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Unified Bernstein and Bleimann-Butzer-Hahn basis and its properties

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Abstract

In this paper we introduce the unification of Bernstein and Bleimann-Butzer-Hahn basis via the generating function. We give the representation of this unified family in terms of Apostol-type polynomials and Stirling numbers of the second kind. More generating functions of trigonometric type are also obtained to this unification.

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1 Introduction

In this paper, we introduce a two-parameter generating function, which generates not only the Bernstein basis polynomials, but also the Bleimann-Butzer-Hahn basis functions. The generating function that we propose is given by

$$\mathcal{G}_{a,b}(t,x;k,m) := \left[\frac{2^{1-k}x^k t^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} e^{t[\frac{1+bx}{1+ax}]} = \sum_{n=0}^{\infty} \mathcal{P}_n^{(a,b)}(x;k,m)\frac{t^n}{n!}, \tag{1}$$

where $k, m \in \mathbb{Z}^+ := \{1, 2, \ldots\}$, $a, b \in \mathbb{R}$, $t \in \mathbb{C}$. Here, $x \in I$ where I is a subinterval of $\mathbb{R}$ such that the expansion in (1) is valid. The following two cases will be important for us.

1. The case $a = 0$, $b = -1$. In this case, we let $x \in [0, 1]$ and we see that

$$\mathcal{G}_{0,-1}(t,x;k,m) = \left[2^{1-k}x^k t^k\right]^m \frac{1}{(mk)!} e^{t[1-x]} = \sum_{n=0}^{\infty} \mathcal{P}_n^{(0,-1)}(x;k,m)\frac{t^n}{n!}$$

generates the unifying Bernstein basis polynomials $\mathcal{P}_n^{(0,-1)}(x;k,m) := \mathcal{B}_n(mk,x)$ which were introduced and investigated in [1]. We should note further that $\mathcal{G}_{0,-1}(t,x;1,m)$ gives

$$\mathcal{G}_{0,-1}(t,x;1,m) = [xt]^m \frac{1}{m!} e^{t[1-x]} = \sum_{n=0}^{\infty} \mathcal{B}_n(m,x)\frac{t^n}{n!}$$

which generates the celebrated Bernstein basis polynomials (see [2–8])

$$\mathcal{B}_n(m,x) := B_m^n(x) = \binom{n}{m} x^k (1-x)^{n-m}.$$

Note that the Bernstein operators $B_n : C[0,1] \to C[0,1]$ are given by

$$B_n(f;x) = \sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m} x^k (1-x)^{n-m}, \quad n \in \mathbb{N} := \{1,2,\ldots\}$$

and by the Korovkin theorem, it is known that $B_n(f;x) \rightrightarrows f(x)$ for all $f \in C[0,1]$, where $C[0,1]$ denotes the space of continuous functions defined on $[0,1]$, and the notation '$\rightrightarrows$' denotes the uniform convergence with respect to the usual supremum norm on $C[0,1]$. Very recently, interesting properties of Bernstein polynomials were discussed in [7, 9–11] and [12].

2. The case $a = 1$, $b = 0$. In this case, we let $x \in [0,\infty)$ and define

$$\begin{aligned}\mathcal{G}_{1,0}(t,x;k,m) &:= \left[\frac{2^{1-k}x^k t^k}{(1+x)^k}\right]^m \frac{1}{(mk)!} e^{t[\frac{1}{1+x}]} \\ &= \sum_{n=0}^{\infty} \mathcal{P}_n^{(1,0)}(x;k,m)\frac{t^n}{n!}.\end{aligned}$$

We will see that this generating function produces the generalized Bleimann-Butzer-Hahn basis functions $\mathcal{P}_n^{(1,0)}(x;k,m) := \mathcal{H}_n(mk,x)$. Furthermore, the special case

$$\begin{aligned}\mathcal{G}_{1,0}(t,x;1,m) &= \left[\frac{xt}{(1+x)}\right]^m \frac{1}{(mk)!} e^{t[\frac{1}{1+x}]} \\ &= \sum_{n=0}^{\infty} \mathcal{H}_n(m,x)\frac{t^n}{n!}\end{aligned}$$

generates the well-known Bleimann-Butzer-Hahn basis functions:

$$\mathcal{H}_n(m,x) := H_m^n(x) = \binom{n}{m}\frac{x^m}{(1+x)^n}.$$

The Bleimann-Butzer-Hahn operators were introduced in [5] and defined by

$$L_n(f;x) = \frac{1}{(1+x)^n}\sum_{m=0}^{n} f\left(\frac{m}{n}\right)\binom{n}{m} x^m; \quad x \in [0,\infty), n \in \mathbb{N}.$$

Denoting $C_B[0,\infty)$ by the space of real-valued bounded continuous functions defined on $[0,\infty)$, they proved that $L_n(f) \to f$ as $n \to \infty$. On the other hand, the convergence is uniform on each compact subset of $[0,\infty)$, where the norm is the usual supremum norm of $C_B[0,\infty)$. For the review of the results concerning the Bleimann-Butzer-Hahn operators obtained in the period 1980-2009, we refer to [13].

The following theorem gives the explicit representation of the basis family defined in (1). Note that throughout the paper, we let $\mathcal{P}_n^{(a,b)}(x;k,m) := 0$ for $n \leq mk$.

Theorem 1 *If $n \geq mk$, we have*

$$\mathcal{P}_n^{(a,b)}(x;k,m) = 2^{(1-k)m} x^{mk}\binom{n}{mk}\frac{(1+bx)^{n-mk}}{(1+ax)^n}.$$

Proof Direct calculations give

$$\begin{aligned}\mathcal{G}_{a,b}(t,x;k,m) &= \left[\frac{2^{1-k}x^k t^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} e^{t[\frac{1+bx}{1+ax}]} \\ &= \frac{2^{(1-k)m}}{(mk)!}\left(\frac{xt}{1+ax}\right)^{mk} \sum_{n=0}^{\infty}\left(\frac{1+bx}{1+ax}\right)^n \frac{t^n}{n!} \\ &= 2^{(1-k)m} x^{mk} \sum_{n=mk}^{\infty} \binom{n}{mk} \frac{(1+bx)^{n-mk}}{(1+ax)^n} \frac{t^n}{n!}. \end{aligned} \tag{2}$$

Comparing (1) and (2), we get the result. □

Corollary 2 *By taking $a = 0$, $b = -1$ in Theorem 1, we obtain the explicit representation of the unifying Bernstein basis polynomials* [1]:

$$\mathcal{P}_n^{(0,-1)}(x;k,m) := \mathcal{B}_n(mk,x) = 2^{(1-k)m} x^{mk} \binom{n}{mk} (1-x)^{n-mk}.$$

Furthermore, $\mathcal{B}_n(m,x) = B_m^n(x)$ is the well-known Bernstein basis.

Corollary 3 *Taking $a = 1$, $b = 0$ in Theorem 1, we get the explicit representation of the generalized Bleimann-Butzer-Hahn basis*:

$$\mathcal{P}_n^{(1,0)}(x;k,m) := \mathcal{H}_n(mk,x) = 2^{(1-k)m} x^{mk} \binom{n}{mk} \frac{1}{(1+x)^n}.$$

Moreover, $\mathcal{H}_n(m,x) = H_m^n(x)$ is the Bleimann-Butzer-Hahn basis function.

We organize the paper as follows. In Section 2, we obtain the representation of this unified family in terms of Apostol-type polynomials and Stirling numbers of the second kind. In Section 3, we give more trigonometric generating functions for this unification and obtain a certain summation formula. All the special cases are listed at the end of each theorem.

2 Representation in terms of Apostol-type polynomials and Stirling numbers

Recently [14], the first author introduced the unification of the Apostol-Bernoulli, Euler and Genocchi polynomials by

$$\begin{aligned}\mathcal{P}_{a,b}^{(\alpha)}(x;t;k,\beta) &:= \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} Q_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!} \\ &\big(k \in \mathbb{N}_0; a,b \in \mathbb{R}\backslash\{0\}; \alpha,\beta \in \mathbb{C}\big). \end{aligned} \tag{3}$$

For the convergence of the series in (3), we refer to [14, p.2453].

Some of the well-known polynomials included by $Q_{n,\beta}^{(\alpha)}(x;k,a,b)$ are listed below.

Remark 4 Having $k = a = b = 1$ and $\beta = \lambda$ in (3), we get

$$Q_{n,\lambda}^{(\alpha)}(x;1,1,1) = \mathcal{B}_n^{(\alpha)}(x;\lambda).$$

Note that $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ are the generalized Apostol-Bernoulli polynomials defined through the following generating relation:

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$$
$$\bigl(|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log\lambda| \text{ when } \lambda \neq 1\bigr),$$

where α and λ are arbitrary real or complex parameters and $x \in \mathbb{R}$. Note that when $\lambda \neq 1$, the order α should be restricted to nonnegative integer values. These polynomials were introduced by Luo and Srivastava [15] and investigated in [16, 17] and [18]. The Apostol-Bernoulli polynomials and numbers are obtained by the generalized Apostol-Bernoulli polynomials, respectively, as follows:

$$B_n(x;\lambda) = \mathcal{B}_n^{(1)}(x;\lambda), \qquad B_n(\lambda) = B_n(0;\lambda) \quad (n \in \mathbb{N}_0).$$

Taking $\lambda = 1$ in the above relations, we obtain the classical Bernoulli polynomials $B_n(x)$ and Bernoulli numbers B_n.

Remark 5 Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in (3), we get

$$Q_{n,\frac{\lambda}{2}}^{(\alpha)}\left(x;1,\frac{-1}{2},1\right) = \mathcal{G}_n^{\alpha}(x;\lambda),$$

the Apostol-Genocchi polynomial of order α (arbitrary real or complex) which was defined by [19, 20]. Here the parameter λ is arbitrary real or complex. These polynomials are given as follows:

$$\left(\frac{2t}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{\alpha}(x;\lambda)\frac{t^n}{n!}$$
$$\bigl(|t| < \pi \text{ when } \lambda = 1; |t| < \bigl|\log(-\lambda)\bigr| \text{ when } \lambda \neq 1\bigr).$$

Note that when $\lambda \neq -1$, the order α should be restricted to nonnegative integer values. The Apostol-Genocchi polynomials and numbers are respectively given by

$$G_n(x;\lambda) = \mathcal{G}_n^{1}(x;\lambda), \qquad G_n(\lambda) = G_n(0;\lambda).$$

When $\lambda = 1$, the above relations give the classical Genocchi polynomials $G_n(x)$ and Genocchi numbers G_n.

Although our results do not contain the Apostol-Euler polynomials, for the sake of completeness, we give their definitions as a special case of the polynomial family $Q_{n,\beta}^{(\alpha)}(x;k,a,b)$.

Remark 6 Setting $k + 1 = -a = b = 1$ and $\beta = \lambda$ in (3), we get

$$Q_{n,\lambda}^{(\alpha)}(x;0,-1,1) = \mathcal{E}_n^{(\alpha)}(x;\lambda).$$

Recall that the Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ are generalized by Luo [21] and given by the generating relation

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{E}_n^{\alpha}(x;\lambda)\frac{t^n}{n!}$$
$$\bigl(|t|<\pi \text{ when } \lambda=1; |t|<\bigl|\log(-\lambda)\bigr| \text{ when } \lambda\neq 1; 1^{\alpha}:=1\bigr)$$

for arbitrary real or complex parameters α and λ and $x\in\mathbb{R}$. The Apostol-Euler polynomials and numbers are given respectively by

$$E_n(x;\lambda)=\mathcal{E}_n^1(x;\lambda), \qquad E_n(\lambda)=E_n(1;\lambda).$$

When $\lambda=1$, the above relations give the classical Euler polynomials $E_n(x)$ and Euler numbers E_n.

Now, recall that the Stirling numbers of the second kind are denoted by $S(j,i)$ and defined by (see [22, p.58 (15)])

$$\left(e^t-1\right)^i = i!\sum_{j=i}^{\infty} S(j,i)\frac{t^j}{j!}.$$

The following theorem states an interesting explicit representation of the unified basis in terms of Apostol-type polynomials and relation between Stirling numbers of the second kind.

Theorem 7 *The following representation*:

$$\mathcal{P}_n^{(a,b)}(x;k,m) = \frac{1}{(mk)!}\left(\frac{x}{1+ax}\right)^{mk}\sum_{i=0}^{m}\binom{m}{i}\left(\beta^d-c^d\right)^{m-i}\beta^{id}i! \times \sum_{j=i}^{n}\binom{n}{j}S(j,i)Q_{n-j,\beta}^{(m)}\left(\frac{1+bx}{1+ax};k,c,d\right)$$

holds true between the unified Bernstein and Bleimann-Butzer-Hahn basis and Apostol-type polynomials.

Proof We get, using (1), that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;k,m)\frac{t^n}{n!}
&= \mathcal{G}_{a,b}(t,x;k,m) \\
&= \left[\frac{2^{1-k}x^k t^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!}e^{t[\frac{1+bx}{1+ax}]} \\
&= \frac{1}{(mk)!}\left(\frac{x}{1+ax}\right)^{mk}\left[\frac{2^{1-k}t^k}{\beta^d e^t-c^d}\right]^m e^{t[\frac{1+bx}{1+ax}]}\left(\beta^d e^t-c^d\right)^m \\
&= \frac{1}{(mk)!}\left(\frac{x}{1+ax}\right)^{mk}\left[\frac{2^{1-k}t^k}{\beta^d e^t-c^d}\right]^m e^{t[\frac{1+bx}{1+ax}]}\left(\beta^d-c^d+\beta^d\left[e^t-1\right]\right)^m.
\end{aligned} \tag{4}$$

On the other hand, since

$$\begin{aligned}\left(\beta^d - c^d + \beta^d\left[e^t - 1\right]\right)^m &= \sum_{i=0}^{m} \binom{m}{i} \left(\beta^d - c^d\right)^{m-i} \beta^{id} \left[e^t - 1\right]^i \\ &= \sum_{i=0}^{m} \binom{m}{i} \left(\beta^d - c^d\right)^{m-i} \beta^{id} i! \sum_{j=i}^{\infty} S(j,i) \frac{t^j}{j!},\end{aligned}$$

we can write from (4) that

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{P}_n^{(a,b)}(x;k,m) \frac{t^n}{n!} \\ &\quad= \frac{1}{(mk)!} \left(\frac{x}{1+ax}\right)^{mk} \left[\frac{2^{1-k} t^k}{\beta^b e^t - a^b}\right]^m e^{t[\frac{1+bx}{1+ax}]} \\ &\qquad \times \sum_{i=0}^{m} \binom{m}{i} \left(\beta^b - a^b\right)^{m-i} \beta^{ib} i! \sum_{j=i}^{\infty} S(j,i) \frac{t^j}{j!}.\end{aligned}$$

Now, using (3) in the above relation, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathcal{P}_n^{(a,b)}(x;k,m) \frac{t^n}{n!} \\ &\quad= \frac{1}{(mk)!} \left(\frac{x}{1+ax}\right)^{mk} \sum_{n=0}^{\infty} Q_{n,\beta}^{(m)}\left(\frac{1+bx}{1+ax};k,c,d\right) \frac{t^n}{n!} \\ &\qquad \times \sum_{i=0}^{m} \binom{m}{i} \left(\beta^d - c^d\right)^{m-i} \beta^{id} i! \sum_{j=i}^{\infty} S(j,i) \frac{t^j}{j!} \\ &\quad= \frac{1}{(mk)!} \left(\frac{x}{1+ax}\right)^{mk} \sum_{n=0}^{\infty} \left\{ \sum_{i=0}^{m} \binom{m}{i} \left(\beta^d - c^d\right)^{m-i} \beta^{id} i! \right. \\ &\qquad \left. \times \sum_{j=i}^{n} \binom{n}{j} S(j,i) Q_{n-j,\beta}^{(m)}\left(\frac{1+bx}{1+ax};k,c,d\right) \right\} \frac{t^n}{n!}.\end{aligned}$$

Whence the result. □

Now, we list some important corollaries of the above theorem.

Corollary 8 *Since* $\mathcal{P}_n^{(0,-1)}(x;1,m) = B_m^n(x)$ *and* $Q_{n,\lambda}^{(\alpha)}(x;1,1,1) = \mathcal{B}_n^{(\alpha)}(x;\lambda)$, *we obtain the following* [1]:

$$B_m^n(x) = \frac{x^m}{m!} \sum_{i=0}^{m} \binom{m}{i} (\lambda - 1)^{m-i} \lambda^i i! \sum_{j=i}^{n} \binom{n}{j} S(j,i) \mathcal{B}_{n-j}^{(m)}(1-x;\lambda).$$

Furthermore, for $\lambda = 1$, *we have the following known relation*:

$$B_m^n(x) = x^m \sum_{j=m}^{n} \binom{n}{j} S(j,m) B_{n-j}^{(m)}(1-x).$$

Corollary 9 *Since* $\mathcal{P}_n^{(0,-1)}(x;1,m) = B_m^n(x)$ *and* $Q_{n,\frac{\lambda}{2}}^{(\alpha)}(x;1,\frac{-1}{2},1) = \mathcal{G}_n^{\alpha}(x;\lambda)$, *we get*

$$B_m^n(x) = \frac{x^m}{2^m m!}\sum_{i=0}^{m}\binom{m}{i}(\lambda+1)^{m-i}\lambda^i i!\sum_{j=i}^{n}\binom{n}{j}S(j,i)\mathcal{G}_{n-j}^m(1-x;\lambda).$$

Corollary 10 *Since* $\mathcal{P}_n^{(1,0)}(x;1,m) = H_m^n(x)$ *and* $Q_{n,\lambda}^{(\alpha)}(x;1,1,1) = \mathcal{B}_n^{(\alpha)}(x;\lambda)$, *we obtain*

$$\begin{aligned}H_m^n(x) &= \frac{1}{m!}\left(\frac{x}{1+x}\right)^m\sum_{i=0}^{m}\binom{m}{i}(\lambda-1)^{m-i}\lambda^i i! \\ &\quad\times\sum_{j=i}^{n}\binom{n}{j}S(j,i)\mathcal{B}_{n-j}^{(m)}\left(\frac{1}{1+x};\lambda\right).\end{aligned}$$

Furthermore, when $\lambda = 1$, *we have the following*:

$$H_m^n(x) = \left(\frac{x}{1+x}\right)^m\sum_{j=m}^{n}\binom{n}{j}S(j,m)B_{n-j}^{(m)}\left(\frac{1}{1+x}\right).$$

Corollary 11 *Since* $\mathcal{P}_n^{(1,0)}(x;1,m) = H_m^n(x)$ *and* $Q_{n,\frac{\lambda}{2}}^{(\alpha)}(x;1,\frac{-1}{2},1) = \mathcal{G}_n^{\alpha}(x;\lambda)$, *we get*

$$\begin{aligned}H_m^n(x) &= \frac{1}{2^m m!}\left(\frac{x}{1+x}\right)^m\sum_{i=0}^{m}\binom{m}{i}(\lambda-1)^{m-i}\lambda^i i! \\ &\quad\times\sum_{j=i}^{n}\binom{n}{j}S(j,i)\mathcal{G}_{n-j}^{m}\left(\frac{1}{1+x};\lambda\right).\end{aligned}$$

3 Generating functions of trigonometric type

In this section, we obtain a trigonometric generating relation for the unified Bernstein and Bleimann-Butzer-Hahn basis. Furthermore, we give a certain summation formula for this unification. We start with the following theorem.

Theorem 12 *For the unified family, we have the following implicit summation formulae*:

$$\begin{aligned}\left[\frac{2^{1-2l}x^{2l}}{(1+ax)^{2l}}\right]^m\frac{(-t^2)^{lm}}{(2lm)!}\cos t\left(\frac{1+bx}{1+ax}\right) &= \sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l,m)\frac{t^{2n}}{(2n)!}, \\ \left[\frac{2^{1-2l}x^{2l}}{(1+ax)^{2l}}\right]^m\frac{(-t^2)^{lm}}{(2lm)!}\sin t\left(\frac{1+bx}{1+ax}\right) &= \sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l,m)\frac{t^{2n+1}}{(2n+1)!}\end{aligned} \tag{5}$$

and

$$\begin{aligned}\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\cos t\left(\frac{1+bx}{1+ax}\right) &= \sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l+1,2j)\frac{t^{2n}}{(2n)!}, \\ \left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\sin t\left(\frac{1+bx}{1+ax}\right) &= \sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l+1,2j)\frac{t^{2n+1}}{(2n+1)!}.\end{aligned} \tag{6}$$

Finally,

$$\begin{aligned}
&\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\sin t\left(\frac{1+bx}{1+ax}\right)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l+1,2j+1)\frac{t^{2n}}{(2n)!},\\
&\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\cos t\left(\frac{1+bx}{1+ax}\right)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l+1,2j+1)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}\tag{7}$$

Proof Writing $k=2l$ ($l\in\mathbb{N}_0$) in (1), we get

$$\left[\frac{2^{1-2l}x^{2l}t^{2l}}{(1+ax)^{2l}}\right]^m\frac{1}{(2lm)!}e^{t[\frac{1+bx}{1+ax}]}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l,m)\frac{t^n}{n!}.$$

Letting $t\to it$, we get

$$\left[\frac{2^{1-2l}x^{2l}}{(1+ax)^{2l}}\right]^m\frac{(it)^{2lm}}{(2lm)!}e^{it[\frac{1+bx}{1+ax}]}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l,m)\frac{(it)^n}{n!}$$

and hence

$$\begin{aligned}
&\left[\frac{2^{1-2l}x^{2l}}{(1+ax)^{2l}}\right]^m\frac{(-t^2)^{lm}}{(2lm)!}\left\{\cos t\left(\frac{1+bx}{1+ax}\right)+i\sin t\left(\frac{1+bx}{1+ax}\right)\right\}\\
&\quad=\sum_{n=0}^{\infty}\mathcal{P}_{2n}^{(a,b)}(x;2l,m)\frac{(it)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\mathcal{P}_{2n+1}^{(a,b)}(x;2l,m)\frac{(it)^{2n+1}}{(2n+1)!}\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l,m)\frac{t^{2n}}{(2n)!}\\
&\qquad+i\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l,m)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Equating real and imaginary parts, we get (5).

Now, taking $k=2l+1$and $m=2j$ ($l,j\in\mathbb{N}_0$) in (1), we obtain

$$\left[\frac{2^{1-(2l+1)}x^{2l+1}t^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j}\frac{1}{(2j(2l+1))!}e^{t[\frac{1+bx}{1+ax}]}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l+1,2j)\frac{t^n}{n!}.$$

Putting $t\to it$,

$$\left[\frac{2^{-2l}x^{2l+1}(it)^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j}\frac{1}{(2j(2l+1))!}e^{it[\frac{1+bx}{1+ax}]}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l+1,2j)\frac{(it)^n}{n!}.$$

Therefore, we get

$$\begin{aligned}&\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\left\{\cos t\left(\frac{1+bx}{1+ax}\right)+i\sin t\left(\frac{1+bx}{1+ax}\right)\right\}\\&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l+1,2j)\frac{t^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l+1,2j)\frac{t^{2n+1}}{(2n+1)!},\end{aligned}$$

which is precisely (6).

Finally, for $k=2l+1$, $m=2j+1$,

$$\left[\frac{2^{-2l}x^{2l+1}t^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j+1}\frac{e^{t[\frac{1+bx}{1+ax}]}}{[(2j+1)(2l+1)]!}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l+1,2j+1)\frac{t^n}{n!}.$$

Taking $t\to it$,

$$\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j+1}\frac{(it)^{(2l+1)(2j+1)}e^{it[\frac{1+bx}{1+ax}]}}{[(2j+1)(2l+1)]!}=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;2l+1,2j+1)\frac{(it)^n}{n!}.$$

Thus,

$$\begin{aligned}&\left[\frac{2^{-2l}x^{2l+1}}{(1+ax)^{2l+1}}\right]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}\left[-t\sin t\left(\frac{1+bx}{1+ax}\right)+it\cos t\left(\frac{1+bx}{1+ax}\right)\right]\\&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n}^{(a,b)}(x;2l+1,2j+1)\frac{t^{2n}}{(2n)!}\\&\qquad+i\sum_{n=0}^{\infty}(-1)^n\mathcal{P}_{2n+1}^{(a,b)}(x;2l+1,2j+1)\frac{t^{2n+1}}{(2n+1)!}.\end{aligned}$$

Equating real and imaginary parts we get (7). □

Since we obtain the unified Bernstein family in the case $a=0$, $b=-1$, we have the following corollary at once.

Corollary 13 *For the unified Bernstein family, we have the following implicit summation formulae*:

$$\begin{aligned}&\left(2^{1-2l}x^{2l}\right)^m\frac{(-t^2)^{lm}}{(2lm)!}\cos t(1-x)=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n}(2lm,x)\frac{t^{2n}}{(2n)!},\\&\left(2^{1-2l}x^{2l}\right)^m\frac{(-t^2)^{lm}}{(2lm)!}\sin t(1-x)=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n+1}(2lm,x)\frac{t^{2n+1}}{(2n+1)!}\end{aligned}$$

and

$$\begin{aligned}&\left(2^{-2l}x^{2l+1}\right)^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\cos t(1-x)=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n}\big((2l+1)(2j),x\big)\frac{t^{2n}}{(2n)!},\\&\left(2^{-2l}x^{2l+1}\right)^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\sin t(1-x)=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n+1}\big((2l+1)(2j),x\big)\frac{t^{2n+1}}{(2n+1)!}.\end{aligned}\tag{8}$$

Finally,

$$\begin{aligned}
&\bigl[2^{-2l}x^{2l+1}\bigr]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\sin t(1-x)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n}\bigl((2l+1)(2j+1),x\bigr)\frac{t^{2n}}{(2n)!},\\
&\bigl[2^{-2l}x^{2l+1}\bigr]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\cos t(1-x)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_{2n+1}\bigl((2l+1)(2j+1),x\bigr)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}\tag{9}$$

On the other hand, taking $l = 0$ *in* (8) *and* (9), *we get the following relations for the Bernstein basis*:

$$\begin{aligned}
&x^{2j}\frac{(-t^2)^j}{(2j)!}\cos t(1-x)=\sum_{n=0}^{\infty}(-1)^nB_{2j}^{2n}(x)\frac{t^{2n}}{(2n)!},\\
&x^{2j}\frac{(-t^2)^j}{(2j)!}\sin t(1-x)=\sum_{n=0}^{\infty}(-1)^nB_{2j}^{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}
\end{aligned}$$

and

$$\begin{aligned}
&x^{2j+1}\frac{(-t^2)^j}{(2j+1)!}t\sin t(1-x)=\sum_{n=0}^{\infty}(-1)^nB_{2j+1}^{2n}(x)\frac{t^{2n}}{(2n)!},\\
&x^{2j+1}\frac{(-t^2)^j}{(2j+1)!}t\cos t(1-x)=\sum_{n=0}^{\infty}(-1)^nB_{2j+1}^{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Since the case $a = 1$, $b = 0$ gives the unified Bleimann-Butzer-Hahn family, we immediately obtain the following corollary.

Corollary 14 *For the unified Bleimann-Butzer-Hahn family, we have the following implicit summation formulae*:

$$\begin{aligned}
&\left[\frac{2^{1-2l}x^{2l}}{(1+x)^{2l}}\right]^m\frac{(-t^2)^{lm}}{(2lm)!}\cos\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n}(2lm,x)\frac{t^{2n}}{(2n)!},\\
&\left[\frac{2^{1-2l}x^{2l}}{(1+x)^{2l}}\right]^m\frac{(-t^2)^{lm}}{(2lm)!}\sin\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n+1}(2lm,x)\frac{t^{2n+1}}{(2n+1)!}
\end{aligned}$$

and

$$\begin{aligned}
&\left[\frac{2^{-2l}x^{2l+1}}{(1+x)^{2l+1}}\right]^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\cos\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n}\bigl((2l+1)(2j),x\bigr)\frac{t^{2n}}{(2n)!},\\
&\left[\frac{2^{-2l}x^{2l+1}}{(1+x)^{2l+1}}\right]^{2j}\frac{(-t^2)^{(2l+1)j}}{(2j(2l+1))!}\sin\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n+1}\bigl((2l+1)(2j),x\bigr)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned}\tag{10}$$

Finally,

$$\left[\frac{2^{-2l}x^{2l+1}}{(1+x)^{2l+1}}\right]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\sin\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n}\big((2l+1)(2j+1),x\big)\frac{t^{2n}}{(2n)!},$$

$$\left[\frac{2^{-2l}x^{2l+1}}{(1+x)^{2l+1}}\right]^{2j+1}\frac{(-t^2)^{(2lj+l+j)}}{[(2j+1)(2l+1)]!}t\cos\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^n\mathcal{H}_{2n+1}\big((2l+1)(2j+1),x\big)\frac{t^{2n+1}}{(2n+1)!}. \tag{11}$$

Taking $l = 0$ in (10) *and* (11)*, we get the following relations for the Bleimann-Butzer-Hahn basis*:

$$\left[\frac{x}{1+x}\right]^{2j}\frac{(-t^2)^j}{(2j)!}\cos\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^nH_{2j}^{2n}(x)\frac{t^{2n}}{(2n)!},$$

$$\left[\frac{x}{1+x}\right]^{2j}\frac{(-t^2)^j}{(2j)!}\sin\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^nH_{2j}^{2n+1}\frac{t^{2n+1}}{(2n+1)!}.$$

Finally,

$$\left[\frac{x}{1+x}\right]^{2j+1}\frac{(-t^2)^j}{(2j+1)!}t\sin\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^nH_{2j+1}^{2n}(x)\frac{t^{2n}}{(2n)!},$$

$$\left[\frac{x}{1+x}\right]^{2j+1}\frac{(-t^2)^j}{(2j+1)!}t\cos\left(\frac{t}{1+x}\right)=\sum_{n=0}^{\infty}(-1)^nH_{2j+1}^{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}.$$

Finally, we obtain a summation formula for the unified Bernstein and Bleimann-Butzer-Hahn basis as follows.

Theorem 15 *For all $n,l\in\mathbb{N}_0$; $a,b\in\mathbb{R}$, the following implicit summation formula holds true*:

$$\mathcal{P}_{n+l}^{(a,b)}(y;k,m)=\sum_{p,r=0}^{l,n}\binom{n}{r}\binom{l}{p}\mathcal{P}_{n+l-r-p}^{(a,b)}(x;k,m)\left[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}\right]^{r+p}.$$

Proof Letting $t\to t+u$ in (1) and then using the fact that

$$\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}A(l,n)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}A(l,n-l), \tag{12}$$

we get

$$\begin{aligned}\left[\frac{2^{1-k}x^k(t+u)^k}{(1+ax)^k}\right]^m\frac{1}{(mk)!}e^{(t+u)[\frac{1+bx}{1+ax}]}&=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;k,m)\frac{(t+u)^n}{n!}\\&=\sum_{n=0}^{\infty}\mathcal{P}_n^{(a,b)}(x;k,m)\sum_{l=0}^{n}\frac{t^{n-l}u^l}{l!(n-l)!}\\&=\sum_{n,l=0}^{\infty}\mathcal{P}_{n+l}^{(a,b)}(x;k,m)\frac{t^nu^l}{n!l!}\end{aligned} \tag{13}$$

and hence

$$\left[\frac{2^{1-k}x^k(t+u)^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} = e^{-(t+u)[\frac{1+bx}{1+ax}]} \sum_{n,l=0}^{\infty} \mathcal{P}_{n+l}^{(a,b)}(x;k,m)\frac{t^n u^l}{n!l!}.$$

Multiplying both sides by $e^{(t+u)[\frac{1+by}{1+ay}]}$ and then expanding the function $e^{(t+u)[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}]}$, we get, after using (12) twice, that

$$\begin{aligned}
&\left[\frac{2^{1-k}x^k(t+u)^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} e^{(t+u)[\frac{1+by}{1+ay}]} \\
&= e^{(t+u)[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}]} \sum_{n,l=0}^{\infty} \mathcal{P}_{n+l}^{(a,b)}(x;k,m)\frac{t^n u^l}{n!l!} \\
&= \sum_{n,l=0}^{\infty}\sum_{r=0}^{\infty} \mathcal{P}_{n+l}^{(a,b)}(x;k,m)\frac{[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}]^r}{r!}(t+u)^r\frac{t^n u^l}{n!l!} \\
&= \sum_{n,l,p,r=0}^{\infty} \mathcal{P}_{n+l}^{(a,b)}(x;k,m)\left[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}\right]^{r+p}\frac{t^{n+r}u^{p+l}}{n!l!r!p!}.
\end{aligned}$$

Now, using (12) with the index pairs (n,r) and (l,p), we get

$$\begin{aligned}
&\left[\frac{2^{1-k}x^k(t+u)^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} e^{(t+u)[\frac{1+by}{1+ay}]} \\
&= \sum_{n,l=0}^{\infty}\sum_{p,r=0}^{l,n} \binom{n}{r}\binom{l}{p} \mathcal{P}_{n+l-r-p}^{(a,b)}(x;k,m)\left[\frac{1+by}{1+ay}-\frac{1+bx}{1+ax}\right]^{r+p}\frac{t^n u^l}{n!l!}. \qquad (14)
\end{aligned}$$

Since the left-hand side is equal by (13) to

$$\left[\frac{2^{1-k}x^k(t+u)^k}{(1+ax)^k}\right]^m \frac{1}{(mk)!} e^{(t+u)[\frac{1+by}{1+ay}]} = \sum_{n,l=0}^{\infty} \mathcal{P}_{n+l}^{(a,b)}(y;k,m)\frac{t^n u^l}{n!l!}, \qquad (15)$$

the proof is completed by comparing the coefficients of $\frac{t^n u^l}{n!l!}$ in (14) and (15). □

In the case $a = 0$, $b = -1$, we obtain the following result for the unified Bernstein family at once.

Corollary 16 *For all* $n,l \in \mathbb{N}_0$, *the following implicit summation formula*:

$$\mathcal{B}_{n+l}(mk,y) = \sum_{p,r=0}^{l,n} \binom{n}{r}\binom{l}{p} \mathcal{B}_{n+l-r-p}(mk,x)[x-y]^{r+p} \qquad (16)$$

holds true for the unified Bernstein family. Taking $k = 1$ *in* (16), *we get the following relation for the Bernstein basis*:

$$B_m^{n+l}(y) = \sum_{p,r=0}^{l,n} \binom{n}{r}\binom{l}{p} B_m^{n+l-r-p}(x)[x-y]^{r+p}.$$

Since the case $a = 1$, $b = 0$ gives the unified Bleimann-Butzer-Hahn family, we have the following result.

Corollary 17 *For all $n, l \in \mathbb{N}_0$, the following implicit summation formula*:

$$\mathcal{H}_{n+l}(mk, y) = \sum_{p,r=0}^{l,n} \binom{n}{r}\binom{l}{p}\mathcal{H}_{n+l-r-p}(mk, x)[x-y]^{r+p} \tag{17}$$

holds true for the unified Bleimann-Butzer-Hahn family. Upon taking $k = 1$ in (17), *we get the following relation for the Bleimann-Butzer-Hahn basis*:

$$H_m^{n+l}(y) = \sum_{p,r=0}^{l,n} \binom{n}{r}\binom{l}{p} H_m^{n+l-r-p}(x)[x-y]^{r+p}.$$

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The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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