Diblík and Halfarová *Advances in Difference Equations* 2013, **2013**:50
http://www.advancesindifferenceequations.com/content/2013/1/50

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Explicit general solution of planar linear discrete systems with constant coefficients and weak delays

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Abstract
In this paper, planar linear discrete systems with constant coefficients and two delays

$$x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n)$$

are considered where $k \in \mathbb{Z}_0^\infty := \{0, 1, \dots, \infty\}$, $x\colon \mathbb{Z}_0^\infty \to \mathbb{R}^2$, $m > n > 0$ are fixed integers and $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are constant 2×2 matrices. It is assumed that the considered system is one with weak delays. The characteristic equations of such systems are identical with those for the same systems but without delayed terms. In this case, after several steps, the space of solutions with a given starting dimension $2(m+1)$ is pasted into a space with a dimension less than the starting one. In a sense, this situation is analogous to one known in the theory of linear differential systems with constant coefficients and weak delays when the initially infinite dimensional space of solutions on the initial interval turns (after several steps) into a finite dimensional set of solutions. For every possible case, explicit general solutions are constructed and, finally, results on the dimensionality of the space of solutions are obtained.

AMS Subject Classification: 39A06; 39A12

Keywords: discrete equation; weak delays; spectrum; explicit solution; dimension of the solutions space

Introduction

Preliminary notions and properties

We use the following notation: for integers s, q, $s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$, where $s = -\infty$ or $q = \infty$ are admitted, too. Throughout this paper, using notation $\mathbb{Z}_s^q$, we always assume $s \le q$. In this paper, we deal with the discrete planar systems

$$x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n), \qquad (1)$$

where $m > n > 0$ are fixed integers, $k \in \mathbb{Z}_0^\infty$, $A = (a_{ij})$ and $B = (b_{ij})$, $C = (c_{ij})$ are constant 2×2 matrices, and $x\colon \mathbb{Z}_{-m}^\infty \to \mathbb{R}^2$. Throughout the paper, we assume that

$$B \ne \Theta, \qquad C \ne \Theta, \qquad (2)$$

where Θ is a 2×2 zero matrix. Together with equation (1), we consider the initial (Cauchy) problem

$$x(k) = \varphi(k), \tag{3}$$

where $k = -m, -m+1, \dots, 0$ with $\varphi \colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^2$. The *existence* and *uniqueness* of the solution of initial problem (1), (3) on $\mathbb{Z}_{-m}^{\infty}$ is obvious. We recall that the *solution* $x \colon \mathbb{Z}_{-m}^{\infty} \to \mathbb{R}^2$ of (1), (3) is defined as an *infinite sequence*

$$\bigl\{x(-m) = \varphi(-m), x(-m+1) = \varphi(-m+1), \dots, x(0) = \varphi(0), x(1), x(2), \dots, x(k), \dots\bigr\}$$

such that, for any $k \in \mathbb{Z}_0^{\infty}$, equality (1) holds.

The space of all initial data (3) with $\varphi \colon \mathbb{Z}_{-m}^{0} \to \mathbb{R}^2$ is obviously $2(m+1)$-dimensional. Below, we describe the fact that among systems (1), there are such systems that their space of solutions, being initially $2(m+1)$-dimensional, on a reduced interval turns into a space having a dimension less than $2(m+1)$.

Systems with weak delays

We consider system (1) and look for a solution having the form $x(k) = \xi\lambda^k$, where $k \in \mathbb{Z}_{-m}^{\infty}$, $\lambda = \text{const}$ with $\lambda \neq 0$ and $\xi = (\xi_1, \xi_2)^T$ is a nonzero constant vector. The usual procedure leads to the characteristic equation

$$D := \det\bigl(A + \lambda^{-m}B + \lambda^{-n}C - \lambda I\bigr) = 0, \tag{4}$$

where I is the unit 2×2 matrix. Together with (1), we consider a system with the terms containing delays omitted

$$x(k+1) = Ax(k) \tag{5}$$

and the characteristic equation

$$\det(A - \lambda I) = 0. \tag{6}$$

Definition 1 System (1) is called a system with weak delays if characteristic equations (4), (6) corresponding to systems (1) and (5) are equal, *i.e.*, if for every $\lambda \in \mathbb{C} \setminus \{0\}$,

$$D = \det\bigl(A + \lambda^{-m}B + \lambda^{-n}C - \lambda I\bigr) = \det(A - \lambda I). \tag{7}$$

We consider a linear transformation

$$x(k) = \mathcal{S}y(k) \tag{8}$$

with a nonsingular 2×2 matrix $\mathcal{S}$. Then the discrete system for y is

$$y(k+1) = A_{\mathcal{S}}y(k) + B_{\mathcal{S}}y(k-m) + C_{\mathcal{S}}y(k-n) \tag{9}$$

with $A_{\mathcal{S}} = \mathcal{S}^{-1}A\mathcal{S}$, $B_{\mathcal{S}} = \mathcal{S}^{-1}B\mathcal{S}$, $C_{\mathcal{S}} = \mathcal{S}^{-1}C\mathcal{S}$. We show that a system's property of being one with weak delays is preserved by every nonsingular linear transformation.

Lemma 1 *If system* (1) *is a system with weak delays, then its arbitrary linear nonsingular transformation* (8) *again leads to a system with weak delays* (9).

Proof It is easy to show that

$$\det\left(A_{\mathcal{S}} + \lambda^{-m} B_{\mathcal{S}} + \lambda^{-n} C_{\mathcal{S}} - \lambda I\right) = \det(A_{\mathcal{S}} - \lambda I)$$

holds since

$$\begin{aligned}
\det\left(A_{\mathcal{S}} + \lambda^{-m} B_{\mathcal{S}} + \lambda^{-n} C_{\mathcal{S}} - \lambda I\right) &= \det\left(\mathcal{S}^{-1} A \mathcal{S} + \lambda^{-m} \mathcal{S}^{-1} B \mathcal{S} + \lambda^{-n} \mathcal{S}^{-1} C \mathcal{S} - \lambda \mathcal{S}^{-1} I \mathcal{S}\right) \\
&= \det\left[\mathcal{S}^{-1}\left(A + \lambda^{-m} B + \lambda^{-n} C - \lambda I\right)\mathcal{S}\right] \\
&= \det\left(A + \lambda^{-m} B + \lambda^{-n} C - \lambda I\right),
\end{aligned}$$

$$\begin{aligned}
\det(A_{\mathcal{S}} - \lambda I) &= \det\left(\mathcal{S}^{-1} A \mathcal{S} - \lambda \mathcal{S}^{-1} I \mathcal{S}\right) \\
&= \det\left[\mathcal{S}^{-1}(A - \lambda I)\mathcal{S}\right] \\
&= \det(A - \lambda I)
\end{aligned}$$

and the equality

$$\det\left(A + \lambda^{-m} B + \lambda^{-n} C - \lambda I\right) = \det(A - \lambda I)$$

is assumed. □

Necessary and sufficient conditions determining the weak delays

In the below theorem, we give conditions, in terms of determinants, indicating whether a system is one with weak delays.

Theorem 1 *System* (1) *is a system with weak delays if and only if the following seven conditions hold simultaneously*:

$$b_{11} + b_{22} = 0, \tag{10}$$

$$c_{11} + c_{22} = 0, \tag{11}$$

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0, \tag{12}$$

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0, \tag{13}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{14}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{15}$$

$$\begin{vmatrix} b_{11} & b_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0. \tag{16}$$

Proof We start with computing determinant (4). We get

$$D = \begin{vmatrix} a_{11} + b_{11}\lambda^{-m} + c_{11}\lambda^{-n} - \lambda & a_{12} + b_{12}\lambda^{-m} + c_{12}\lambda^{-n} \\ a_{21} + b_{21}\lambda^{-m} + c_{21}\lambda^{-n} & a_{22} + b_{22}\lambda^{-m} + c_{22}\lambda^{-n} - \lambda \end{vmatrix}.$$

Expand the determinant on the right-hand side along the first column:

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} & a_{12} + b_{12}\lambda^{-m} + c_{12}\lambda^{-n} \\ a_{21} & a_{22} + b_{22}\lambda^{-m} + c_{22}\lambda^{-n} - \lambda \end{vmatrix} + \lambda^{-m} \begin{vmatrix} b_{11} & a_{12} + b_{12}\lambda^{-m} + c_{12}\lambda^{-n} \\ b_{21} & a_{22} + b_{22}\lambda^{-m} + c_{22}\lambda^{-n} - \lambda \end{vmatrix} \\ & + \lambda^{-n} \begin{vmatrix} c_{11} & a_{12} + b_{12}\lambda^{-m} + c_{12}\lambda^{-n} \\ c_{21} & a_{22} + b_{22}\lambda^{-m} + c_{22}\lambda^{-n} - \lambda \end{vmatrix} + \lambda \begin{vmatrix} -1 & a_{12} + b_{12}\lambda^{-m} + c_{12}\lambda^{-n} \\ 0 & a_{22} + b_{22}\lambda^{-m} + c_{22}\lambda^{-n} - \lambda \end{vmatrix}. \end{aligned}$$

Expanding each of the determinants along the second column, we will have

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \lambda^{-m} \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \lambda^{-n} \begin{vmatrix} a_{11} & c_{12} \\ a_{21} & c_{22} \end{vmatrix} + \lambda \begin{vmatrix} a_{11} & 0 \\ a_{21} & -1 \end{vmatrix} \\ & + \lambda^{-m} \left[\begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} + \lambda^{-m} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \lambda^{-n} \begin{vmatrix} b_{11} & c_{12} \\ b_{21} & c_{22} \end{vmatrix} + \lambda \begin{vmatrix} b_{11} & 0 \\ b_{21} & -1 \end{vmatrix} \right] \\ & + \lambda^{-n} \left[\begin{vmatrix} c_{11} & a_{12} \\ c_{21} & a_{22} \end{vmatrix} + \lambda^{-m} \begin{vmatrix} c_{11} & b_{12} \\ c_{21} & b_{22} \end{vmatrix} + \lambda^{-n} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} + \lambda \begin{vmatrix} c_{11} & 0 \\ c_{21} & -1 \end{vmatrix} \right] \\ & + \lambda \left[\begin{vmatrix} -1 & a_{12} \\ 0 & a_{22} \end{vmatrix} + \lambda^{-m} \begin{vmatrix} -1 & b_{12} \\ 0 & b_{22} \end{vmatrix} + \lambda^{-n} \begin{vmatrix} -1 & c_{12} \\ 0 & c_{22} \end{vmatrix} + \lambda \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \right]. \end{aligned}$$

After simplification, we get

$$\begin{aligned} D = {} & \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} - \lambda^{-m+1}(b_{11} + b_{22}) - \lambda^{-n+1}(c_{11} + c_{22}) \\ & + \lambda^{-m} \left[\begin{vmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix} \right] + \lambda^{-n} \left[\begin{vmatrix} a_{11} & a_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ a_{21} & a_{22} \end{vmatrix} \right] \\ & + \lambda^{-m-n} \left[\begin{vmatrix} b_{11} & b_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ b_{21} & b_{22} \end{vmatrix} \right] + \lambda^{-2m} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \lambda^{-2n} \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}. \end{aligned}$$

Now we see that for (7) to hold, *i.e.*,

$$D = \det\bigl(A + \lambda^{-m}B + \lambda^{-n}C - \lambda I\bigr) = \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix},$$

conditions (10)-(16) are both necessary and sufficient. □

Lemma 2 *Conditions* (10)-(16) *are equivalent to*

$$\operatorname{tr} B = \det B = 0, \tag{17}$$

$$\det(A + B) = \det A, \tag{18}$$

$$\operatorname{tr} C = \det C = 0, \tag{19}$$

$$\det(A + C) = \det A, \tag{20}$$

$$\det(B + C) = 0. \tag{21}$$

Proof We show that assumptions (10)-(16) imply (17)-(21). It is obvious that condition (17) is equivalent to (10), (12) and condition (19) is equivalent to (11), (13). Now we consider

$$\det(A + B) = \begin{vmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{vmatrix}.$$

Expanding the determinant on the right-hand side along the first column and then expanding each of the determinants along the second column, we have

$$\begin{aligned} \det(A + B) &= \begin{vmatrix} a_{11} & a_{12} + b_{12} \\ a_{21} & a_{22} + b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} + b_{12} \\ b_{21} & a_{22} + b_{22} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \\ &= \bigl[\text{due to (12) and (14)}\bigr] = \det A. \end{aligned} \tag{22}$$

Similarly, using (13) and (15), we get (20), *i.e.*, $\det(A + C) = \det A$. Now we consider

$$\det(B + C) = \begin{vmatrix} b_{11} + c_{11} & b_{12} + c_{12} \\ b_{21} + c_{21} & b_{22} + c_{22} \end{vmatrix}.$$

Expanding the determinant on the right-hand side along the first column and then expanding each of the determinants along the second column, we have

$$\begin{aligned} \det(B + C) &= \begin{vmatrix} b_{11} & b_{12} + c_{12} \\ b_{21} & b_{22} + c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & b_{12} + c_{12} \\ c_{21} & b_{22} + c_{22} \end{vmatrix} \\ &= \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & c_{12} \\ b_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & b_{12} \\ c_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \\ &= \bigl[\text{due to (12), (13) and (16)}\bigr] = 0. \end{aligned} \tag{23}$$

Now we prove that assumptions (17) and (21) imply (10) and (16). Due to equivalence (10)-(13) with (17) and (19), it remains to be shown that (18), (20) and (21) imply (14)-(16).

If (18) holds, then, from computations (22), we see that

$$\begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} \\ b_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = 0,$$

i.e., because of (17), we get (14). Similarly, formula (15) can be proved with the aid of (20) and by (19).

Finally, we show that (21) implies (16). From (23), using (17) and (19), we get

$$\det(B+C) = \begin{vmatrix} b_{11} & c_{12} \\ b_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & b_{12} \\ c_{21} & b_{22} \end{vmatrix} = 0,$$

i.e., (16) holds. □

Problem under consideration

The aim of this paper is to show that after several steps, the dimension of the space of all solutions, being initially equal to the dimension $2(m+1)$ of the space of initial data (3) generated by discrete functions φ, is reduced to a dimension less than the initial one on an interval of the form $\mathbb{Z}_s^\infty$ with an $s > 0$. In other words, we will show that the $2(m+1)$-dimensional space of all solutions of (1) is reduced to a less-dimensional space of solutions on $\mathbb{Z}_s^\infty$. This problem is solved directly by explicitly computing the corresponding solutions of the Cauchy problems with each of the arising cases being considered. The underlying idea for such investigation is simple. If (1) is a system with weak delays, then the corresponding characteristic equation has only two eigenvalues instead of $2(m+1)$ eigenvalues in the case of systems with non-weak delays. This explains why the dimension of the space of solutions becomes less than the initial one. The final results (Theorems 7-10) provide the dimension of the space of solutions. Our results generalize the results in [1] where system (1) with $C = \Theta$ is analyzed.

Auxiliary formula

For the reader's convenience, we recall one explicit formula (see, *e.g.*, [2]) for the solutions of linear scalar discrete non-delayed equations used in this paper. We consider the first-order linear discrete nonhomogeneous equation

$$w(k+1) = aw(k) + g(k), \qquad w(k_0) = w_0, \quad k \in \mathbb{Z}_{k_0}^\infty$$

with $a \in \mathbb{C}$ and $g\colon \mathbb{Z}_{k_0}^\infty \to \mathbb{C}$. Then it is easy to verify that

$$w(k) = a^{k-k_0} w_0 + \sum_{r=k_0}^{k-1} a^{k-1-r} g(r), \quad k \in \mathbb{Z}_{k_0+1}^\infty. \tag{24}$$

Throughout the paper, we adopt the customary notation for the sum: $\sum_{i=\ell+s}^{\ell} \mathcal{F}(i) = 0$, where ℓ is an integer, s is a positive integer and '$\mathcal{F}$' denotes the function considered independently of whether it is defined for indicated arguments or not.

Note that formula (24) is many times used in recent literature to analyze asymptotic properties of solutions of various classes of difference equations, including nonlinear equations. We refer, *e.g.*, to [3–9] and to relevant references therein.

Results

If (7) holds, then equations (4) and (6) have only two (and the same) roots simultaneously. In order to prove the properties of the family of solutions of (1) formulated in the Introduction, we will discuss each combination of roots, *i.e.*, the cases of two real and distinct roots, a pair of complex conjugate roots and, finally, a double real root.

Jordan forms of matrix *A* and corresponding solutions of problem (1), (3)

It is known that for every matrix A, there exists a nonsingular matrix S transforming it to the corresponding Jordan matrix form Λ. This means that

$$\Lambda = S^{-1}AS,$$

where Λ has the following possible forms depending on the roots of characteristic equation (6), *i.e.*, on the roots of

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0. \tag{25}$$

If (25) has two real distinct roots λ_1, λ_2, then

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{26}$$

if the roots are complex conjugate, *i.e.*, $\lambda_{1,2} = p \pm iq$ with $q \neq 0$, then

$$\Lambda = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \tag{27}$$

and, finally, in the case of one double real root $\lambda_{1,2} = \lambda$, we have either

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \tag{28}$$

or

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{29}$$

The transformation $y(k) = S^{-1}x(k)$ transforms (1) into the system

$$y(k+1) = \Lambda y(k) + B^* y(k-m) + C^* y(k-n), \quad k \in \mathbb{Z}_0^\infty \tag{30}$$

with $B^* = S^{-1}BS$, $B^* = (b^*_{ij})$, $C^* = S^{-1}CS$, $C^* = (c^*_{ij})$, $i, j = 1, 2$. Together with (30), we consider the initial problem

$$y(k) = \varphi^*(k), \tag{31}$$

$k \in \mathbb{Z}_{-m}^0$ with $\varphi^* : \mathbb{Z}_{-m}^0 \to \mathbb{R}^2$, where $\varphi^*(k) = S^{-1}\varphi(k)$ is the initial function corresponding to the initial function φ in (3).

Below, we consider all four possible cases (26)-(29) separately.

We define

$$\Phi_1(k) := \big(0, \varphi_1^*(k)\big)^T, \qquad \Phi_2(k) := \big(\varphi_2^*(k), 0\big)^T, \quad k \in \mathbb{Z}_{-m}^0. \tag{32}$$

Assuming that (1) is a system with weak delays, by Lemma 1, system (30) is one with weak delays again.

The case (26) *of two real distinct roots*

In this case, we have $\Lambda^k = \operatorname{diag}(\lambda_1^k, \lambda_2^k)$. The necessary and sufficient conditions (10)-(16) for (30) turn into

$$b^*_{11} + b^*_{22} = 0, \tag{33}$$

$$c^*_{11} + c^*_{22} = 0, \tag{34}$$

$$\begin{vmatrix} b^*_{11} & b^*_{12} \\ b^*_{21} & b^*_{22} \end{vmatrix} = b^*_{11}b^*_{22} - b^*_{12}b^*_{21} = 0, \tag{35}$$

$$\begin{vmatrix} c^*_{11} & c^*_{12} \\ c^*_{21} & c^*_{22} \end{vmatrix} = c^*_{11}c^*_{22} - c^*_{12}c^*_{21} = 0, \tag{36}$$

$$\begin{vmatrix} \lambda_1 & 0 \\ b^*_{21} & b^*_{22} \end{vmatrix} + \begin{vmatrix} b^*_{11} & b^*_{12} \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 b^*_{22} + \lambda_2 b^*_{11} = 0, \tag{37}$$

$$\begin{vmatrix} \lambda_1 & 0 \\ c^*_{21} & c^*_{22} \end{vmatrix} + \begin{vmatrix} c^*_{11} & c^*_{12} \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 c^*_{22} + \lambda_2 c^*_{11} = 0, \tag{38}$$

$$\begin{vmatrix} b^*_{11} & b^*_{12} \\ c^*_{21} & c^*_{22} \end{vmatrix} + \begin{vmatrix} c^*_{11} & c^*_{12} \\ b^*_{21} & b^*_{22} \end{vmatrix} = 0. \tag{39}$$

Since $\lambda_1 \neq \lambda_2$, equations (33), (37) yield $b^*_{11} = b^*_{22} = 0$ and equations (34), (38) yield $c^*_{11} = c^*_{22} = 0$. Then, from (35), we get $b^*_{12}b^*_{21} = 0$ so that either $b^*_{21} = 0$ or $b^*_{12} = 0$, and from (36), we get $c^*_{12}c^*_{21} = 0$, which means that either $c^*_{21} = 0$ or $c^*_{12} = 0$. In view of assumptions $B \neq \Theta$ and $C \neq \Theta$, we conclude that only the following cases (I), (II) are possible:

$$\begin{aligned}
&\text{(I)} \quad b^*_{11} = b^*_{22} = b^*_{21} = 0, \qquad b^*_{12} \neq 0,\\
&\phantom{\text{(I)}} \quad c^*_{11} = c^*_{22} = c^*_{21} = 0, \qquad c^*_{12} \neq 0,\\
&\text{(II)} \quad b^*_{11} = b^*_{22} = b^*_{12} = 0, \qquad b^*_{21} \neq 0,\\
&\phantom{\text{(II)}} \quad c^*_{11} = c^*_{22} = c^*_{12} = 0, \qquad c^*_{21} \neq 0.
\end{aligned}$$

Theorem 2 *Let* (1) *be a system with weak delays and let* (25) *admit two real distinct roots* λ_1, λ_2. *If the conditions* (I) *hold, then the solution of initial problem* (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}^{\infty}_{-m}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases}
\varphi^*(k) \quad \text{if } k \in \mathbb{Z}^0_{-m}, \\
\Lambda^k \varphi^*(0) + b^*_{12} \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \Phi_2(r-m) \\
\quad + c^*_{12} \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \Phi_2(r-n) \\
\quad \text{if } k \in \mathbb{Z}^{n+1}_1, \\
\Lambda^k \varphi^*(0) + b^*_{12} \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \Phi_2(r-m) \\
\quad + c^*_{12} [\sum_{r=0}^{n} \lambda_1^{k-1-r} \Phi_2(r-n) + \Phi_2(0) \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-n}] \\
\quad \text{if } k \in \mathbb{Z}^{m+1}_{n+2}, \\
\Lambda^k \varphi^*(0) + b^*_{12} [\sum_{r=0}^{m} \lambda_1^{k-1-r} \Phi_2(r-m) + \Phi_2(0) \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m}] \\
\quad + c^*_{12} [\sum_{r=0}^{n} \lambda_1^{k-1-r} \Phi_2(r-n) + \Phi_2(0) \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-n}] \\
\quad \text{if } k \in \mathbb{Z}^{\infty}_{m+2}.
\end{cases} \tag{40}$$

If the conditions of case (II) *are true, then the solution of initial problem* (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ \Lambda^k \varphi^*(0) + b_{21}^* \sum_{r=0}^{k-1} \lambda_2^{k-1-r} \Phi_1(r-m) \\ \quad + c_{21}^* \sum_{r=0}^{k-1} \lambda_2^{k-1-r} \Phi_1(r-n) \\ \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \Lambda^k \varphi^*(0) + b_{21}^* \sum_{r=0}^{k-1} \lambda_2^{k-1-r} \Phi_1(r-m) \\ \quad + c_{21}^* [\sum_{r=0}^{n} \lambda_2^{k-1-r} \Phi_1(r-n) + \Phi_1(0) \sum_{r=n+1}^{k-1} \lambda_1^{r-n} \lambda_2^{k-1-r}] \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \Lambda^k \varphi^*(0) + b_{21}^* [\sum_{r=0}^{m} \lambda_2^{k-1-r} \Phi_1(r-m) + \Phi_1(0) \sum_{r=m+1}^{k-1} \lambda_1^{r-m} \lambda_2^{k-1-r}] \\ \quad + c_{21}^* [\sum_{r=0}^{n} \lambda_2^{k-1-r} \Phi_1(r-n) + \Phi_1(0) \sum_{r=n+1}^{k-1} \lambda_1^{r-n} \lambda_2^{k-1-r}] \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}. \end{cases} \tag{41}$$

Proof If the conditions (I) are true, then the transformed system (30) takes the form

$$y_1(k+1) = \lambda_1 y_1(k) + b_{12}^* y_2(k-m) + c_{12}^* y_2(k-n), \tag{42}$$

$$y_2(k+1) = \lambda_2 y_2(k), \quad k \in \mathbb{Z}_0^{\infty} \tag{43}$$

and if the conditions (II) hold, then (30) takes the form

$$y_1(k+1) = \lambda_1 y_1(k), \tag{44}$$

$$y_2(k+1) = \lambda_2 y_2(k) + b_{21}^* y_1(k-m) + c_{21}^* y_1(k-n), \quad k \in \mathbb{Z}_0^{\infty}. \tag{45}$$

We investigate only initial problem (42), (43), (31) since initial problem (44), (45), (31) can be examined in a similar way. From (43), (31), we get

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ \lambda_2^k \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{46}$$

Then (42) becomes

$$y_1(k+1) = \begin{cases} \lambda_1 y_1(k) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \varphi_2^*(k-n) & \text{if } k \in \mathbb{Z}_0^{n}, \\ \lambda_1 y_1(k) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \lambda_2^{k-n} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{n+1}^{m}, \\ \lambda_1 y_1(k) + b_{12}^* \lambda_2^{k-m} \varphi_2^*(0) + c_{12}^* \lambda_2^{k-n} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{m+1}^{\infty}. \end{cases} \tag{47}$$

First, we solve this equation for $k \in \mathbb{Z}_0^n$. This means that we consider the problem

$$\begin{cases} y_1(k+1) = \lambda_1 y_1(k) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \varphi_2^*(k-n), \quad k \in \mathbb{Z}_0^{n}, \\ y_1(0) = \varphi_1^*(0). \end{cases}$$

With the aid of formula (24), we get

$$y_1(k) = \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda_1^{k-1-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\big], \quad k \in \mathbb{Z}_1^{n+1}. \tag{48}$$

Now we solve equation (47) for $k \in \mathbb{Z}_{n+1}^m$, *i.e.*, we consider the problem (with initial data deduced from (48))

$$\begin{cases} y_1(k+1) = \lambda_1 y_1(k) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \lambda_2^{k-n} \varphi_2^*(0), \quad k \in \mathbb{Z}_{n+1}^m, \\ y_1(n+1) = \lambda_1^{n+1} \varphi_1^*(0) + \sum_{r=0}^n \lambda_1^{n-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)]. \end{cases}$$

Applying formula (24) yields (for $k \in \mathbb{Z}_{n+2}^{m+1}$)

$$\begin{aligned} y_1(k) &= \lambda_1^{k-(n+1)} y_1(n+1) \\ &\quad + \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0)\big] \\ &= \lambda_1^{k-n-1} \left[\lambda_1^{n+1} \varphi_1^*(0) + \sum_{r=0}^{n} \lambda_1^{n-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\big] \right] \\ &\quad + \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0)\big] \\ &= \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{n} \lambda_1^{k-1-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\big] \\ &\quad + \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0)\big]. \end{aligned} \tag{49}$$

Now we solve equation (47) for $k \in \mathbb{Z}_{m+1}^\infty$, *i.e.*, we consider the problem (with initial data deduced from (49))

$$\begin{cases} y_1(k+1) = \lambda_1 y_1(k) + \varphi_2^*(0)[b_{12}^* \lambda_2^{k-m} + c_{12}^* \lambda_2^{k-n}], \quad k \in \mathbb{Z}_{m+1}^\infty, \\ y_1(m+1) = \lambda_1^{m+1} \varphi_1^*(0) + \sum_{r=0}^n \lambda_1^{m-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)] \\ \qquad\qquad + \sum_{r=n+1}^m \lambda_1^{m-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0)]. \end{cases}$$

Applying formula (24) yields (for $k \in \mathbb{Z}_{m+2}^\infty$)

$$\begin{aligned} y_1(k) &= \lambda_1^{k-(m+1)} y_1(m+1) \\ &\quad + \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \varphi_2^*(0)\big[b_{12}^* \lambda_2^{r-m} + c_{12}^* \lambda_2^{r-n}\big] \\ &= \lambda_1^{k-m-1} \left[\lambda_1^{m+1} \varphi_1^*(0) + \sum_{r=0}^{n} \lambda_1^{m-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\big] \right. \\ &\quad \left. + \sum_{r=n+1}^{m} \lambda_1^{m-r}\big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0)\big] \right] \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \varphi_2^*(0) \big[b_{12}^* \lambda_2^{r-m} + c_{12}^* \lambda_2^{r-n} \big] \\
&= \lambda_1^k \varphi_1^*(0) + \sum_{r=0}^{n} \lambda_1^{k-1-r} \big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n) \big] \\
&+ \sum_{r=n+1}^{m} \lambda_1^{k-1-r} \big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda_2^{r-n} \varphi_2^*(0) \big] \\
&+ \varphi_2^*(0) \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \big[b_{12}^* \lambda_2^{r-m} + c_{12}^* \lambda_2^{r-n} \big]. \qquad (50)
\end{aligned}
$$

Summing up all particular cases (48), (49), (50), we have

$$
y_1(k) = \begin{cases}
\varphi_1^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^0, \\
\lambda_1^k \varphi_1^*(0) + b_{12}^* \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \varphi_2^*(r-m) \\
\quad + c_{12}^* \sum_{r=0}^{k-1} \lambda_1^{k-1-r} \varphi_2^*(r-n) \\
\quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\
\lambda_1^k \varphi_1^*(0) + b_{12}^* [\sum_{r=0}^{n} \lambda_1^{k-1-r} \varphi_2^*(r-m) + \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r} \varphi_2^*(r-m)] \\
\quad + c_{12}^* [\sum_{r=0}^{n} \lambda_1^{k-1-r} \varphi_2^*(r-n) + \varphi_2^*(0) \sum_{r=n+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-n}] \\
\quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\
\lambda_1^k \varphi_1^*(0) + b_{12}^* [\sum_{r=0}^{n} \lambda_1^{k-1-r} \varphi_2^*(r-m) + \sum_{r=n+1}^{m} \lambda_1^{k-1-r} \varphi_2^*(r-m) \\
\quad + \varphi_2^*(0) \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-m}] + c_{12}^* [\sum_{r=0}^{n} \lambda_1^{k-1-r} \varphi_2^*(r-n) \\
\quad + \varphi_2^*(0) [\sum_{r=n+1}^{m} \lambda_1^{k-1-r} \lambda_2^{r-n} + \sum_{r=m+1}^{k-1} \lambda_1^{k-1-r} \lambda_2^{r-n}]] \\
\quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}.
\end{cases} \qquad (51)
$$

Now, taking into account (32), formula (40) is a consequence of (46) and (51). Formula (41) can be proved in a similar way.

Finally, we note that both formulas (40), (41) remain valid for $b_{12}^* = b_{21}^* = 0$, $c_{12}^* = c_{21}^* = 0$ as well. In this case, the transformed system (1) reduces to a system without delays. This possibility is excluded by conditions (2). □

The case (27) *of two complex conjugate roots*

The necessary and sufficient conditions (10)-(16) for (30) to take the forms (33)-(36), (39) and

$$
\begin{vmatrix} p & q \\ b_{21}^* & b_{22}^* \end{vmatrix} + \begin{vmatrix} b_{11}^* & b_{12}^* \\ -q & p \end{vmatrix} = p\big(b_{11}^* + b_{22}^*\big) + q\big(b_{12}^* - b_{21}^*\big) = 0 \qquad (52)
$$

and

$$
\begin{vmatrix} p & q \\ c_{21}^* & c_{22}^* \end{vmatrix} + \begin{vmatrix} c_{11}^* & c_{12}^* \\ -q & p \end{vmatrix} = p\big(c_{11}^* + c_{22}^*\big) + q\big(c_{12}^* - c_{21}^*\big) = 0. \qquad (53)
$$

The system of conditions (33), (35) and (52) gives $b^*_{12} = b^*_{21}$, $(b^*_{11})^2 = -(b^*_{12})^2$ and admits only one possibility, namely,

$$b^*_{11} = b^*_{22} = b^*_{12} = b^*_{21} = 0.$$

For the system of conditions (34), (36) and (53), we have $c^*_{12} = c^*_{21}$, $(c^*_{11})^2 = -(c^*_{12})^2$ and we get only one possibility as well, namely,

$$c^*_{11} = c^*_{22} = c^*_{12} = c^*_{21} = 0.$$

Consequently, $B^* = \Theta$, $B = \Theta$ and $C^* = \Theta$, $C = \Theta$. Initial problem (1), (3) reduces to a problem without delay

$$\begin{cases} x(k+1) = Ax(k), \\ x(k) = \varphi(k), \quad k \in \mathbb{Z}^0_{-m} \end{cases}$$

and, obviously,

$$x(k) = \begin{cases} \varphi(k) & \text{if } k \in \mathbb{Z}^0_{-m}, \\ A^k\varphi(0) & \text{if } k \in \mathbb{Z}^\infty_1. \end{cases} \tag{54}$$

From this discussion, the next theorem follows.

Theorem 3 *There exists no system* (1) *with weak delays if* Λ *has the form* (27).

Finally, we note that assumptions (2) alone exclude this case.

The case (28) *of double real root*

We have $\Lambda^k = \operatorname{diag}(\lambda^k, \lambda^k)$. For (30), the necessary and sufficient conditions (10)-(16) are reduced to (33)-(36), (39) and

$$\begin{vmatrix} \lambda & 0 \\ b^*_{21} & b^*_{22} \end{vmatrix} + \begin{vmatrix} b^*_{11} & b^*_{12} \\ 0 & \lambda \end{vmatrix} = \lambda\left(b^*_{11} + b^*_{22}\right) = 0 \tag{55}$$

and

$$\begin{vmatrix} \lambda & 0 \\ c^*_{21} & c^*_{22} \end{vmatrix} + \begin{vmatrix} c^*_{11} & c^*_{12} \\ 0 & \lambda \end{vmatrix} = \lambda\left(c^*_{11} + c^*_{22}\right) = 0. \tag{56}$$

From (33), (35) and (55), we get $b^*_{12}b^*_{21} = -(b^*_{11})^2$. Then, from (34), (36) and (56), we have $c^*_{12}c^*_{21} = -(c^*_{11})^2$. From condition (39), we get

$$b^*_{11}c^*_{22} - b^*_{12}c^*_{21} + b^*_{22}c^*_{11} - b^*_{21}c^*_{12} = 0. \tag{57}$$

Multiplying (57) by $b^*_{12}c^*_{12}$, we have

$$b^*_{11}c^*_{22}b^*_{12}c^*_{12} - \left(b^*_{12}\right)^2 c^*_{21}c^*_{12} + b^*_{22}c^*_{11}b^*_{12}c^*_{12} - b^*_{21}\left(c^*_{12}\right)^2 b^*_{12} = 0. \tag{58}$$

Substituting $b^*_{12}b^*_{21} = -(b^*_{11})^2$, $c^*_{12}c^*_{21} = -(c^*_{11})^2$ into (58) and using (33), (34), we obtain

$$-b^*_{11}c^*_{11}b^*_{12}c^*_{12} + \left(b^*_{12}\right)^2\left(c^*_{11}\right)^2 - b^*_{11}c^*_{11}b^*_{12}c^*_{12} + \left(b^*_{11}\right)^2\left(c^*_{12}\right)^2 = 0. \tag{59}$$

Equation (59) can be written as

$$\left(b^*_{12}c^*_{11} - c^*_{12}b^*_{11}\right)^2 = 0$$

and

$$b^*_{12}c^*_{11} = c^*_{12}b^*_{11}. \tag{60}$$

Now we will analyze the two possible cases: $b^*_{12}b^*_{21} = 0$ and $b^*_{12}b^*_{21} \neq 0$.

For the case $b^*_{12}b^*_{21} = 0$, we have from (33), (35) that $b^*_{11} = b^*_{22} = 0$ and $b^*_{12} = 0$ or $b^*_{21} = 0$. For $b^*_{12} = 0$ and $b^*_{21} \neq 0$, condition (37) gives $c^*_{12} = 0$. Then, from (34), (36), we get $c^*_{11} = c^*_{22} = 0$ and $c^*_{21} \neq 0$.

Now we discuss the case $b^*_{12}b^*_{21} \neq 0$. From conditions (33), (35), we have $b^*_{12}b^*_{21} = -(b^*_{11})^2$ and $b^*_{11}b^*_{22} \neq 0$. This yields $b^*_{11} \neq 0$, $b^*_{22} \neq 0$, and from (60), we have $c^*_{11} \neq 0$, $c^*_{12} \neq 0$. By conditions (34), (36), we get $c^*_{22} \neq 0$, $c^*_{21} \neq 0$.

From the assumptions $B \neq \Theta$ and $C \neq \Theta$, we conclude that only the following cases (I), (II), (III) are possible:

$$\begin{aligned}
&\text{(I)} && b^*_{11} = b^*_{22} = b^*_{21} = 0, && b^*_{12} \neq 0,\\
& && c^*_{11} = c^*_{22} = c^*_{21} = 0, && c^*_{12} \neq 0,\\
&\text{(II)} && b^*_{11} = b^*_{22} = b^*_{12} = 0, && b^*_{21} \neq 0,\\
& && c^*_{11} = c^*_{22} = c^*_{12} = 0, && c^*_{21} \neq 0,\\
&\text{(III)} && b^*_{12}b^*_{21} \neq 0, && c^*_{12}c^*_{21} \neq 0.
\end{aligned}$$

The case $b^*_{12}b^*_{21} = 0$, $c^*_{12}c^*_{21} = 0$

Theorem 4 *Let* (1) *be a system with weak delay,* (25) *admit a two-fold root* $\lambda_{1,2} = \lambda$, $b^*_{12}b^*_{21} = 0$ *and the matrix* Λ *have the form* (28). *Then the solution of initial problem* (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}^{\infty}_{-m}$, *where in the case* $b^*_{21} = 0$, $c^*_{21} = 0$, $y(k)$ *has the form*

$$y(k) = \begin{cases}
\varphi^*(k) \quad \text{if } k \in \mathbb{Z}^0_{-m},\\
\Lambda^k\varphi^*(0) + b^*_{12}\sum_{r=0}^{k-1}\lambda^{k-1-r}\Phi_2(r-m)\\
\quad + c^*_{12}\sum_{r=0}^{k-1}\lambda^{k-1-r}\Phi_2(r-n)\\
\quad \text{if } k \in \mathbb{Z}^{n+1}_{1},\\
\Lambda^k\varphi^*(0) + b^*_{12}\sum_{r=0}^{k-1}\lambda^{k-1-r}\Phi_2(r-m)\\
\quad + c^*_{12}[\sum_{r=0}^{n}\lambda^{k-1-r}\Phi_2(r-n) + (k-1-n)\lambda^{k-1-n}\Phi_2(0)]\\
\quad \text{if } k \in \mathbb{Z}^{m+1}_{n+2},\\
\Lambda^k\varphi^*(0) + b^*_{12}[\sum_{r=0}^{m}\lambda^{k-1-r}\Phi_2(r-m) + (k-1-m)\lambda^{k-1-m}\Phi_2(0)]\\
\quad + c^*_{12}[\sum_{r=0}^{n}\lambda^{k-1-r}\Phi_2(r-n) + (k-1-n)\lambda^{k-1-n}\Phi_2(0)]\\
\quad \text{if } k \in \mathbb{Z}^{\infty}_{m+2}.
\end{cases} \tag{61}$$

If $b^*_{12} = 0$, $c^*_{12} = 0$ *is true, then the solution of initial problem* (1), (3) *is* $x(k) = Sy(k)$, $k \in \mathbb{Z}^{\infty}_{-m}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if } k \in \mathbb{Z}^0_{-m}, \\ \Lambda^k \varphi^*(0) + b^*_{21} \sum_{r=0}^{k-1} \lambda^{k-1-r} \Phi_1(r-m) \\ \quad + c^*_{21} \sum_{r=0}^{k-1} \lambda^{k-1-r} \Phi_1(r-n) \\ \quad \text{if } k \in \mathbb{Z}^{n+1}_1, \\ \Lambda^k \varphi^*(0) + b^*_{21} \sum_{r=0}^{k-1} \lambda^{k-1-r} \Phi_1(r-m) \\ \quad + c^*_{21} [\sum_{r=0}^{n} \lambda^{k-1-r} \Phi_1(r-n) + (k-1-n)\lambda^{k-1-n} \Phi_1(0)] \\ \quad \text{if } k \in \mathbb{Z}^{m+1}_{n+2}, \\ \Lambda^k \varphi^*(0) + b^*_{21} [\sum_{r=0}^{m} \lambda^{k-1-r} \Phi_1(r-m) + (k-1-m)\lambda^{k-1-m} \Phi_1(0)] \\ \quad + c^*_{21} [\sum_{r=0}^{n} \lambda^{k-1-r} \Phi_1(r-n) + (k-1-n)\lambda^{k-1-n} \Phi_1(0)] \\ \quad \text{if } k \in \mathbb{Z}^{\infty}_{m+2}. \end{cases} \tag{62}$$

Proof Case (I) means that $b^*_{12} \neq 0$, $c^*_{12} \neq 0$. Then (30) turns into the system

$$y_1(k+1) = \lambda y_1(k) + b^*_{12} y_2(k-m) + c^*_{12} y_2(k-n), \quad k \in \mathbb{Z}^{\infty}_0, \tag{63}$$

$$y_2(k+1) = \lambda y_2(k) \tag{64}$$

and if $b^*_{21} \neq 0$, $c^*_{21} \neq 0$, (30) turns into the system

$$y_1(k+1) = \lambda y_1(k), \tag{65}$$

$$y_2(k+1) = \lambda y_2(k) + b^*_{21} y_1(k-m) + c^*_{21} y_1(k-n), \quad k \in \mathbb{Z}^{\infty}_0. \tag{66}$$

System (63), (64) can be solved in much the same way as system (42), (43) if we put $\lambda_1 = \lambda_2 = \lambda$, and the discussion of system (65), (66) goes along the same lines as that of system (44), (45) with $\lambda_1 = \lambda_2 = \lambda$. Formulas (61) and (62) are consequences of (40), (41). □

The case $b^*_{12} b^*_{21} \neq 0$, $c^*_{12} c^*_{21} \neq 0$

For $k \in \mathbb{Z}^0_{-m}$, we define

$$\Phi^*_1(k) := \left(b^*_{11} \left[\varphi^*_1(k) + \frac{b^*_{12}}{b^*_{11}} \varphi^*_2(k) \right], -\frac{(b^*_{11})^2}{b^*_{12}} \left[\varphi^*_1(k) + \frac{b^*_{12}}{b^*_{11}} \varphi^*_2(k) \right] \right)^T, \tag{67}$$

$$\Phi^*_2(k) := \left(c^*_{11} \left[\varphi^*_1(k) + \frac{b^*_{12}}{b^*_{11}} \varphi^*_2(k) \right], -\frac{(c^*_{11})^2}{c^*_{12}} \left[\varphi^*_1(k) + \frac{b^*_{12}}{b^*_{11}} \varphi^*_2(k) \right] \right)^T. \tag{68}$$

Theorem 5 *Let system* (1) *be one with weak delay,* (25) *admit two repeated roots* $\lambda_{1,2} = \lambda$, $b^*_{12} b^*_{21} \neq 0$, $c^*_{12} c^*_{21} \neq 0$ *and the matrix* Λ *have the form* (28). *Then the solution of initial* *problem* (1), (3) *is given by* $x(k) = Sy(k)$, $k \in \mathbb{Z}_{-m}^{\infty}$, *where* $y(k)$ *has the form*

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ \Lambda^k \varphi^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r}[\Phi_1^*(r-m) + \Phi_2^*(r-n)] \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \Lambda^k \varphi^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r}[\Phi_1^*(r-m) + \Phi_2^*(r-n)] \\ \quad + \sum_{r=n+1}^{k-1} \lambda^{k-1-r}\Phi_1^*(r-m) + (k-1-n)\lambda^{k-1-n}\Phi_2^*(0) \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \Lambda^k \varphi^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r}[\Phi_1^*(r-m) + \Phi_2^*(r-n)] \\ \quad + \sum_{r=n+1}^{m} \lambda^{k-1-r}\Phi_1^*(r-m) \\ \quad + (k-1-m)\lambda^{k-1-m}\Phi_1^*(0) + (k-1-n)\lambda^{k-1-n}\Phi_2^*(0) \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}. \end{cases} \tag{69}$$

Proof In this case, all the entries of B^*, C^* are nonzero, and from (33)-(36) and (55), (56), we get

$$B^* = \begin{pmatrix} b_{11}^* & b_{12}^* \\ -(b_{11}^*)^2/b_{12}^* & -b_{11}^* \end{pmatrix},$$
$$C^* = \begin{pmatrix} c_{11}^* & c_{12}^* \\ -(c_{11}^*)^2/c_{12}^* & -c_{11}^* \end{pmatrix}.$$

Then system (30) reduces to

$$y_1(k+1) = \lambda y_1(k) + b_{11}^* y_1(k-m) + b_{12}^* y_2(k-m) + c_{11}^* y_1(k-n) + c_{12}^* y_2(k-n), \tag{70}$$

$$y_2(k+1) = \lambda y_2(k) - \frac{(b_{11}^*)^2}{b_{12}^*} y_1(k-m) - b_{11}^* y_2(k-m) - \frac{(c_{11}^*)^2}{c_{12}^*} y_1(k-n) \\ - c_{11}^* y_2(k-n), \tag{71}$$

where $k \in \mathbb{Z}_0^{\infty}$. It is easy to see (multiplying (71) by b_{12}^*/b_{11}^* and summing both equations) that

$$y_1(k+1) + \frac{b_{12}^*}{b_{11}^*} y_2(k+1) = \lambda\left[y_1(k) + \frac{b_{12}^*}{b_{11}^*} y_2(k)\right], \quad k \in \mathbb{Z}_0^{\infty}. \tag{72}$$

Equation (72) is a homogeneous equation with respect to the unknown expression

$$y_1(k) + \left(b_{12}^*/b_{11}^*\right) y_2(k).$$

Then, using (24), we obtain

$$y_1(k) + \frac{b_{12}^*}{b_{11}^*} y_2(k) = \begin{cases} \varphi_1^*(k) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ \lambda^k[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{73}$$

With the aid of (73), we rewrite system (70), (71) as follows:

$$y_1(k+1) = \begin{cases} \lambda y_1(k) + b^*_{11}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \quad + c^*_{11}[\varphi^*_1(k-n) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-n)] \quad \text{if } k \in \mathbb{Z}^n_0, \\ \lambda y_1(k) + b^*_{11}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \quad + c^*_{11}\lambda^{k-n}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \quad \text{if } k \in \mathbb{Z}^m_{n+1}, \\ \lambda y_1(k) + b^*_{11}\lambda^{k-m}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \\ \quad + c^*_{11}\lambda^{k-n}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \quad \text{if } k \in \mathbb{Z}^\infty_{m+1}, \end{cases} \tag{74}$$

$$y_2(k+1) = \begin{cases} \lambda y_2(k) - \frac{(b^*_{11})^2}{b^*_{12}}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \quad - \frac{(c^*_{11})^2}{c^*_{12}}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \quad \text{if } k \in \mathbb{Z}^n_0, \\ \lambda y_2(k) - \frac{(b^*_{11})^2}{b^*_{12}}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \quad - \frac{(c^*_{11})^2}{c^*_{12}}\lambda^{k-n}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \quad \text{if } k \in \mathbb{Z}^m_{n+1}, \\ \lambda y_2(k) - \frac{(b^*_{11})^2}{b^*_{12}}\lambda^{k-m}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \\ \quad - \frac{(c^*_{11})^2}{c^*_{12}}\lambda^{k-n}[\varphi^*_1(0) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(0)] \quad \text{if } k \in \mathbb{Z}^\infty_{m+1}. \end{cases} \tag{75}$$

First, we solve this system for $k \in \mathbb{Z}^n_0$ and consider the problems

$$\begin{cases} y_1(k+1) = \lambda y_1(k) + b^*_{11}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \qquad\qquad + c^*_{11}[\varphi^*_1(k-n) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-n)], \quad k \in \mathbb{Z}^n_0, \\ y_1(0) = \varphi^*_1(0), \end{cases}$$

$$\begin{cases} y_2(k+1) = \lambda y_2(k) - \frac{(b^*_{11})^2}{b^*_{12}}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)] \\ \qquad\qquad - \frac{(c^*_{11})^2}{c^*_{12}}[\varphi^*_1(k-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k-m)], \quad k \in \mathbb{Z}^n_0, \\ y_2(0) = \varphi^*_2(0). \end{cases}$$

With the aid of formula (24), we get

$$y_1(k) = \lambda^k\varphi^*_1(0) + \sum_{r=0}^{k-1}\lambda^{k-1-r}\left(b^*_{11}\left[\varphi^*_1(r-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(r-m)\right] + c^*_{11}\left[\varphi^*_1(r-n) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(r-n)\right]\right), \quad k \in \mathbb{Z}^{n+1}_1, \tag{76}$$

$$y_2(k) = \lambda^k\varphi^*_2(0) - \sum_{r=0}^{k-1}\lambda^{k-1-r}\left(\frac{(b^*_{11})^2}{b^*_{12}}\left[\varphi^*_1(r-m) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(r-m)\right] + \frac{(c^*_{11})^2}{c^*_{12}}\left[\varphi^*_1(r-n) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(r-n)\right]\right), \quad k \in \mathbb{Z}^{n+1}_1. \tag{77}$$

Now we solve system (74), (75) for $k \in \mathbb{Z}_{n+1}^{m}$, *i.e.*, we consider the problem (with initial data deduced from (76), (77))

$$\begin{cases} y_1(k+1) = \lambda y_1(k) + b_{11}^*[\varphi_1^*(k-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(k-m)] \\ \qquad + c_{11}^*\lambda^{k-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \quad \text{if } k \in \mathbb{Z}_{n+1}^{m}, \\ y_1(n+1) = \lambda^{n+1}\varphi_1^*(0) + \sum_{r=0}^{n}\lambda_1^{n-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \qquad + c_{11}^*[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]), \end{cases}$$

$$\begin{cases} y_2(k+1) = \lambda y_2(k) - \frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(k-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(k-m)] \\ \qquad - \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{k-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \quad \text{if } k \in \mathbb{Z}_{n+1}^{m}, \\ y_2(n+1) = \lambda^{n+1}\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{n-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \qquad + \frac{(c_{11}^*)^2}{c_{12}^*}[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]). \end{cases}$$

Formula (24) yields (for $k \in \mathbb{Z}_{n+2}^{m+1}$)

$$\begin{aligned} y_1(k) &= \lambda^{k-(n+1)}y_1(n+1) + \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right. \\ &\quad \left. + c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right) \\ &= \lambda^{k-n-1}\left[\lambda^{n+1}\varphi_1^*(0) + \sum_{r=0}^{n}\lambda_1^{n-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\right. \\ &\quad \left.\left. + c_{11}^*\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right)\right] \\ &\quad + \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right. \\ &\quad \left. + c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right) \\ &= \lambda^{k}\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right. \\ &\quad \left. + c_{11}^*\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right) \\ &\quad + \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right. \\ &\quad \left. + c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right), \end{aligned} \tag{78}$$

$$\begin{aligned} y_2(k) &= \lambda^{k-(n+1)}y_2(n+1) - \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right. \\ &\quad \left. + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{k-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right) \end{aligned}$$

$$\begin{aligned}
&= \lambda^{k-n-1}\Bigg[\lambda^{n+1}\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{n-r}\bigg(\frac{(b_{11}^*)^2}{b_{12}^*}\bigg[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\bigg] \\
&\quad + \frac{(c_{11}^*)^2}{c_{12}^*}\bigg[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\bigg]\bigg)\Bigg] - \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\bigg(\frac{(b_{11}^*)^2}{b_{12}^*}\bigg[\varphi_1^*(r-m) \\
&\quad + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\bigg] + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\bigg[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\bigg]\bigg) \\
&= \lambda^{k}\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{k-1-r}\bigg(\frac{(b_{11}^*)^2}{b_{12}^*}\bigg[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\bigg] \\
&\quad + \frac{(c_{11}^*)^2}{c_{12}^*}\bigg[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\bigg]\bigg) - \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\bigg(\frac{(b_{11}^*)^2}{b_{12}^*}\bigg[\varphi_1^*(r-m) \\
&\quad + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\bigg] + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\bigg[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\bigg]\bigg).
\end{aligned} \tag{79}$$

Now we solve equations (74), (75) for $k \in \mathbb{Z}_{m+1}^{\infty}$, *i.e.*, we consider the problem (with initial data deduced from (78), (79))

$$\begin{cases}
y_1(k+1) = \lambda y_1(k) + b_{11}^*\lambda^{k-m}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\
\qquad\qquad + c_{11}^*\lambda^{k-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \quad \text{if } k \in \mathbb{Z}_{m+1}^{\infty}, \\
y_1(m+1) = \lambda^{m+1}\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{m-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\
\qquad\qquad + c_{11}^*[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\
\qquad\qquad + \sum_{r=n+1}^{m}\lambda^{m-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\
\qquad\qquad + c_{11}^*\lambda^{r-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)]),
\end{cases}$$

$$\begin{cases}
y_2(k+1) = \lambda y_2(k) - \frac{(b_{11}^*)^2}{b_{12}^*}\lambda^{k-m}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\
\qquad\qquad - \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{k-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \quad \text{if } k \in \mathbb{Z}_{m+1}^{\infty}, \\
y_2(m+1) = \lambda^{m+1}\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{m-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\
\qquad\qquad + \frac{(c_{11}^*)^2}{c_{12}^*}[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\
\qquad\qquad - \sum_{r=n+1}^{m}\lambda^{m-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\
\qquad\qquad + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)]).
\end{cases}$$

Applying formula (24) yields (for $k \in \mathbb{Z}_{m+2}^{\infty}$)

$$\begin{aligned}
y_1(k) &= \lambda^{k-(m+1)}y_1(m+1) + \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\bigg(b_{11}^*\lambda^{r-m}\bigg[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\bigg] \\
&\quad + c_{11}^*\lambda^{r-n}\bigg[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\bigg]\bigg) \\
&= \lambda^{k-m-1}\Bigg[\lambda^{m+1}\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{m-r}\bigg(b_{11}^*\bigg[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ c_{11}^*\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right) + \sum_{r=n+1}^{m}\lambda^{m-r}\left(b_{11}^*\left[\varphi_1^*(r-m)\right.\right.\\
&\left.\left.+ \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right] + c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right)\Bigg]\\
&+ \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^*\lambda^{r-m}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right] + c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right)\\
&= \lambda^k\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\\
&\left.+ c_{11}^*\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right) + \sum_{r=n+1}^{m}\lambda^{k-1-r}\left(b_{11}^*\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\\
&\left.+ c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right) + \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\left(b_{11}^*\lambda^{r-m}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right.\\
&\left.+ c_{11}^*\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right),
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
y_2(k) &= \lambda^{k-(m+1)}y_2(m+1) - \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\lambda^{r-m}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right.\\
&\left.+ \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right)\\
&= \lambda^{k-m-1}\Bigg[\lambda^{m+1}\varphi_1^*(0) - \sum_{r=0}^{n}\lambda^{m-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\\
&\left.+ \frac{(c_{11}^*)^2}{c_{12}^*}\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right) - \sum_{r=n+1}^{m}\lambda^{m-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\left[\varphi_1^*(r-m)\right.\right.\\
&\left.\left.+ \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right] + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right)\Bigg]\\
&- \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\lambda^{r-m}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right] + \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right)\\
&= \lambda^k\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\\
&\left.+ \frac{(c_{11}^*)^2}{c_{12}^*}\left[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)\right]\right)\\
&- \sum_{r=n+1}^{m}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\left[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)\right]\right.\\
&\left.+ \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right) - \sum_{r=m+1}^{k-1}\lambda^{k-1-r}\left(\frac{(b_{11}^*)^2}{b_{12}^*}\lambda^{r-m}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right.\\
&\left.+ \frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{r-n}\left[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)\right]\right).
\end{aligned}
\tag{81}
$$

Summing up all particular cases (76), (78) and (80), we have

$$y_1(k) = \begin{cases} \varphi_1^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^0, \\ \lambda^k\varphi_1^*(0) + \sum_{r=0}^{k-1}\lambda^{k-1-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + c_{11}^*[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \lambda^k\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{k-1-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + c_{11}^*[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad + \sum_{r=n+1}^{k-1}\lambda^{k-1-r}b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + (k-1-n)c_{11}^*\lambda^{k-1-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \lambda^k\varphi_1^*(0) + \sum_{r=0}^{n}\lambda^{k-1-r}(b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + c_{11}^*[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad + \sum_{r=n+1}^{m}\lambda^{k-1-r}b_{11}^*[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + (k-1-m)b_{11}^*\lambda^{k-1-m}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad + (k-1-n)c_{11}^*\lambda^{k-1-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty} \end{cases} \tag{82}$$

and from cases (77), (79) and (81), we conclude

$$y_2(k) = \begin{cases} \varphi_2^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^0, \\ \lambda^k\varphi_2^*(0) - \sum_{r=0}^{k-1}\lambda^{k-1-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + \frac{(c_{11}^*)^2}{c_{12}^*}[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \lambda^k\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{k-1-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + \frac{(c_{11}^*)^2}{c_{12}^*}[\varphi_1^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad - \sum_{r=n+1}^{k-1}\lambda^{k-1-r}\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad - (k-1-n)\frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{k-1-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \lambda^k\varphi_2^*(0) - \sum_{r=0}^{n}\lambda^{k-1-r}(\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad + \frac{(c_{11}^*)^2}{c_{12}^*}[\varphi^*(r-n) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-n)]) \\ \quad - \sum_{r=n+1}^{m}\lambda^{k-1-r}\frac{(b_{11}^*)^2}{b_{12}^*}[\varphi_1^*(r-m) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(r-m)] \\ \quad - (k-1-m)\frac{(b_{11}^*)^2}{b_{12}^*}\lambda^{k-1-m}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad - (k-1-n)\frac{(c_{11}^*)^2}{c_{12}^*}\lambda^{k-1-n}[\varphi_1^*(0) + \frac{b_{12}^*}{b_{11}^*}\varphi_2^*(0)] \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}. \end{cases} \tag{83}$$

Formula (69) is now a direct consequence of (82), (83) and (67), (68). □

The case (29) *of a double real root*
If the matrix Λ has the form (29), the necessary and sufficient conditions (10)-(16) for (30) are reduced to (33)-(36), (39) and

$$\begin{vmatrix} \lambda & 1 \\ b_{21}^* & b_{22}^* \end{vmatrix} + \begin{vmatrix} b_{11}^* & b_{12}^* \\ 0 & \lambda \end{vmatrix} = \lambda\left(b_{11}^* + b_{22}^*\right) - b_{21}^* = 0 \tag{84}$$

and

$$\begin{vmatrix} \lambda & 1 \\ c_{21}^* & c_{22}^* \end{vmatrix} + \begin{vmatrix} c_{11}^* & c_{12}^* \\ 0 & \lambda \end{vmatrix} = \lambda\left(c_{11}^* + c_{22}^*\right) - c_{21}^* = 0. \tag{85}$$

Then (33), (35) and (84) give $b_{11}^* = b_{22}^* = b_{21}^* = 0$, and from (34), (36) and (85), we have $c_{11}^* = c_{22}^* = c_{21}^* = 0$.

Theorem 6 *Let* (1) *be a system with weak delay,* (25) *admit a double root* $\lambda_{1,2} = \lambda$ *and the matrix* Λ *have the form* (29). *Then* $b_{11}^* = b_{22}^* = b_{21}^* = 0$, $c_{11}^* = c_{22}^* = c_{21}^* = 0$ *and the solution of initial problem* (1), (3) *is* $x(k) = \mathcal{S}y(k)$, $y(k) = (y_1(k), y_2(k))^T$ *and*

$$y_1(k) = \begin{cases} \varphi_1^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^0, \\ \lambda^k\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) + b_{12}^* \sum_{r=0}^{k-1} \lambda^{k-1-r}\varphi_2^*(r-m) \\ \quad + c_{12}^* \sum_{r=0}^{k-1} \lambda^{k-1-r}\varphi_2^*(r-n) \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \lambda^k\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) + b_{12}^* \sum_{r=0}^{k-1} \lambda^{k-1-r}\varphi_2^*(r-m) \\ \quad + c_{12}^* \sum_{r=0}^{n} \lambda^{k-1-r}\varphi_2^*(r-n) \\ \quad + (k-1-n)c_{12}^*\lambda^{k-1-n}\varphi_2^*(0) \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \lambda^k\varphi_1^*(0) + k\lambda^{k-1}\varphi_2^*(0) + b_{12}^* \sum_{r=0}^{m} \lambda^{k-1-r}\varphi_2^*(r-m) \\ \quad + c_{12}^* \sum_{r=0}^{n} \lambda_1^{k-1-r}\varphi_2^*(r-n) \\ \quad + (k-1-m)b_{12}^*\lambda^{k-1-m}\varphi_2^*(0) + (k-1-n)c_{12}^*\lambda_1^{k-1-n}\varphi_2^*(0) \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}, \end{cases} \tag{86}$$

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m}^0, \\ \lambda^k\varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{87}$$

Proof System (30) can be written as

$$y_1(k+1) = \lambda y_1(k) + y_2(k) + b_{12}^* y_2(k-m) + c_{12}^* y_2(k-n), \tag{88}$$

$$y_2(k+1) = \lambda y_2(k), \quad k \in \mathbb{Z}_0^{\infty}. \tag{89}$$

Solving (89), we get

$$y_2(k) = \begin{cases} \varphi_2^*(k) & \text{if } k \in \mathbb{Z}_{-m}^0, \\ \lambda^k\varphi_2^*(0) & \text{if } k \in \mathbb{Z}_1^{\infty}. \end{cases} \tag{90}$$

Then (88) turns into

$$y_1(k+1) = \begin{cases} \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \varphi_2^*(k-n) & \text{if } k \in \mathbb{Z}_0^m, \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \lambda^{k-n} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{n+1}^m, \\ \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \lambda^{k-m} \varphi_2^*(0) + c_{12}^* \lambda^{k-n} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{m+1}^\infty. \end{cases} \tag{91}$$

Equation (91) can be solved in a way similar to that of equation (47) in the proof of Theorem 2 using (24).

First, we solve equation (91) for $k \in \mathbb{Z}_0^n$. This means that we consider the problem

$$\begin{cases} y_1(k+1) = \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \varphi_2^*(k-n), & k \in \mathbb{Z}_0^n, \\ y_1(0) = \varphi_1^*(0). \end{cases}$$

With the aid of formula (24), we get

$$\begin{aligned} y_1(k) &= \lambda^k \varphi_1^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} \big[\lambda^r \varphi_2^*(0) + b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n) \big] \\ &= \lambda^k \varphi_1^*(0) + k \lambda^{k-1} \varphi_2^*(0) \\ &\quad + \sum_{r=0}^{k-1} \lambda^{k-1-r} \big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n) \big], \quad k \in \mathbb{Z}_1^{n+1}. \end{aligned} \tag{92}$$

Now we solve equation (91) for $k \in \mathbb{Z}_{n+1}^m$, *i.e.*, we consider the problem (with initial data deduced from (92))

$$\begin{cases} y_1(k+1) = \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \varphi_2^*(k-m) + c_{12}^* \lambda^{k-n} \varphi_2^*(0) & \text{if } k \in \mathbb{Z}_{n+1}^m, \\ y_1(n+1) = \lambda^{n+1} \varphi_1^*(0) + (n+1)\lambda^n \varphi_2^*(0) + \sum_{r=0}^n \lambda^{n-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)]. \end{cases}$$

Applying formula (24) yields (for $k \in \mathbb{Z}_{n+2}^{m+1}$)

$$\begin{aligned} y_1(k) &= \lambda^{k-n-1} y_1^*(n+1) + \sum_{r=n+1}^{k-1} \lambda^{k-1-r} \big[\lambda^r \varphi_2^*(0) + b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda^{r-n} \varphi_2^*(0) \big] \\ &= \lambda^{k-n-1} \left[\lambda^{n+1} \varphi_1^*(0) + (n+1)\lambda^n \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{n-r} \big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n) \big] \right] \\ &\quad + \sum_{r=n+1}^{k-1} \lambda^{k-1-r} \big[\lambda^r \varphi_2^*(0) + b_{12}^* \varphi_2^*(r-m) + c_{12}^* \lambda^{r-n} \varphi_2^*(0) \big] \\ &= \lambda^k \varphi_1^*(0) + k \lambda^{k-1} \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r} \big[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n) \big] \\ &\quad + \sum_{r=n+1}^{k-1} \lambda^{k-1-r} b_{12}^* \varphi_2^*(r-m) + (k-1-n) c_{12}^* \lambda^{k-1-n} \varphi_2^*(0). \end{aligned} \tag{93}$$

Now we solve equation (91) for $k \in \mathbb{Z}_{m+1}^{\infty}$, *i.e.*, we consider the problem (with initial data deduced from (93))

$$\begin{cases} y_1(k+1) = \lambda y_1(k) + \lambda^k \varphi_2^*(0) + b_{12}^* \lambda^{k-m} \varphi_2^*(0) + c_{12}^* \lambda^{k-n} \varphi_2^*(0) \quad \text{if } k \in \mathbb{Z}_{m+1}^{\infty}, \\ y_1(m+1) = \lambda^{m+1} \varphi_1^*(0) + (m+1)\lambda^m \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{m-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)] \\ \qquad\qquad + \sum_{r=n+1}^{m} \lambda^{m-r} b_{12}^* \varphi_2^*(r-m) + (m-n) c_{12}^* \lambda^{m-n} \varphi_2^*(0). \end{cases}$$

Applying formula (24) yields (for $k \in \mathbb{Z}_{m+2}^{\infty}$)

$$\begin{aligned} y_1(k) &= \lambda^{k-m-1} y_1^*(m+1) + \sum_{r=m+1}^{k-1} \lambda^{k-1-r} \bigl[\lambda^r \varphi_2^*(0) + b_{12}^* \lambda^{r-m} \varphi_2^*(0) + c_{12}^* \lambda^{r-n} \varphi_2^*(0)\bigr] \\ &= \lambda^{k-m-1} \Biggl[\lambda^{m+1} \varphi_1^*(0) + (m+1)\lambda^m \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{m-r} \bigl[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\bigr] \\ &\quad + \sum_{r=n+1}^{m} \lambda^{m-r} b_{12}^* \varphi_2^*(r-m) + (m-n) c_{12}^* \lambda^{m-n} \varphi_2^*(0) \Biggr] \\ &\quad + \sum_{r=m+1}^{k-1} \lambda^{k-1-r} \bigl[\lambda^r \varphi_2^*(0) + b_{12}^* \lambda^{r-m} \varphi_2^*(0) + c_{12}^* \lambda^{r-n} \varphi_2^*(0)\bigr] \\ &= \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r} \bigl[b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)\bigr] \\ &\quad + \sum_{r=n+1}^{m} \lambda^{k-1-r} b_{12}^* \varphi_2^*(r-m) + (k-1-m) b_{12}^* \lambda^{k-1-m} \varphi_2^*(0) \\ &\quad + (k-1-n) c_{12}^* \lambda^{k-1-n} \varphi_2^*(0). \end{aligned} \tag{94}$$

Summing up all particular cases (92), (93), (94), we get

$$y_1(k) = \begin{cases} \varphi_1^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^{0}, \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) + \sum_{r=0}^{k-1} \lambda^{k-1-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)] \\ \quad \text{if } k \in \mathbb{Z}_{1}^{n+1}, \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)] \\ \quad + \sum_{r=n+1}^{k-1} \lambda^{k-1-r} b_{12}^* \varphi_2^*(r-m) + (k-1-n) c_{12}^* \lambda^{k-1-n} \varphi_2^*(0) \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \lambda^k \varphi_1^*(0) + k\lambda^{k-1} \varphi_2^*(0) + \sum_{r=0}^{n} \lambda^{k-1-r} [b_{12}^* \varphi_2^*(r-m) + c_{12}^* \varphi_2^*(r-n)] \\ \quad + \sum_{r=n+1}^{m} \lambda^{k-1-r} b_{12}^* \varphi_2^*(r-m) + (k-1-m) b_{12}^* \lambda^{k-1-m} \varphi_2^*(0) \\ \quad + (k-1-n) c_{12}^* \lambda^{k-1-n} \varphi_2^*(0) \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}. \end{cases} \tag{95}$$

Formulas (86) and (87) are consequences of (95), (90). □

Example

The set of 2×2 matrices satisfying conditions (10)-(16) is sufficiently large. Consider, *e.g.*, matrices A, B and C, satisfying conditions (10)-(16), defined as

$$A = \begin{pmatrix} 1 & 1 \\ -1 & \frac{3}{2}\sqrt{2}+1 \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{3}{2}\sqrt{2} & -\frac{3}{2} \\ 3 & -\frac{3}{2}\sqrt{2} \end{pmatrix}, \qquad C = \begin{pmatrix} \sqrt{2} & -1 \\ 2 & -\sqrt{2} \end{pmatrix}$$

and a planar linear discrete systems with weak delays

$$x(k+1) = Ax(k) + Bx(k-m) + Cx(k-n), \tag{96}$$

where $k \in \mathbb{Z}_0^\infty := \{0, 1, \dots, \infty\}$, $x \colon \mathbb{Z}_0^\infty \to \mathbb{R}^2$, $m > n > 0$ are fixed integers, *i.e.*,

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) + \frac{3}{2}\sqrt{2}x_1(k-m) - \frac{3}{2}x_2(k-m) + \sqrt{2}x_1(k-n) - x_2(k-n), \\ x_2(k+1) = -x_1(k) + (\frac{3}{2}\sqrt{2}+1)x_2(k) + 3x_1(k-m) \\ \qquad\qquad - \frac{3}{2}\sqrt{2}x_2(k-m) + 2x_1(k-n) - \sqrt{2}x_2(k-n). \end{cases}$$

Together with (96), we consider the initial problem

$$x(k) = \varphi(k), \tag{97}$$

where $k = -m, -m+1, \dots, 0$ with $\varphi \colon \mathbb{Z}_{-m}^0 \to \mathbb{R}^2$.

Characteristic equation (6) has two real distinct roots $\lambda_1 = 1 + (\sqrt{2})/2$ and $\lambda_2 = \sqrt{2}+1$. Then the Jordan form Λ of the matrix A

$$\Lambda = \begin{pmatrix} 1 + \frac{\sqrt{2}}{2} & 0 \\ 0 & 1+\sqrt{2} \end{pmatrix}.$$

Then

$$S = \begin{pmatrix} 2 & -1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

and the transformation

$$y(k) = S^{-1}x(k) = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 1 & -\sqrt{2} \end{pmatrix} x(k)$$

transforms (96) into the system

$$y(k+1) = \Lambda y(k) + B^{*} y(k-m) + C^{*} y(k-n), \quad k \in \mathbb{Z}_0^\infty \tag{98}$$

with

$$B^{*} = S^{-1}BS = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{3}{2}\sqrt{2} & -\frac{3}{2} \\ 3 & -\frac{3}{2}\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{3}{2}\sqrt{2} & 0 \end{pmatrix}$$

and

$$C^* = S^{-1}CS = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & -1 \\ 2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}.$$

Together with (98), we consider the initial problem

$$y(k) = \varphi^*(k),$$

$k \in \mathbb{Z}_{-m}^0$ with $\varphi^* : \mathbb{Z}_{-m}^0 \to \mathbb{R}^2$, where

$$\varphi^*(k) = S^{-1}\varphi(k) = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 1 & -\sqrt{2} \end{pmatrix} \varphi(k)$$

is the initial function corresponding to the initial function φ in (97).

All the assumptions of Theorem 2, case (II), hold. Therefore by formula (41) we get that the solution of initial problem (96), (97) is

$$x(k) = Sy(k) = \begin{pmatrix} 2 & -1 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} y(k),$$

$k \in \mathbb{Z}_{-m}^\infty$, where $y(k)$ has the form

$$y(k) = \begin{cases} \varphi^*(k) \quad \text{if } k \in \mathbb{Z}_{-m}^0, \\ \left(\begin{smallmatrix} 1+\frac{1}{2}\sqrt{2} & 0 \\ 0 & \sqrt{2}+1 \end{smallmatrix}\right)^k \varphi^*(0) - \frac{3}{2}\sqrt{2}\sum_{r=0}^{k-1}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-m) \\ \quad - \sqrt{2}\sum_{r=0}^{k-1}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-n) \quad \text{if } k \in \mathbb{Z}_1^{n+1}, \\ \left(\begin{smallmatrix} 1+\frac{1}{2}\sqrt{2} & 0 \\ 0 & \sqrt{2}+1 \end{smallmatrix}\right)^k \varphi^*(0) - \frac{3}{2}\sqrt{2}\sum_{r=0}^{k-1}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-m) \\ \quad - \sqrt{2}[\sum_{r=0}^{n}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-n) \\ \quad + \Phi_1(0)\sum_{r=n+1}^{k-1}(1+\frac{1}{2}\sqrt{2})^{r-n}(\sqrt{2}+1)^{k-1-r}] \\ \quad \text{if } k \in \mathbb{Z}_{n+2}^{m+1}, \\ \left(\begin{smallmatrix} 1+\frac{1}{2}\sqrt{2} & 0 \\ 0 & \sqrt{2}+1 \end{smallmatrix}\right)^k \varphi^*(0) - \frac{3}{2}\sqrt{2}[\sum_{r=0}^{m}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-m) \\ \quad + \Phi_1(0)\sum_{r=m+1}^{k-1}(1+\frac{1}{2}\sqrt{2})^{r-m}(\sqrt{2}+1)^{k-1-r}] \\ \quad - \sqrt{2}[\sum_{r=0}^{n}(\sqrt{2}+1)^{k-1-r}\Phi_1(r-n) \\ \quad + \Phi_1(0)\sum_{r=n+1}^{k-1}(1+\frac{1}{2}\sqrt{2})^{r-n}(\sqrt{2}+1)^{k-1-r}] \\ \quad \text{if } k \in \mathbb{Z}_{m+2}^{\infty}, \end{cases}$$

$\Phi_1(k) := (0, \varphi_1^*(k))^T$ and $\Phi_2(k) := (\varphi_2^*(k), 0)^T$ with $k \in \mathbb{Z}_{-m}^0$.

Dimension of the set of solutions

Since all the possible cases of planar system (1) with weak delay have been analyzed, we are ready to formulate results concerning the dimension of the space of solutions of (1) assuming that initial conditions (3) are variable. The below theorems remain valid if coefficients b_{ij}^* are replaced with coefficients c_{ij}^*, $i, j = 1, 2$.

Theorem 7 *Let* (1) *be a system with weak delay and* (25) *have both roots different from zero. Then the space of solutions, being initially* $2(m+1)$*-dimensional, becomes on* $\mathbb{Z}_{m+2}^{\infty}$ *only*

(1) $(m+2)$*-dimensional if equation* (25) *has*
  (a) *two real distinct roots and* $(b_{12}^*)^2 + (b_{21}^*)^2 > 0$;
  (b) *a double real root,* $b_{12}^* b_{21}^* = 0$ *and* $(b_{12}^*)^2 + (b_{21}^*)^2 > 0$;
  (c) *a double real root and* $b_{12}^* b_{21}^* \neq 0$.

(2) 2*-dimensional if equation* (25) *has*
  (a) *two real distinct roots and* $b_{12}^* = b_{21}^* = 0$;
  (b) *a pair of complex conjugate roots*;
  (c) *a double real root and* $b_{12}^* = b_{21}^* = 0$.

Proof We will carefully go through all the theorems considered (Theorems 2-6) adding the case of a pair of complex conjugate roots and our conclusion will hold at least on $\mathbb{Z}_{m+2}^{\infty}$ (some of the statements hold on a larger interval).

(a) Analyzing the statement of Theorem 2 (the case (26) of two real distinct roots), we obtain the following subcases:

(a1) If $b_{11}^* = b_{22}^* = b_{21}^* = 0$, $b_{12}^* \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m+2}^{\infty}$ equals $m+2$ since the last formula in (40) uses only $m+2$ arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(-m), \varphi_2^*(-m+1), \dots, \varphi_2^*(0).$$

(a2) If $b_{11}^* = b_{22}^* = b_{12}^* = 0$, $b_{21}^* \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m+2}^{\infty}$ equals $m+2$ since the last formula in (41) uses only $m+2$ arbitrary parameters

$$\varphi_1^*(-m), \varphi_1^*(-m+1), \dots, \varphi_1^*(0), \varphi_2^*(0).$$

(a3) If $b_{11}^* = b_{22}^* = b_{12}^* = b_{21}^* = 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m+2}^{\infty}$ equals 2 since the last formula in (40) and in (41) uses only 2 arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(0).$$

This means that all the cases considered are covered by conclusions (1)(a) and (2)(a) of Theorem 7.

(b) In the case (27) of two complex conjugate roots, we have $b_{11}^* = b_{22}^* = b_{12}^* = b_{21}^* = 0$ and formula (54) uses only 2 arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(0)$$

for every $k \in \mathbb{Z}_1^{\infty}$. This is covered by case (2)(b) of Theorem 7.

(c) Analyzing the statement of Theorem 4 and Theorem 5 (the case (28) of a double real root), we obtain the following subcases:

(c1) If $b_{21}^* = 0$, $b_{12}^* \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}_{m+2}^{\infty}$ equals $m+2$ since the last formula in (61) uses only $m+2$ arbitrary parameters

$$\varphi_1^*(0), \varphi_2^*(-m), \varphi_2^*(-m+1), \dots, \varphi_2^*(0).$$

(c2) If $b^*_{12} = 0$, $b^*_{21} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}^{\infty}_{m+2}$ equals $m + 2$ since the last formula in (62) uses only $m + 2$ arbitrary parameters

$$\varphi^*_1(-m), \varphi^*_1(-m+1), \dots, \varphi^*_1(0), \varphi^*_2(0).$$

(c3) If $b^*_{12} = b^*_{21} = 0$, then the dimension of the space of solutions on $\mathbb{Z}^{\infty}_{m+2}$ equals 2 since the last formula in (61) and in (62) uses only 2 arbitrary parameters

$$\varphi^*_1(0), \varphi^*_2(0).$$

(c4) If $b^*_{12}b^*_{21} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}^{\infty}_{m+2}$ equals $m + 2$ since the last formula in (69) uses only $m + 2$ arbitrary parameters

$$C(-m), C(-m+1), \dots, C(0), \varphi^*_1(0),$$

where

$$C(k) := \left[\varphi^*_1(k) + \frac{b^*_{12}}{b^*_{11}}\varphi^*_2(k)\right], \quad k \in \mathbb{Z}^0_{-m}.$$

The parameter $\varphi^*_2(0)$ cannot be seen as independent since it depends on the independent parameters $\varphi^*_1(0)$ and $C(0)$.

All the cases considered are covered by conclusions (1)(b), (1)(c) and (2)(c) of Theorem 7.

(d) Analyzing the statement of Theorem 6 (the case (29) of a double real root), we obtain the following subcases:

(d1) If $b^*_{11} = b^*_{22} = b^*_{21} = 0$, $b^*_{12} \neq 0$, then the dimension of the space of solutions on $\mathbb{Z}^{\infty}_{m+2}$ equals $m + 2$ since the last formula in (86) uses only $m + 2$ arbitrary parameters

$$\varphi^*_1(0), \varphi^*_2(-m), \varphi^*_2(-m+1), \dots, \varphi^*_2(0)$$

and the last formula in (87) provides no new information.

(d2) If $b^*_{11} = b^*_{22} = b^*_{21} = b^*_{12} = 0$, then the dimension of the space of solutions on $\mathbb{Z}^{\infty}_{m+2}$ equals 2 since, as follows from (86), (87), there are only 2 arbitrary parameters

$$\varphi^*_1(0), \varphi^*_2(0).$$

Both cases are covered by conclusions (1)(b) and (2)(c) of Theorem 7.

Since there are no cases other than the above cases (a)-(d), the proof is finished. □

Theorem 7 can be formulated simply as follows.

Theorem 8 *Let* (1) *be a system with weak delay and let* (25) *have both roots different from zero. Then the space of solutions, being initially* $2(m+1)$*-dimensional, is on* $\mathbb{Z}^{\infty}_{m+2}$ *only*

(1) $(m+2)$*-dimensional if* $(b^*_{12})^2 + (b^*_{21})^2 > 0$.

(2) 2*-dimensional if* $b^*_{12} = b^*_{21} = 0$.

We omit the proofs of the following two theorems since, again, they are much the same as those of Theorems 2-6.

Theorem 9 *Let* (1) *be a system with weak delay and let* (25) *have a simple root* $\lambda = 0$. *Then the space of solutions, being initially* $2(m+1)$*-dimensional, is either* $(m+1)$*-dimensional or* 1*-dimensional on* $\mathbb{Z}_{m+2}^{\infty}$.

Theorem 10 *Let* (1) *be a system with weak delay and let* (25) *have a double root* $\lambda = 0$. *Then the space of solutions, being initially* $2(m+1)$*-dimensional, turns into a* 0*-dimensional space on* $\mathbb{Z}_1^{\infty}$*, namely, into the zero solution.*

Concluding remarks

To our best knowledge, weak delay was first defined in [10] for systems of linear delayed differential systems with constant coefficients and in [1] for planar linear discrete systems with a single delay. The systems with weak delays analyzed in this paper can be simplified and their solutions can be found in explicit analytical forms. Consequently, analytical forms of solutions can be used directly to solve several problems for systems with weak delays, *e.g.*, problems of asymptotical behavior of their solutions or some problems of control theory (using different methods, such problems have recently been investigated, *e.g.*, in [11–20]). For an alternative approach to differential-difference equations using the variational iteration method, see, *e.g.*, [21].

In the case of discrete systems of two equations investigated in this paper, to obtain the corresponding eigenvalues, it is sufficient to solve only a second-order polynomial equation rather than a polynomial equation of order $2m$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

Acknowledgements
The first author is supported by the Operational Programme Research and Development for Innovations, No. CZ.1.05/2.1.00/03.0097 (AdMaS).

Received: 7 December 2012 Accepted: 14 February 2013 Published: 6 March 2013

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doi:10.1186/1687-1847-2013-50
Cite this article as: Diblík and Halfarová: **Explicit general solution of planar linear discrete systems with constant coefficients and weak delays.** *Advances in Difference Equations* 2013 **2013**:50.