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Existence results for anti-periodic boundary value problems of fractional differential equations

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Abstract

In this paper, the author is concerned with the fractional equation

$$^{C}D^{\alpha}_{0+}u(t) = f\big(t, u(t), {}^{C}D^{\alpha_1}_{0+}u(t), {}^{C}D^{\alpha_2}_{0+}u(t)\big), \quad t \in (0,1),$$

with the anti-periodic boundary value conditions

$$u(0) = -u(1), \qquad t^{\beta_1-1}{}^{C}D^{\beta_1}_{0+}u(t)_{|t\to 0} = -t^{\beta_1-1}{}^{C}D^{\beta_1}_{0+}u(t)_{|t=1},$$
$$t^{\beta_2-2}{}^{C}D^{\beta_2}_{0+}u(t)_{|t\to 0} = -t^{\beta_2-2}{}^{C}D^{\beta_2}_{0+}u(t)_{|t=1},$$

where ${}^{C}D^{\gamma}_{0+}$ denotes the Caputo fractional derivative of order γ, the constants α, α_1, α_2, β_1, β_2 satisfy the conditions $2 < \alpha \le 3$, $0 < \alpha_1 \le 1 < \alpha_2 \le 2$, $0 < \beta_1 < 1 < \beta_2 < 2$. Different from the recent studies, the function f involves the Caputo fractional derivative ${}^{C}D^{\alpha_1}_{0+}u(t)$ and ${}^{C}D^{\alpha_2}_{0+}u(t)$. In addition, the author put forward new anti-periodic boundary value conditions, which are more suitable than those studied in the recent literature. By applying the Banach contraction mapping principle and the Leray-Schauder degree theory, some existence results of solutions are obtained.
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1 Introduction

In the present paper, we are concerned with the existence of solutions for the fractional differential equation

$$^{C}D^{\alpha}_{0+}u(t) = f\big(t, u(t), {}^{C}D^{\alpha_1}_{0+}u(t), {}^{C}D^{\alpha_2}_{0+}u(t)\big), \quad t \in (0,1), \tag{1.1}$$

with anti-periodic boundary value conditions

$$\begin{cases} u(0) = -u(1), \qquad t^{\beta_1-1}{}^{C}D^{\beta_1}_{0+}u(t)_{|t\to 0} = -t^{\beta_1-1}{}^{C}D^{\beta_1}_{0+}u(t)_{|t=1}, \\ t^{\beta_2-2}{}^{C}D^{\beta_2}_{0+}u(t)_{|t\to 0} = -t^{\beta_2-2}{}^{C}D^{\beta_2}_{0+}u(t)_{|t=1}, \end{cases} \tag{1.2}$$

where ${}^{C}D_{0+}^{\gamma}$ denotes the Caputo fractional derivative of order γ, the constants α, α_1, α_2, β_1, β_2 satisfy conditions $2 < \alpha \leq 3$, $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2$, $0 < \beta_1 < 1 < \beta_2 < 2$, and f is a given continuous function.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, *etc.* (see [1–5]). There has been a significant development in the study of fractional differential equations and inclusions in recent years; see the monographs of Podlubny [5], Kilbas *et al.* [6], Lakshmikantham *et al.* [7], Samko *et al.* [8], Diethelm [9], and the survey by Agarwal *et al.* [10]. For some recent contributions on fractional differential equations, see [11–30] and the references therein.

Anti-periodic boundary value problems occur in the mathematic modeling of a variety of physical processes and have recently received considerable attention. For examples and details of anti-periodic fractional boundary conditions, see [16–22]. In [16], Agarwal and Ahmad studied the solvability of the following anti-periodic boundary value problem for nonlinear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)), & t \in (0, T), 3 < \alpha \leq 4, \\ u(0) = -u(T), & u'(0) = -u'(T), \\ u''(0) = -u''(T), & u'''(0) = -u'''(T), \end{cases} \tag{1.3}$$

where D_{0+}^{α} denotes the Caputo fractional derivative of order α. The existence results are obtained by nonlinear alternative theorem.

In [17], Wang, Ahmad, Zhang investigated the following impulsive anti-periodic fractional boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)), & 2 < \alpha \leq 3, t \in J', \\ \Delta u(t_k) = Q_k(u(t_k)), & k = 1, 2, \ldots, p, \\ \Delta u'(t_k) = I_k(u(t_k)), & k = 1, 2, \ldots, p, \\ \Delta u''(t_k) = I_k^{*}(u(t_k)), & k = 1, 2, \ldots, p, \\ u(0) = -u(1), \quad u'(0) = -u'(1), \quad u''(0) = -u''(1), \end{cases} \tag{1.4}$$

where D_{0+}^{α} denotes the Caputo fractional derivative of order α. By applying some well-known fixed point principles, some existence and uniqueness results are obtained.

In [18], Ahmad, Nieto studied the following anti-periodic fractional boundary value problem:

$$\begin{aligned} &{}^{C}D^{q}x(t) = f\big(t, x(t)\big), \quad t \in [0, T], 1 < q \leq 2, \\ &x(0) = -x(T), \qquad {}^{C}D^{p}x(0) = -{}^{C}D^{p}x(T), \quad 0 < p < 1, \end{aligned} \tag{1.5}$$

where ${}^{C}D^{q}$ denotes the Caputo fractional derivative of order q. By applying some standard fixed point principles, some existence and uniqueness results are obtained.

In [19], Wang and Liu considered the following anti-periodic fractional boundary value problem:

$$\begin{aligned} &{}^{C}D^{\alpha}x(t) = f\big(t, x(t), {}^{C}D^{q}x(t)\big), \quad t \in [0,T], 0 < q < 1, 1 \le \alpha - q, \\ &x(0) = -x(T), \qquad {}^{C}D^{p}x(0) = -{}^{C}D^{p}x(T), \quad 0 < p < 1. \end{aligned} \tag{1.6}$$

By using Schauder's fixed point theorem and the contraction mapping principle, some existence and uniqueness results are obtained.

By careful analysis, we have found that the anti-periodic boundary value condition ${}^{C}D^{p}x(0) = -{}^{C}D^{p}x(T)$ $(0 < p < 1)$ in equations (1.5) and (1.6) actually is equivalent to the boundary value condition ${}^{C}D^{p}x(T) = 0$ $(0 < p < 1)$. It means that, in a sense, in (1.5)-(1.6), the feature of anti-periodicity partially disappears. So, in the present paper, we put forward new anti-periodic boundary value conditions (1.2) so that the anti-periodicity is expressed. In fact, when $\beta_1 \to 1$, $\beta_2 \to 2$, the anti-periodic boundary value conditions in (1.2) are changed into the boundary value conditions

$$u(0) = -u(1), \qquad u'(0) = -u'(1), \qquad u''(0) = -u''(1),$$

which are coincident with anti-periodic boundary value conditions (1.3) and (1.4) mentioned above. So, the anti-periodic boundary value conditions in (1.2) in the present paper are more suitable than those in (1.5) and (1.6). Moreover, different from the literature mentioned above, the function f in (1.1) involves the Caputo fractional derivative ${}^{C}D_{0+}^{\alpha_1}u(t)$ and ${}^{C}D_{0+}^{\alpha_2}u(t)$, which brings more difficulty to the study. To investigate the existence, researchers often equip a Banach space with the norm $\|u\| = \|u\|_0 + \|{}^{C}D_{0+}^{\alpha_1}u\|_0 + \|{}^{C}D_{0+}^{\alpha_2}u\|_0$. However, if such a norm was taken in the study, the introduced conditions would get more complex. So, we take the norm $\|u\| = \max\{\|u'\|_0, \|u''\|_0\}$ by finding some implicit relations. As a result, the conditions introduced are quite simple. By applying the Banach contraction mapping principle and the Leray-Schauder degree theory, some existence results of solutions are obtained in this paper.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. In Section 3, we put forward and prove our main results. Finally, we give two examples to demonstrate our main results.

2 Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper.

Let $\mathbb{N}$ be the set of positive integers, $\mathbb{R}$ be the set of real numbers.

Definition 2.1 ([6]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (a,b] \to \mathbb{R}$ is given by

$$I_{a+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds, \quad t \in (a,b].$$

Definition 2.2 ([6]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (a,b] \to \mathbb{R}$ is given by

$$D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad t \in (a,b],$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Definition 2.3 ([6]) The Caputo fractional derivative of order $\alpha > 0$ of a function y on $(a,b]$ is defined via the above Riemann-Liouville derivatives by

$$\left({}^C D_{a+}^{\alpha} y\right)(x) = \left(D_{a+}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x), \quad x \in (a,b].$$

Lemma 2.1 ([6]) *Let $\alpha > 0$ and $y \in C[a,b]$. Then*

$$\left({}^C D_{a+}^{\alpha} I_{a+}^{\alpha} y\right)(x) = y(x)$$

holds on $[a,b]$.

Lemma 2.2 ([6]) *If $0 < \alpha \notin \mathbb{N}$ and $y \in AC^n[a,b]$, then*

$$D_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}\, ds, \quad t \in [a,b],$$

where $AC^n[a,b] = \{u : [a,b] \to \mathbb{R}$ and $u^{(n-1)} \in AC[a,b]\}$.

Lemma 2.3 ([13]) *Let $n \in \mathbb{N}$ with $n \geq 2$, $\alpha \in (n-1,n]$. If $y \in C^{n-1}[a,b]$ and ${}^C D_{a+}^{\alpha} y \in C(a,b)$, then*

$$I_{a+}^{\alpha}\, {}^C D_{a+}^{\alpha} y(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k$$

holds on (a,b).

Let $X = \{u | u \in C^2[0,1], u(0) = -u(1)\}$. It is well known that X is a Banach space endowed with the norm $\|u\|_2 = \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$, where $\|u^{(i)}\|_0 = \max_{t\in[0,1]} |u^{(i)}|$.

For any $u \in X$, from $u(0) = -u(1)$, there exists a $t_u \in (0,1)$ such that $u(t_u) = 0$. So, from the fact that $u(t) = \int_{t_u}^t u'(s)\, ds$, $t \in [0,1]$, it follows that $|u(t)| \leq |\int_{t_u}^t |u'(s)|\, ds| \leq \|u'\|_0$, $t \in [0,1]$. Thus, $\|u\|_0 \leq \|u'\|_0$, and so $\|u\|_2 = \max\{\|u'\|_0, \|u''\|_0\}$.

In what follows, we regard X as the Banach space with the norm

$$\|u\|_2 = \max\left\{\left\|u'\right\|_0, \left\|u''\right\|_0\right\}.$$

We have the following lemma.

Lemma 2.4 *For a given $h \in C[0,1]$, the function u is a solution of the following anti-periodic boundary value problem*:

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = h(t), & t \in (0,1), \\ u(0) = -u(1), & t^{\beta_1 - 1 C} D_{0+}^{\beta_1} u(t)_{|t\to 0} = -t^{\beta_1 - 1 C} D_{0+}^{\beta_1} u(t)_{|t=1}, \\ t^{\beta_2 - 2 C} D_{0+}^{\beta_2} u(t)_{|t\to 0} = -t^{\beta_2 - 2 C} D_{0+}^{\beta_2} u(t)_{|t=1}. \end{cases} \tag{2.1}$$

If and only if $u \in C^2[0,1]$ is a solution of the integral equation

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \int_0^1 \biggl[-\frac{1}{2\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{(1-2t)\Gamma(2-\beta_1)}{4\Gamma(\alpha-\beta_1)}(1-s)^{\alpha-\beta_1-1} \\ & + \frac{1-\beta_1+2t-2(2-\beta_1)t^2}{8(2-\beta_1)} \cdot \frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)}(1-s)^{\alpha-\beta_2-1} \biggr] h(s)\,ds. \end{aligned} \tag{2.2}$$

Proof Let $u \in C^2[0,1]$ be a solution of (2.1). Then, by Lemma 2.3, we have

$$u(t) = c_0 + c_1 t + c_2 t^2 + I_{0+}^{\alpha} h(t), \quad t \in [0,1], \tag{2.3}$$

for some $c_0, c_1, c_2 \in \mathbb{R}$. Furthermore,

$${}^C D_{0+}^{\beta_1} u(t) = \frac{c_1}{\Gamma(2-\beta_1)} t^{1-\beta_1} + \frac{2c_2}{\Gamma(3-\beta_1)} t^{2-\beta_1} + I_{0+}^{\alpha-\beta_1} h(t), \quad t \in [0,1], \tag{2.4}$$

$${}^C D_{0+}^{\beta_2} u(t) = \frac{2c_2}{\Gamma(3-\beta_2)} t^{2-\beta_2} + I_{0+}^{\alpha-\beta_2} h(t), \quad t \in [0,1]. \tag{2.5}$$

From (2.4)-(2.5), we have

$$t^{\beta_1 - 1 C} D_{0+}^{\beta_1} u(t) = \frac{c_1}{\Gamma(2-\beta_1)} + \frac{2c_2}{\Gamma(3-\beta_1)} t + t^{\beta_1 - 1} I_{0+}^{\alpha-\beta_1} h(t), \quad t \in (0,1), \tag{2.6}$$

$$t^{\beta_2 - 2 C} D_{0+}^{\beta_2} u(t) = \frac{2c_2}{\Gamma(3-\beta_2)} + t^{\beta_2 - 2} I_{0+}^{\alpha-\beta_2} h(t), \quad t \in (0,1). \tag{2.7}$$

Now, we show that

$$\lim_{t\to 0+} t^{\beta_1-1} I_{0+}^{\alpha-\beta_1} h(t) = 0, \qquad \lim_{t\to 0+} t^{\beta_2-2} I_{0+}^{\alpha-\beta_2} h(t) = 0.$$

In fact, since $h \in C[0,1]$, there exists an $M > 0$ such that $|h(t)| \le M$ for all $t \in [0,1]$. Then, from the fact that $0 < \beta_1 < 1 < \beta_2 < 2 < \alpha \le 3$, we have

$$\begin{aligned} \bigl| I_{0+}^{\alpha-\beta_1} h(t) \bigr| &= \frac{1}{\Gamma(\alpha-\beta_1)} \biggl| \int_0^t (t-s)^{\alpha-\beta_1-1} h(s)\,ds \biggr| \\ &\le \frac{1}{\Gamma(\alpha-\beta_1)} \int_0^t (t-s)^{\alpha-\beta_1-1} \bigl| h(s) \bigr|\,ds \\ &\le \frac{M}{\Gamma(\alpha-\beta_1)} \int_0^t (t-s)^{\alpha-\beta_1-1}\,ds \\ &= \frac{M}{\Gamma(\alpha-\beta_1+1)} t^{\alpha-\beta_1}, \quad t \in (0,1]. \end{aligned}$$

Thus,

$$\left|t^{\beta_1-1}I_{0+}^{\alpha-\beta_1}h(t)\right| \le \frac{M}{\Gamma(\alpha-\beta_1+1)}t^{\alpha-1}, \quad t\in(0,1],$$

and so

$$\lim_{t\to 0+} t^{\beta_1-1}I_{0+}^{\alpha-\beta_1}h(t) = 0,$$

noting that $2<\alpha\le 3$.

Similarly, we also have

$$\left|t^{\beta_2-2}I_{0+}^{\alpha-\beta_2}h(t)\right| \le \frac{M}{\Gamma(\alpha-\beta_2+1)}t^{\alpha-2}, \quad t\in(0,1],$$

and so

$$\lim_{t\to 0+} t^{\beta_2-2}I_{0+}^{\alpha-\beta_2}h(t) = 0,$$

noting that $2<\alpha\le 3$.

So, from (2.6)-(2.7), we have

$$\begin{cases} t^{\beta_1-1C}D_{0+}^{\beta_1}u(t)_{|t\to 0} = \frac{c_1}{\Gamma(2-\beta_1)}, \\ t^{\beta_1-1C}D_{0+}^{\beta_1}u(t)_{|t=1} = \frac{c_1}{\Gamma(2-\beta_1)} + \frac{2c_2}{\Gamma(3-\beta_1)} + I_{0+}^{\alpha-\beta_1}h(1), \end{cases} \tag{2.8}$$

$$\begin{cases} t^{\beta_2-2C}D_{0+}^{\beta_2}u(t)_{|t\to 0} = \frac{2c_2}{\Gamma(3-\beta_2)}, \\ t^{\beta_2-2C}D_{0+}^{\beta_2}u(t)_{|t=1} = \frac{2c_2}{\Gamma(3-\beta_2)} + I_{0+}^{\alpha-\beta_2}h(1). \end{cases} \tag{2.9}$$

Thus, by the boundary value condition in (2.1), combined with (2.3), (2.8)-(2.9), we have

$$2c_0 + c_1 + c_2 + I_{0+}^{\alpha}h(1) = 0, \tag{2.10}$$

$$\frac{2c_1}{\Gamma(2-\beta_1)} + \frac{2c_2}{\Gamma(3-\beta_1)} + I_{0+}^{\alpha-\beta_1}h(1) = 0, \tag{2.11}$$

$$\frac{4c_2}{\Gamma(3-\beta_2)} + I_{0+}^{\alpha-\beta_2}h(1) = 0. \tag{2.12}$$

From (2.10)-(2.12), we have

$$c_0 = -\frac{1}{2}I_{0+}^{\alpha}h(1) + \frac{\Gamma(2-\beta_1)}{4}I_{0+}^{\alpha-\beta_1}h(1) - \frac{(\beta_1-1)\Gamma(3-\beta_2)}{8(2-\beta_1)}I_{0+}^{\alpha-\beta_2}h(1), \tag{2.13}$$

$$c_1 = -\frac{\Gamma(2-\beta_1)}{2}I_{0+}^{\alpha-\beta_1}h(1) + \frac{\Gamma(3-\beta_2)}{4(2-\beta_1)}I_{0+}^{\alpha-\beta_2}h(1), \tag{2.14}$$

$$c_2 = -\frac{\Gamma(3-\beta_2)}{4}I_{0+}^{\alpha-\beta_2}h(1). \tag{2.15}$$

Substituting (2.13)-(2.15) into (2.3), we obtain

$$\begin{aligned} u(t) = {} & I_{0+}^{\alpha}h(t) - \frac{1}{2}I_{0+}^{\alpha}h(1) + \frac{(1-2t)}{4}\Gamma(2-\beta_1)I_{0+}^{\alpha-\beta_1}h(1) \\ & + \frac{1-\beta_1+2t-2(2-\beta_1)t^2}{8(2-\beta_1)}\Gamma(3-\beta_2)I_{0+}^{\alpha-\beta_2}h(1), \quad t \in [0,1]. \end{aligned}$$

That is, u satisfies (2.2).

Conversely, if u is a solution of the fractional integral equation (2.2), then by finding the second derivative for both sides of (2.2), we have

$$\begin{aligned} u''(t) = {} & \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}h(s)\,ds \\ & - \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2)}\int_0^1 (1-s)^{\alpha-\beta-1}h(s)\,ds, \quad t \in [0,1]. \end{aligned} \tag{2.16}$$

Noting that $h \in C[0,1]$, it follows from (2.16) that $u \in C^2[0,1]$. Again, by Lemma 2.1 and (2.2), a direct computation shows that the solution given by (2.2) satisfies (2.1). This completes the proof. □

Now, we define the operator A as

$$\begin{aligned} (Ah) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds + \int_0^1 \biggl[-\frac{1}{2\Gamma(\alpha)}(1-s)^{\alpha-1} + \frac{(1-2t)\Gamma(2-\beta_1)}{4\Gamma(\alpha-\beta_1)}(1-s)^{\alpha-\beta_1-1} \\ & + \frac{1-\beta_1+2t-2(2-\beta_1)t^2}{8(2-\beta_1)} \cdot \frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)}(1-s)^{\alpha-\beta_2-1}\biggr]h(s)\,ds \end{aligned} \tag{2.17}$$

for $h \in C[0,1]$.

From the proof of Lemma 2.4, we know that the operator A maps $C[0,1]$ into X.

Now, we establish the following lemma, which will play an important role in the forthcoming analysis.

Lemma 2.5 *For any $u \in X$ and $0 < \alpha_1 \le 1 < \alpha_2 \le 2$, we have*

(i) $\|u\|_0 \le \|u'\|_0$, *and so* $\|u\|_0 \le \|u\|_2$;
(ii) $\|{}^CD_{0+}^{\alpha_1}u\|_0 \le \frac{1}{\Gamma(2-\alpha_1)}\|u'\|_0$, *and so* $\|{}^CD_{0+}^{\alpha_1}u\|_0 \le \frac{1}{\Gamma(2-\alpha_1)}\|u\|_2$;
(iii) $\|{}^CD_{0+}^{\alpha_2}u\|_0 \le \frac{1}{\Gamma(3-\alpha_2)}\|u''\|_0$, *and so* $\|{}^CD_{0+}^{\alpha_2}u\|_0 \le \frac{1}{\Gamma(3-\alpha_2)}\|u\|_2$.

Proof Conclusion (i) has been proved as before. We come to show that conclusions (ii)-(iii) are true. Obviously, when $\alpha_1 = 1$, $\alpha_2 = 2$, the conclusions are true. So, we only consider the case $0 < \alpha_1 < 1 < \alpha_2 < 2$. In fact, by Lemma 2.2, for any $u \in X$, we have

$$\begin{aligned} \bigl|{}^CD_{0+}^{\alpha_1}u(t)\bigr| & = \frac{1}{\Gamma(1-\alpha_1)}\left|\int_0^t (t-s)^{-\alpha_1}u'(s)\,ds\right| \\ & \le \|u'\|_0 \frac{1}{\Gamma(1-\alpha_1)}\int_0^t (t-s)^{-\alpha_1}\,ds \\ & \le \frac{1}{\Gamma(2-\alpha_1)}\|u'\|_0, \quad t \in [0,1]. \end{aligned}$$

So,

$$\left\|{}^C D_{0+}^{\alpha_1} u\right\|_0 \le \frac{1}{\Gamma(2-\alpha_1)} \left\|u'\right\|_0.$$

Similarly, we have that $\|{}^C D_{0+}^{\alpha_2} u\|_0 \le \frac{1}{\Gamma(3-\alpha_2)} \|u''\|_0$. □

We also need the following lemmas.

Lemma 2.6 ([31]) *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$, and $T : \bar{\Omega} \to X$ is a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\bar{\Omega}$.

Lemma 2.7 (Leray-Schauder [31]) *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \mu Tu, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

3 Main results

We list the following hypotheses which will be used in the sequel:

(H_1) $f \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.
(H_2) $2 < \alpha \le 3$, $0 < \alpha_1 \le 1 < \alpha_2 \le 2$, $0 < \beta_1 < 1 < \beta_2 < 2$.
(H_3) There exist constants L_1, L_2, and L_3 such that

$$\left|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)\right| \le L_1|x_2 - x_1| + L_2|y_2 - y_1| + L_3|z_2 - z_1|$$

for $x_i, y_i, z_i \in \mathbb{R}$, $i = 1,2,3$, and $t \in [0,1]$.

First, we establish the following lemma to obtain our main results.

Lemma 3.1 *Assume that* (H_1)-(H_2) *hold. Then the operator $T : X \to X$ is completely continuous, where T is defined by*

$$(Tu)(t) = (AFu)(t), \qquad (Fu)(t) = f\big(t, u(t), {}^C D_{0+}^{\alpha_1} u(t), {}^C D_{0+}^{\alpha_2} u(t)\big), \quad t \in [0,1],$$

and the operator A is given by (2.17).

Proof For any $u \in X$, we have that ${}^C D_{0+}^{\alpha_1} u, {}^C D_{0+}^{\alpha_2} u \in C[0,1]$. Then $Fu \in C[0,1]$ from the hypothesis (H_1). Thus, $Tu \in X$ from (2.17) and the proof of Lemma 2.4.

First, by a direct computation, we know that the following relations hold:

$$\begin{aligned}(Tu)'(t) = {} & \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Fu(s)\,ds - \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1)} \int_0^1 (1-s)^{\alpha-\beta_1-1} Fu(s)\,ds \\ & + \frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)} \int_0^1 \frac{1-2(2-\beta_1)t}{4(2-\beta_1)} (1-s)^{\alpha-\beta_2-1} Fu(s)\,ds, \qquad (3.1)\end{aligned}$$

$$(Tu)''(t) = \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)}(Fu)(s)\,ds - \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2)}\int_0^1 (1-s)^{\alpha-\beta_2-1}(Fu)(s)\,ds. \tag{3.2}$$

Now, we show that T is a compact operator.

Let V be an arbitrary bounded set in X. Then there exists an $L > 0$ such that $\|u\|_2 \le L$. Thus, by Lemma 2.5, it follows that $|u(t)| \le L$, $|{}^C D_{0+}^{\alpha_1} u(t)| \le \frac{1}{\Gamma(2-\alpha_1)} L$, $|{}^C D_{0+}^{\alpha_2} u(t)| \le \frac{1}{\Gamma(3-\alpha_2)} L$ for all $u \in V$ and $t \in [0,1]$.

So, by the hypothesis (H_1), there exists an $M > 0$ such that

$$\bigl|(Fu)(t)\bigr| \le M, \quad t \in [0,1], \text{ for all } u \in V. \tag{3.3}$$

Consequently, by (3.1), (3.3) and observing that $|1 - 2(2-\beta_1)t| \le 2(2-\beta_1)$, $t \in [0,1]$, we have

$$\bigl|(Tu)'(t)\bigr| \le \left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right)M,$$

and so

$$\bigl\|(Tu)'\bigr\|_0 \le \left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right)M. \tag{3.4}$$

Similarly, by (3.2), (3.3), we have

$$\bigl\|(Tu)''\bigr\|_0 \le \left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right)M. \tag{3.5}$$

Thus,

$$\|Tu\|_2 \le \left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right)M.$$

That is, TV is uniformly bounded.

Now, we show that TV is equicontinuous.

In fact, for any $u \in V$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, since $|Fu(t)| \le M$, $t \in [0,1]$, from (3.1) it follows that

$$\begin{aligned}\bigl|(Tu)'(t_2) - (Tu)'(t_1)\bigr| &\le \frac{M}{\Gamma(\alpha)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) + M\int_0^1 \frac{(t_2-t_1)\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2)}(1-s)^{\alpha-\beta_2-1}\,ds \\ &= M\left[\frac{1}{\Gamma(\alpha)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}(t_2-t_1)\right],\end{aligned} \tag{3.6}$$

and

$$\bigl|(Tu)''(t_2) - (Tu)''(t_1)\bigr| \le \frac{M}{\Gamma\alpha-1)}\bigl[t_2^{\alpha-2} - t_1^{\alpha-2} + (t_2-t_1)^{\alpha-2}\bigr]. \tag{3.7}$$

So, inequalities (3.6)-(3.7) imply that TV is equicontinuous. By the Arzela-Ascoli theorem, T is a compact operator.

Finally, we prove that T is continuous.

Assume that $\{u_n\}$ is an arbitrary sequence in X with $u_n \to u_0$, $u_0 \in X$. Then there is an $L > 0$ such that $\|u_n\|_0 \leq L$, $\|u_n'\|_0 \leq L$, $\|u_n''\|_0 \leq L$, $n = 0,1,2,\ldots$, and so

$$\left\|{}^CD_{0+}^{\alpha_1}u_n\right\|_0 \leq \frac{L}{\Gamma(2-\alpha_1)}, \qquad \left\|{}^CD_{0+}^{\alpha_2}u_n\right\|_0 \leq \frac{L}{\Gamma(3-\alpha_2)}, \quad n = 0,1,2,\ldots,$$

from Lemma 2.5.

On the other hand, for an arbitrary $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left|f(t,x_2,y_2,z_2) - f(t,x_1,y_1,z_1)\right| < \varepsilon \tag{3.8}$$

for all $t \in [0,1]$, $x_i \in [-L,L]$, $y_i \in [-\frac{L}{\Gamma(2-\alpha_1)}, \frac{L}{\Gamma(2-\alpha_1)}]$, $z_i \in [-\frac{L}{\Gamma(3-\alpha_2)}, \frac{L}{\Gamma(3-\alpha_2)}]$ with $|x_2 - x_1| < \delta$, $|y_2 - y_1| < \delta$, $|z_2 - z_1| < \delta$, because of the uniform continuity of f on

$$[0,1] \times [-L,L] \times \left[-\frac{L}{\Gamma(2-\alpha_1)}, \frac{L}{\Gamma(2-\alpha_1)}\right] \times \left[-\frac{L}{\Gamma(3-\alpha_2)}, \frac{L}{\Gamma(3-\alpha_2)}\right].$$

In view of the fact that $u_n \to u_0$, there is an $N \geq 1$ such that

$$\|u_n - u_0\|_0 < \delta, \qquad \left\|{}^CD_{0+}^{\alpha_1}u_n - {}^CD_{0+}^{\alpha_1}u_0\right\|_0 < \delta, \qquad \left\|{}^CD_{0+}^{\alpha_2}u_n - {}^CD_{0+}^{\alpha_2}u_0\right\|_0 < \delta \tag{3.9}$$

when $n \geq N$.

Thus, from (3.1)-(3.2) together with (3.8)-(3.9), by a similar deducing as (3.4)-(3.5), we have

$$\left\|(Tu_n)' - (Tu_0)'\right\|_0 \leq \left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]\varepsilon$$

and

$$\left\|(Tu_n)'' - (Tu_0)''\right\|_0 \leq \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]\varepsilon.$$

Hence,

$$\|Tu_n - Tu_0\|_2 \leq \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]\varepsilon.$$

That is, T is continuous in X. □

To state our main results in this paper, we first introduce some notations for convenience.

Let $f_0 = \overline{\lim}_{|x|+|y|+|z|\to 0} \max_{t\in[0,1]} \frac{|f(t,x,y,z)|}{|x|+|y|+|z|}$, $f_\infty = \overline{\lim}_{|x|+|y|+|z|\to\infty} \max_{t\in[0,1]} \frac{|f(t,x,y,z)|}{|x|+|y|+|z|}$.

Set $D = \frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}$, $E = 1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)}$, $L = L_1 + \frac{L_2}{\Gamma(2-\alpha_1)} + \frac{L_3}{\Gamma(3-\alpha_2)}$, where L_i $(i = 1,2,3)$ are given in (H_3).

We are in a position to state the first result in the present paper.

Theorem 3.1 *Suppose* (H_1)-(H_3) *hold. If* $DL < 1$, *then BVP* (1.1)-(1.2) *has a unique solution.*

Proof For any $u, v \in X$, by (H_3) and Lemma 2.5, we have

$$\begin{aligned}
&\left|f\left(t,u(t),{}^CD_{0+}^{\alpha_1}u(t),{}^CD_{0+}^{\alpha_2}u(t)\right)-f\left(t,v(t),{}^CD_{0+}^{\alpha_1}v(t),{}^CD_{0+}^{\alpha_2}v(t)\right)\right| \\
&\quad\le L_1\left|u(t)-v(t)\right|+L_2\left|{}^CD_{0+}^{\alpha_1}u(t)-{}^CD_{0+}^{\alpha_1}v(t)\right|+L_3\left|{}^CD_{0+}^{\alpha_2}u(t)-{}^CD_{0+}^{\alpha_2}v(t)\right| \\
&\quad\le L_1\|u-v\|_0+L_2\left\|{}^CD_{0+}^{\alpha_1}(u-v)\right\|_0+L_3\left\|{}^CD_{0+}^{\alpha_2}(u-v)\right\|_0 \\
&\quad\le L_1\|u-v\|_0+\frac{L_2}{\Gamma(2-\alpha_1)}\left\|(u-v)'\right\|_0+\frac{L_3}{\Gamma(3-\alpha_2)}\left\|(u-v)''\right\|_0 \\
&\quad\le\left(L_1+\frac{L_2}{\Gamma(2-\alpha_1)}+\frac{L_3}{\Gamma(3-\alpha_2)}\right)\|u-v\|_2 \\
&\quad=L\|u-v\|_2.
\end{aligned}$$

So, it follows from (3.1) that

$$\begin{aligned}
\left|(Tu)'(t)-(Tv)'(t)\right|\le{}&\left\{\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds+\frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1)}\int_0^1(1-s)^{\alpha-\beta_1-1}\,ds\right. \\
&\left.+\frac{\Gamma(3-\beta_2)}{\Gamma(\alpha-\beta_2)}\int_0^1\frac{|1-2(2-\beta_1)t|}{4(2-\beta_1)}(1-s)^{\alpha-\beta_2-1}\,ds\right\}L\|u-v\|_2.
\end{aligned}$$

Thus, observing that $|1-2(2-\beta_1)t|\le 2(2-\beta_1)$, we have

$$\left\|(Tu)'-(Tv)'\right\|_0\le\left[\frac{1}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)}+\frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]L\|u-v\|_2.$$

Similarly,

$$\left\|(Tu)''-(Tv)''\right\|_0\le\left[\frac{1}{\Gamma(\alpha-1)}+\frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]L\|u-v\|_2.$$

Thus,

$$\begin{aligned}
\|Tu-Tv\|_2&\le\left[\frac{1}{\Gamma(\alpha-1)}+\frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)}+\frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right]L\|u-v\|_2 \\
&=DL\|u-v\|_2.
\end{aligned}$$

As $DL<1$, T is a contraction mapping. So, by the contraction mapping principle, T has a unique fixed point u. That is, u is the unique solution of BVP (1.1)-(1.2) by Lemma 2.4. □

Our next existence result is based on Lemma 2.6.

Theorem 3.2 *Assume that* (H_1)-(H_2) *hold. If* $f_0<\lambda$, *then BVP* (1.1)-(1.2) *has at least one solution, where* $\lambda=(DE)^{-1}$.

Proof By Lemma 3.1, we know that $T: X\to X$ is completely continuous. Again, in view of $f_0<\lambda$, there exists an $r_1>0$ such that

$$\left|f(t,x,y,z)\right|\le\lambda\left(|x|+|y|+|z|\right),\quad t\in[0,1] \tag{3.10}$$

when $|x|+|y|+|z|\le r_1$.

Take $r = E^{-1}r_1$, where $E = (1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)})$. Set $\Omega = \{x \in X : \|x\|_2 < r\}$. For any $u \in \partial\Omega$, we have that $u \in X$ with $\|u\|_2 = r$. Thus, $\|u\|_0 \le r$, $\|u'\|_0 \le r$, and $\|u''\|_0 \le r$. So, Lemma 2.5 ensures that

$$\left\|{}^C D_{0+}^{\alpha_1} u\right\|_0 \le \frac{1}{\Gamma(2-\alpha_1)} r, \qquad \left\|{}^C D_{0+}^{\alpha_2} u\right\|_0 \le \frac{1}{\Gamma(3-\alpha_2)} r.$$

Therefore,

$$\left|u(t)\right| + \left|{}^C D_{0+}^{\alpha_1} u(t)\right| + \left|{}^C D_{0+}^{\alpha_2} u(t)\right| \le \left(1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)}\right) r = r_1, \quad t \in [0,1].$$

Thus, from (3.10), it follows that

$$\begin{aligned} \left|f\left(t, u(t), {}^C D_{0+}^{\alpha_1} u(t), {}^C D_{0+}^{\alpha_2} u(t)\right)\right| &\le \lambda\left(1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)}\right) r \\ &= \lambda E r, \quad t \in [0,1]. \end{aligned}$$

Thus, by a similar deducing to that in (3.4) and (3.5), we have

$$\left\|(Tu)'\right\|_0 \le \lambda\left(\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right) Er$$

and

$$\left\|(Tu)''\right\|_0 \le \lambda\left(\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)}\right) Er.$$

Thus, $\|Tu\|_2 \le \lambda DEr$, where $D = \frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}$. As $\lambda = (DE)^{-1}$, we have that $\|Tu\|_2 \le r = \|u\|_2$.

So, by virtue of Lemma 2.6, T has at least one fixed point u. That is, u is a solution of BVP (1.1)-(1.2) by Lemma 2.4. The proof is complete. □

The last result of this section is based on the Leray-Schauder fixed point theorem, namely Lemma 2.7.

Theorem 3.3 *Suppose that* (H_1)-(H_2) *hold. If* $f_\infty < \lambda$, *then BVP* (1.1)-(1.2) *has at least one solution, where* $\lambda = (DE)^{-1}$.

Proof As before, $T : X \to X$ is completely continuous. From $f_\infty < \lambda$, we can choose a $\varepsilon \in (0, \lambda)$ such that $f_\infty < \lambda - \varepsilon$. Then there is an $R > 0$ such that

$$\left|f(t,x,y,z)\right| < (\lambda - \varepsilon)\left(|x| + |y| + |z|\right)$$

holds when $|x| + |y| + |z| \ge R$ for $t \in [0,1]$.

Let $M = \max_{t\in[0,1],|x|+|y|+|z|\le R} |f(t,x,y,z)|$. Then we always have

$$\left|f(t,x,y,z)\right| \le M + (\lambda - \varepsilon)\left(|x| + |y| + |z|\right). \tag{3.11}$$

Set $V = \{u : u \in X, u = \mu Tu, 0 < \mu < 1\}$. Now, we show that V is a bounded set.

In fact, for any $u \in V$, from Lemma 2.5 and (3.11), it follows that

$$\begin{aligned}
&\left|f\left(t, u(t), {}^C D_{0+}^{\alpha_1} u(t), {}^C D_{0+}^{\alpha_2} u(t)\right)\right| \\
&\quad \leq M + (\lambda - \varepsilon)\left(\left|u(t)\right| + \left|{}^C D_{0+}^{\alpha_1} u(t)\right| + \left|{}^C D_{0+}^{\alpha_2} u(t)\right|\right) \\
&\quad \leq M + (\lambda - \varepsilon)\left(\|u\|_0 + \left\|{}^C D_{0+}^{\alpha_1} u\right\|_0 + \left\|{}^C D_{0+}^{\alpha_2} u\right\|_0\right) \\
&\quad \leq M + (\lambda - \varepsilon)\left(1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)}\right)\|u\|_2 \\
&\quad = M + (\lambda - \varepsilon)E\|u\|_2, \quad \text{for } t \in [0,1].
\end{aligned} \tag{3.12}$$

Thus, by (3.1) combined with (3.12) and observing that $|1 - 2(2-\beta_1)t| \leq 2(2-\beta_1)$, we have immediately

$$\begin{aligned}
\left|(Tu)'(t)\right| &\leq \left\{\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\,ds + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1)}\int_0^1 (1-s)^{\alpha-\beta_1-1}\,ds \right. \\
&\quad \left. + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2)}\int_0^1 (1-s)^{\alpha-\beta_2-1}\,ds\right\}\left[M + (\lambda - \varepsilon)E\|u\|_2\right] \\
&\leq \left[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right] \cdot \left[M + (\lambda - \varepsilon)E\|u\|_2\right].
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\left|(Tu)''(t)\right| \leq \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right] \cdot \left[M + (\lambda - \varepsilon)E\|u\|_2\right]. \tag{3.14}$$

So, from (3.13)-(3.14), we have

$$\begin{aligned}
\|Tu\|_2 &\leq \left[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)}\right] \cdot \left[M + (\lambda - \varepsilon)E\right]\|u\|_2 \\
&= D(M + (\lambda - \varepsilon)E\|u\|_2.
\end{aligned}$$

Now, the relation $u = \mu Tu$ with $0 < \mu < 1$ implies

$$\|u\|_2 = \mu\|Tu\|_2 \leq DM + (\lambda - \varepsilon)DE\|u\|_2.$$

Because $\lambda DE = 1$, the above inequality implies $\|u\|_2 \leq \frac{M}{\varepsilon E}$. That is, V is a bounded set. So, by Lemma 2.7, we have that T has at least one fixed point u. That is, u is a solution of BVP (1.1)-(1.2) by Lemma 2.4. This completes the proof. □

Example 3.1 Consider the following anti-periodic boundary value problem:

$$\begin{cases}
{}^C D_{0+}^{\frac{5}{2}} u(t) = \frac{\lambda}{1+t}\frac{|u(t)|}{1+|u(t)|} - \lambda e^{-t}\ln[1 + ({}^C D_{0+}^{\frac{3}{4}}\, u(t))^2 + ({}^C D_{0+}^{\frac{7}{4}}\, u(t))^2], \\
u(0) = -u(1), \qquad t^{-\frac{1}{2}}\,{}^C D_{0+}^{\frac{1}{2}} u(t)_{|t\to 0} = -t^{-\frac{1}{2}}\,{}^C D_{0+}^{\frac{1}{2}} u(t)_{|t=1}, \\
t^{-\frac{1}{2}}\,{}^C D_{0+}^{\frac{3}{2}} u(t)_{|t\to 0} = -t^{-\frac{1}{2}}\,{}^C D_{0+}^{\frac{3}{2}} u(t)_{|t=1},
\end{cases} \tag{3.15}$$

where $\alpha = \frac{5}{2}$, $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{7}{4}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{3}{2}$, and $0 < \lambda < \frac{8\sqrt{\pi}\Gamma(\frac{1}{4})}{(16+3\pi)(8+\Gamma(\frac{1}{4}))}$. Clearly, the function $f = \frac{\lambda}{t+1}\frac{|x|}{1+|x|} - \lambda e^{-t}\ln(1 + y^2 + z^2)$ satisfies $|f(t, x_2, y_z, z_2) - f(t, x_1, y_1, z_1)| \leq \lambda(|x_2 - x_1| + |y_2 -$

$y_1| + |z_2 - z_1|)$. Further, $D = \frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)} = \frac{(16+3\pi)}{8\sqrt{\pi}}$, $L = L_1 + \frac{L_2}{\Gamma(2-\alpha_1)} + \frac{L_3}{\Gamma(3-\alpha_2)} = \lambda\frac{8+\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})}$. As $DL = \lambda\frac{(16+3\pi)(8+\Gamma(\frac{1}{4}))}{8\sqrt{\pi}\Gamma(\frac{1}{4})} < 1$, all the assumptions of Theorem 3.1 are satisfied. Hence BVP (3.15) has a unique solution.

Example 3.2 Consider the following anti-periodic boundary value problem:

$$\begin{cases} {}^CD_{0+}^{\frac{5}{2}}u(t) = \mu\sin u(t) + e^{\cos t}[({}^CD_{0+}^{\frac{3}{4}}\,u(t))^3 - ({}^CD_{0+}^{\frac{7}{4}}\,u(t))^2], \quad t \in (0,1), \\ u(0) = -u(1), \qquad t^{-\frac{1}{2}}\,{}^CD_{0+}^{\frac{1}{2}}u(t)_{|t\to 0} = -t^{-\frac{1}{2}}\,{}^CD_{0+}^{\frac{1}{2}}u(t)_{|t=1}, \\ t^{-\frac{1}{2}}\,{}^CD_{0+}^{\frac{3}{2}}u(t)_{|t\to 0} = -t^{-\frac{1}{2}}\,{}^CD_{0+}^{\frac{3}{2}}u(t)_{|t=1}, \end{cases} \tag{3.16}$$

where $\alpha = \frac{5}{2}$, $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{7}{4}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{3}{2}$, and $0 < \mu < \frac{8\sqrt{\pi}\Gamma(\frac{1}{4})}{(16+3\pi)(8+\Gamma(\frac{1}{4}))}$. Clearly, the function $f = \mu\sin x + e^{\cos t}(y^3 - z^2)$ satisfies $f_0 \le \mu$, where $f_0 = \overline{\lim}_{|x|+|y|+|z|\to 0}\max_{t\in[0,1]}\frac{f(t,x,y,z)}{|x|+|y|+|z|}$. Further, $D = \frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-\beta_1)}{2\Gamma(\alpha-\beta_1+1)} + \frac{\Gamma(3-\beta_2)}{2\Gamma(\alpha-\beta_2+1)} = \frac{(16+3\pi)}{8\sqrt{\pi}}$, $E = 1 + \frac{1}{\Gamma(2-\alpha_1)} + \frac{1}{\Gamma(3-\alpha_2)} = \frac{8+\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})}$, $\lambda = (DE)^{-1} = \frac{8\sqrt{\pi}\Gamma(\frac{1}{4})}{(16+3\pi)(8+\Gamma(\frac{1}{4}))}$. As $f_0 < \lambda$, all the assumptions of Theorem 3.2 are satisfied. Hence BVP (3.16) has at least one solution.

Competing interests

The author declares that he has no competing interests.

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