Gao et al. *Advances in Difference Equations* 2013, **2013**:43
http://www.advancesindifferenceequations.com/content/2013/1/43

Advances in Difference Equations
a SpringerOpen Journal

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Stability of the solutions for nonlinear fractional differential equations with delays and integral boundary conditions

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Abstract
In this article, we establish sufficient conditions for the existence, uniqueness and stability of solutions for nonlinear fractional differential equations with delays and integral boundary conditions.
MSC: 34A08; 34A30; 34D20

Keywords: Riemann-Liouvile derivatives; nonlinear fractional differential equation; delay; integral boundary conditions; stability

1 Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non-integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. Fractional differential equations appear naturally in a number of fields such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, control theory, *etc.* An excellent account of the study of fractional differential equations can be found in [1–11] and the references therein. Boundary value problems for fractional differential equations have been discussed in [12–22]. By contrast, the development of stability for solutions of fractional differential equations is a bit slow. El-Sayed, Gaafar and Hamadalla [23] discuss the existence, uniqueness and stability of solutions for the non-local non-autonomous system of fractional order differential equations with delays

$$D^{\alpha}x_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t-r_j) + h_i(t), \quad t > 0,$$

where D^{α} denotes the Riemann-Liouville derivative of order α.

We consider nonlinear fractional differential equations with delay and integral boundary conditions of the form

$$D^{\alpha}x(t) = \sum_{j=1}^{n} a_j(t)f\bigl(t, x(t), x(t-\tau_j)\bigr), \quad t > 0, \tag{1.1}$$

$$x(t) = \phi(t) \quad \text{for } t < 0 \quad \text{and} \quad \lim_{t\to 0^-} \phi(t) = 0, \tag{1.2}$$

$$I^{1-\alpha}x(t)|_{t=0} = 0, \tag{1.3}$$

where $\alpha \in (0,1)$, $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $a_j(t)$, $\phi(t)$ are given continuous functions, $\tau_j \geq 0$, $j = 1, 2, \dots, n$ are constants.

In this article our aim is to show the existence of a unique solution for (1.1)-(1.3) and its uniform stability.

2 Preliminaries

In this section, we introduce notation, definitions and preliminary facts which are used throughout this paper.

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right-hand side exists pointwise on $\mathbb{R}^+$. Γ is the gamma function.

For instance, $I^\alpha f$ exists for all $\alpha > 0$ when $f \in C^0(\mathbb{R}^+) \cap L^1_{\mathrm{loc}}(\mathbb{R}^+)$; note also that when $f \in C^0(\mathbb{R}_0^+)$, then $I^\alpha f \in C^0(\mathbb{R}_0^+)$ and moreover $I^\alpha f(0) = 0$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds.$$

Definition 2.3 Let $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and satisfy the Lipschitz conditions

$$\left| f(t, x, y_j) - f(t, u, v_j) \right| \leq k|x-u| + k_j|y_j - v_j|, \quad k > 0, k_j > 0, j = 1, 2, \dots, n$$

for all $x, y_j, u, v_j \in \mathbb{R}$.

3 Existence of a unique solution for nonlinear fractional differential equations (1.1)-(1.3)

Let X be the class of all continuous functions defined on $\mathbb{R}^+$ with the norm

$$\|x\| = \sup_{t \in \mathbb{R}^+} \left\{ e^{-Nt} \left| x(t) \right| \right\}, \quad x \in X.$$

Theorem 3.1 *Let $f : \mathbb{R}^\times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition: if*

$$\frac{\sum_{j=1}^n a_j (k + k_j e^{-N\tau_j})}{N^\alpha} < 1,$$

where $a_j = \max_{t \in \mathbb{R}^+} \{|a_j(t)|\}$, then nonlinear fractional differential equations (1.1)-(1.3) *have a unique positive solution.*

Proof For $t > 0$, equation (1.1) can be written as

$$\frac{d}{dt}I^{1-\alpha}x(t) = \sum_{j=1}^{n} a_j(t)f\big(t, x(t), x(t-\tau_j)\big).$$

Integrating both sides of the above equation, we obtain

$$I^{1-\alpha}x(t) - I^{1-\alpha}x(t)|_{t=0} = \sum_{j=1}^{n}\int_0^t a_j(s)f\big(s, x(s), x(s-\tau_j)\big)\,ds$$

then

$$I^{1-\alpha}x_i(t) = \sum_{j=1}^{n}\int_0^t a_j(s)f\big(s, x(s), x(s-\tau_j)\big)\,ds.$$

Applying the operator by I^{α} on both sides,

$$Ix(t) = \sum_{j=1}^{n} I^{\alpha+1}a_j(t)f\big(t, x(t), x(t-\tau_j)\big)$$

differentiating both sides, we obtain

$$x(t) = \sum_{j=1}^{n} I^{\alpha}a_j(t)f\big(t, x(t), x(t-\tau_j)\big). \tag{3.1}$$

Now, let $F : X \to X$ be defined by

$$Fx = \sum_{j=1}^{n} I^{\alpha}a_j(t)f\big(t, x(t), x(t-\tau_j)\big).$$

Then

$$\begin{aligned}
\big|Fx(t) - Fy(t)\big| &= \left|\sum_{j=1}^{n} I^{\alpha}a_j(t)f\big(t, x(t), x(t-\tau_j)\big) - \sum_{j=1}^{n} I^{\alpha}a_j(t)f\big(t, y(t), y(t-\tau_j)\big)\right| \\
&= \left|\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big\{a_j(s)f\big(s, x(s), x(s-\tau_j)\big) - a_j(s)f\big(s, y(s), y(s-\tau_j)\big)\big\}\,ds\right| \\
&\le \sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|a_j(s)f\big(s, x(s), x(s-\tau_j)\big) - a_j(s)f\big(s, y(s), y(s-\tau_j)\big)\big|\,ds \\
&\le \sum_{j=1}^{n} a_j\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big\{k\big|x(s) - y(s)\big| + k_j\big|x(s-\tau_j) - y(s-\tau_j)\big|\big\}\,ds \\
&\le \sum_{j=1}^{n} a_j k\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|x(s) - y(s)\big|\,ds
\end{aligned}$$

$$+\sum_{j=1}^{n} a_j k_j \int_0^{\tau_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x(s-\tau_j)-y(s-\tau_j)\bigr|\,ds$$
$$+\sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x(s-\tau_j)-y(s-\tau_j)\bigr|\,ds.$$

By conditions (1.2), we have

$$\begin{aligned}\bigl|Fx(t)-Fy(t)\bigr| &\le \sum_{j=1}^{n} a_j k \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x(s)-y(s)\bigr|\,ds \\ &\quad + \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|x(s-\tau_j)-y(s-\tau_j)\bigr|\,ds\end{aligned}$$

and

$$\begin{aligned}
&e^{-Nt}\bigl|Fx(t)-Fy(t)\bigr| \\
&\quad\le \sum_{j=1}^{n} a_j k \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns} \bigl|x(s)-y(s)\bigr|\,ds \\
&\qquad + \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)} \bigl|x(s-\tau_j)-y(s-\tau_j)\bigr|\,ds \\
&\quad\le \sum_{j=1}^{n} a_j k \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)}\,ds \\
&\qquad + \sum_{j=1}^{n} a_j k_j \int_0^{t-\tau_j} \frac{(t-\theta-\tau_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)} e^{-N\theta} \bigl|x(\theta)-y(\theta)\bigr|\,d\theta \\
&\quad\le \sum_{j=1}^{n} a_j k \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \int_0^{Nt} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\,du \\
&\qquad + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \int_0^{t-\tau} \frac{(t-\theta-\tau_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)}\,d\theta \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^{\alpha}}\|x-y\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \int_0^{t-\tau_j} \frac{u^{\alpha-1}}{\Gamma(\alpha)} e^{-Nu} e^{-N\tau_j}\,du \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^{\alpha}}\|x-y\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \frac{e^{-N\tau_j}}{N^{\alpha}} \int_0^{N(t-\tau_j)} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\,du \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^{\alpha}}\|x-y\| + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\bigl\{e^{-Nt}\bigl|x(t)-y(t)\bigr|\bigr\} \frac{e^{-N\tau_j}}{N^{\alpha}} \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^{\alpha}}\|x-y\| + \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^{\alpha}} \sup_{t\in\mathbb{R}^+} e^{-Nt}\bigl|x(t)-y(t)\bigr| \\
&\quad\le \frac{\sum_{j=1}^{n} a_j (k+k_j e^{-N\tau_j})}{N^{\alpha}}\|x-y\|.
\end{aligned}$$

Now, choose N large enough such that $\frac{\sum_{j=1}^{n} a_j(k+k_j e^{-N\tau_j})}{N^\alpha} < 1$. So, the map $F : X \to X$ is a contraction and it has a fixed point $x = Fx$, and hence there exists a unique $x \in X$ which is a solution of integral equation (3.1).

We now prove the equivalence between integral equation (3.1) and nonlinear fractional differential equations (1.1)-(1.3). Indeed, since $x \in X$ and $I^{1-\alpha}x(t) \in C(X)$, applying the operator $I^{1-\alpha}$ on both sides of (3.1), we obtain

$$\begin{aligned} I^{1-\alpha}x(t) &= \sum_{j=1}^{n} I^{1-\alpha} I^\alpha a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr) \\ &= \sum_{j=1}^{n} I a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr). \end{aligned}$$

Differentiating both sides,

$$DI^{1-\alpha}x(t) = \sum_{j=1}^{n} DIa_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr),$$

we get

$$D^\alpha x(t) = \sum_{j=1}^{n} a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr), \quad t > 0,$$

which proves the equivalence of (3.1) and (1.1). We want to prove that $\lim_{t\to 0^+} x = 0$. Since $a_j(t)f(t,x(t),x(t-\tau_j))$ are continuous on $[0,T]$, there exist constants m, M such that $m \le a_j(t)f(t,x(t),x(t-\tau_j)) \le M$. We have

$$I^\alpha a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a_j(s) f\bigl(s, x(s), x(s-\tau_j)\bigr)\, ds,$$

which implies

$$\begin{aligned} m \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds &\le I^\alpha f\bigl(t, x(t), x(t-\tau_j)\bigr) \\ &\le M \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds, \\ nm \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds &\le \sum_{j=1}^{n} I^\alpha a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr) \\ &\le nM \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds, \end{aligned}$$

which in turn implies

$$nm \frac{t^\alpha}{\Gamma(\alpha+1)} \le \sum_{j=1}^{n} I^\alpha a_j(t) f\bigl(t, x(t), x(t-\tau_j)\bigr) \le nM \frac{t^\alpha}{\Gamma(\alpha+1)}$$

and

$$\lim_{t\to 0^+}\sum_{j=1}^{n} I^{\alpha} a_j(t) f\big(t, x(t), x(t-\tau_j)\big) = 0.$$

Then from (3.1) $\lim_{t\to 0^+} x(t) = 0$ and from (1.2), we have $\lim_{t\to 0^+}\phi(t) = 0$. □

Now, for $t \in (-\infty, T]$, $T < \infty$, the solution of nonlinear fractional differential equations (1.1)-(1.3) takes the form

$$x(t) = \begin{cases} \phi(t), & t < 0, \\ 0, & t = 0, \\ \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a_j(s) f(s, x(s), x(s-\tau_j))\, ds, & t > 0. \end{cases}$$

4 Stability of a unique solution for nonlinear fractional differential equations (1.1)-(1.3)

In this section, we study the stability of the solution of nonlinear fractional differential equations (1.1)-(1.3).

The $\widetilde{x}(t)$ is a solution of the nonlinear fractional differential equations

$$(\widetilde{P}) \quad \begin{cases} D^{\alpha} x(t) = \sum_{j=1}^{n} a_j(t) f(t, x(t), x(t-\tau_j)), & t > 0, \\ x(t) = \widetilde{\phi}(t) \quad \text{for } t < 0 \quad \text{and} \quad \lim_{t\to 0^-}\widetilde{\phi}(t) = 0, \\ I^{1-\alpha}\widetilde{x}(t)|_{t=0} = 0. \end{cases}$$

Definition 4.1 The solution of nonlinear fractional differential equation (1.1) is stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any two solutions $x(t)$ and $\widetilde{x}(t)$ of nonlinear fractional differential equations (1.1)-(1.3) and $\widetilde{P}$ respectively, one has $\|\phi(t) - \widetilde{\phi}(t)\| \leq \delta$, then $\|x(t) - \widetilde{x}(t)\| < \epsilon$ for all $t \geq 0$.

Theorem 4.2 *The solution of nonlinear fractional differential equations* (1.1)-(1.3) *is uniformly stable.*

Proof Let $x(t)$ and $\widetilde{x}(t)$ be the solutions of nonlinear fractional differential equations (1.1)-(1.3) and $\widetilde{P}$ respectively, then for $t > 0$, from (3.1), we have

$$\begin{aligned} \big|x(t) - \widetilde{x}(t)\big| &= \left|\sum_{j=1}^{n} I^{\alpha} a_j(t) f\big(t, x(t), x(t-\tau_j)\big) - \sum_{j=1}^{n} I^{\alpha} a_j(t) f\big(t, \widetilde{x}(t), \widetilde{x}(t-\tau_j)\big)\right| \\ &\leq \sum_{j=1}^{n} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|a_j(s) f\big(s, x(s), x(s-\tau_j)\big) - a_j(s) f\big(s, \widetilde{x}(s), \widetilde{x}(s-\tau_j)\big)\big|\, ds \\ &\leq \sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|x(s) - \widetilde{x}(s)\big|\, ds \\ &\quad + \sum_{j=1}^{n} a_j k_j \int_0^{\tau_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|\phi(s-\tau_j) - \widetilde{\phi}(s-\tau_j)\big|\, ds \\ &\quad + \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \big|x(s-\tau_j) - \widetilde{x}(s-\tau_j)\big|\, ds \end{aligned}$$

and

$$
\begin{aligned}
&e^{-Nt}\left|x(t)-\widetilde{x}(t)\right| \\
&\quad\le \sum_{j=1}^{n} a_j k \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} e^{-Ns}\left|x(s)-\widetilde{x}(s)\right| ds \\
&\qquad + \sum_{j=1}^{n} a_j k_j \int_0^{\tau_j} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)}\left|\phi(s-\tau_j)-\widetilde{\phi}(s-\tau_j)\right| ds \\
&\qquad + \sum_{j=1}^{n} a_j k_j \int_{\tau_j}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s+\tau_j)} e^{-N(s-\tau_j)}\left|x(s-\tau_j)-\widetilde{x}(s-\tau_j)\right| ds \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^\alpha}\left\|x(t)-\widetilde{x}(t)\right\| \int_0^{Nt} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\, du \\
&\qquad + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}(t)\right|\right\} \int_{-\tau_j}^{0} \frac{(t-\theta-\tau_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)}\, d\theta \\
&\qquad + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \int_{0}^{t-\tau_j} \frac{(t-\theta-\tau_j)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-\theta)}\, d\theta \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^\alpha}\left\|x(t)-\widetilde{x}(t)\right\| \\
&\qquad + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \int_{N(t-\tau_j)}^{Nt} \frac{u^{\alpha-1}e^{-Nu}}{\Gamma(\alpha)}\, du \\
&\qquad + \sum_{j=1}^{n} a_j k_j \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \frac{e^{-N\tau_j}}{N^\alpha} \int_{0}^{N(t-\tau)} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)}\, du \\
&\quad\le \frac{\sum_{j=1}^{n} a_j k}{N^\alpha}\left\|x(t)-\widetilde{x}(t)\right\| + \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha} \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|x(t)-\widetilde{x}(t)\right|\right\} \\
&\qquad + \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha} \sup_{t\in\mathbb{R}^+}\left\{e^{-Nt}\left|\phi(t)-\widetilde{\phi}j(t)\right|\right\} \\
&\quad\le \frac{\sum_{j=1}^{n} a_j (k+k_j e^{-N\tau_j})}{N^\alpha}\left\|x(t)-\widetilde{x}(t)\right\| + \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha}\left\|\phi(t)-\widetilde{\phi}(t)\right\|.
\end{aligned}
$$

Then

$$
\left[1-\frac{\sum_{j=1}^{n} a_j (k+k_j e^{-N\tau_j})}{N^\alpha}\right]\left\|x(t)-\widetilde{x}(t)\right\| \le \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha}\left\|\phi(t)-\widetilde{\phi}(t)\right\|
$$

and

$$
\left\|x(t)-\widetilde{x}(t)\right\| \le \frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha}\left[1-\frac{\sum_{j=1}^{n} a_j (k+k_j e^{-N\tau_j})}{N^\alpha}\right]^{-1}\left\|\phi(t)-\widetilde{\phi}(t)\right\|;
$$

therefore, for $\epsilon > 0$, we can find $\delta = (\frac{\sum_{j=1}^{n} a_j k_j e^{-N\tau_j}}{N^\alpha})^{-1}[1-\frac{\sum_{j=1}^{n} a_j (k+k_j e^{-N\tau_j})}{N^\alpha}]\epsilon$ such that $\|\phi(t)-\widetilde{\phi}(t)\| < \delta$. Then $\|x(t)-\widetilde{x}(t)\| \le \epsilon$, which proves that the solution $x(t)$ is uniformly stable.

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZG carried out the stability of the solutions for nonlinear fractional differential equations studies and drafted the manuscript. LY and ZL carried out the stability of the solutions for nonlinear fractional differential equations studies. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by the Natural Science Foundation of Hunan Province (13JJ6068, 12JJ9001), Hunan Provincial Science and Technology Department of Science and Technology Project (2012SK3117) and Construct program of the key discipline in Hunan Province.

Received: 7 July 2012 Accepted: 31 January 2013 Published: 25 February 2013

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doi:10.1186/1687-1847-2013-43
Cite this article as: Gao et al.: **Stability of the solutions for nonlinear fractional differential equations with delays and integral boundary conditions.** *Advances in Difference Equations* 2013 **2013**:43.