen and Yi [9] (resp. Li and Gao [12]) conjectured that the condition on the order of growth of $f(z)$ in Theorem B (resp. Theorem C) could be omitted. The same conjecture should be made for Theorems 1.1 and 1.2.

2 Lemmas

Lemma 2.1 ([16]) *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f) = \varsigma < 1$, $c \in \mathbb{C}$, and $\varepsilon > 0$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon)}}\right) = S(r,f),$$

possibly outside of a set of r with a finite logarithmic measure.

The following lemma is a Clunie-type lemma [17] for the difference-differential polynomials of a meromorphic function f, which is a finite sum of products of f, derivatives of f, and of their shifts, with all the coefficients being small functions of f. It can be proved by applying Lemma 2.1 with a similar reasoning as in [18] and stated as follows.

Lemma 2.2 ([18]) *Let $f(z)$ be a meromorphic function of hyper order $\rho_2(f) < 1$ and $P(z,f)$, $Q(z,f)$ be two difference-differential polynomials of f. If*

$$f^n P(z,f) = Q(z,f)$$

holds and if the total degree of $Q(z,f)$ in f and its derivatives and their shifts is $\leq n$, then $m(r, P(z,f)) = S(r,f)$.

3 Proof of Theorem 1.1

Since $f(z)$ and $L := L(z,f)$ share the value a, b, ∞ CM, we have

$$\frac{L-a}{f-a} = e^p \tag{3.1}$$

and

$$\frac{L-b}{f-b} = e^q, \tag{3.2}$$

where $p = p(z)$, $q = q(z)$ are entire functions such that $\max\{\rho(e^p), \rho(e^q)\} \leq \rho_2(f)$.

It follows from (3.1) and (3.2) that

$$\left(e^q - e^p\right)f = a - b + be^q - ae^p. \tag{3.3}$$

If $e^p \equiv e^q$, then from (3.3) we obtain

$$(a-b)\left(1-e^p\right) = 0.$$

Since $a - b \not\equiv 0$, we get $e^p \equiv 1$ and hence finish our proof from (3.1).

Next, we assume that $e^p \not\equiv e^q$ and complete our proof in three steps.

Step 1. We prove the case (i): $L(z,a) - a = L(z,b) - b \equiv 0$. From (3.1), we get from Lemma 2.1 that

$$T\left(r, e^p\right) = m\left(r, e^p\right) = m\left(r, \frac{L-a}{f-a}\right) = m\left(r, \frac{L(z, f-a)}{f-a}\right) = S(r,f). \tag{3.4}$$

Similarly, we have $T(r, e^q) = S(r,f)$. Now, we can deduce a contradiction from (3.3) that

$$\begin{aligned} T(r,f) &= T\left(r, \frac{a-b+be^q-ae^p}{e^q-e^p}\right) \\ &\leq 2\left(T(r,a) + T(r,b) + T\left(r,e^p\right) + T\left(r,e^p\right)\right) + S(r,f) = S(r,f). \end{aligned}$$

Step 2. We prove the case (iii): $\rho(f) \notin \mathbb{N} \cup \{\infty\}$. In this case, $p(z)$, $q(z)$ are polynomials, and we have

$$\max\{\deg p(z), \deg q(z)\} \leq [\rho(f)] < \rho(f). \tag{3.5}$$

From (3.3), we obtain

$$T(r,f) = T\left(r, \frac{a-b+be^q-ae^p}{e^q-e^p}\right) \leq 2T(r,e^p) + 2T(r,e^q) + S(r,f),$$

which gives $\rho(f) \leq \max\{\deg p(z), \deg q(z)\}$, a contradiction to (3.5).

Step 3. We prove the case (ii): $L(z,a) - a \equiv 0$ or $L(z,b) - b \equiv 0$, and $N(r,f) < \lambda T(r,f)$ for some $\lambda \in (0,1)$. Without loss of generality, we assume that $L(z,a) - a \equiv 0$ and hence (3.4) still holds.

Differentiating (3.1) and (3.2), we get

$$L'f - f'L - p'fL = a(L' - f') + a'(f - L) - ap'(f + L) + p'a^2$$

and

$$L'f - f'L - q'fL = b(L' - f') + b'(f - L) - bq'(f + L) + p'b^2.$$

Combining two equations above, we get

$$A_1 fL = A_2(L' - f') + A_3 f + A_4 L + A_5, \tag{3.6}$$

where $A_1 = p' - q'$, $A_2 = b - a$, $A_3 = b' - a' + ap' + bq'$, $A_4 = a' - b'_a p' + bq'$, $A_5 = q'b^2 - p'a^2$.

Notice that the right-hand side of (3.6) is a difference-differential polynomial of f with degree in f, its derivatives and their shifts being ≤ 1. Then from Lemma 2.2 and its remark, we have $m(r,L) = S(r,f)$. Considering this, with (3.1) and (3.4), we obtain

$$m(r,f) = m\left(r, \frac{L-a}{e^p} + a\right) \leq m(r,e^p) + m(r,L) + 2m(r,a) + S(r,f) = S(r,f),$$

and hence $T(r,f) = N(r,f) + m(r,f) = N(r,f) + S(r,f)$, which contradicts the condition $N(r,f) < \lambda T(r,f)$ for some $\lambda \in (0,1)$.

4 Proof of Theorem 1.2

Since $f(z)$ and $L(z,f)$ share a CM, we have

$$\frac{L(z,f) - a}{f(z) - a} = e^p, \tag{4.1}$$

where p is a polynomial such that $\deg p(z) \leq \max\{1, [\rho(f)]\} = m$.

It follows from (4.1) that

$$L'(z,f) - a' = (f'(z) - a')e^p + p'(f(z) - a)e^p. \tag{4.2}$$

For each point z_i, $1 \leq i \leq m$, satisfying the assumption in Theorem 1.2, we get

$$f(z_i) = L(z_i, f) = b(z_i) \neq a(z_i), \tag{4.3}$$

and

$$f'(z_i) - b'(z_i) = L'(z_i, f) - b'(z_i) = 0, \tag{4.4}$$

from (4.1) and (4.3), we see that $e^{p(z_i)} = 1$. Then we can obtain from (4.2) and (4.4) that $p'(z_i) = 0$. By assumption, $p'(z)$ has at least m zeros. This means that $p'(z) \equiv 0$. Therefore,

$$L(z, f) - a = c\big(f(z) - a\big) \tag{4.5}$$

holds for some nonconstant c. For the point z_1 such that $L(z_1, f) = f(z_1) = b(z_1) \neq a(z_1)$, we get from (4.5) that $c = 1$ and hence prove that $f(z) \equiv L(z, f)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors drafted the manuscript, read and approved the final manuscript.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (No. 11226091, No. 11171013).

Received: 30 November 2012 Accepted: 24 February 2013 Published: 18 March 2013

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doi:10.1186/1687-1847-2013-58
Cite this article as: Li and Chen: **Meromorphic functions sharing small functions with their linear difference polynomials.** *Advances in Difference Equations* 2013 **2013**:58.