Advances in Difference Equations
a SpringerOpen Journal

RESEARCH ARTICLE **Open Access**

A new class of fractional boundary value problems

Bashir Ahmad[1*], Ahmed Alsaedi[1], Afrah Assolami[1] and Ravi P Agarwal[1,2]

*Correspondence:
bashirahmad_qau@yahoo.com
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this paper, a fractional boundary value problem with a new boundary condition is studied. This new boundary condition relates the nonlocal value of the unknown function at ξ with its influence due to a sub-strip $(\eta, 1)$, where $0 < \xi < \eta < 1$. The main results are obtained with the aid of some classical fixed point theorems and Leray-Schauder nonlinear alternative. A demonstration of applications of these results is also given.

MSC: 34A12; 34A40

Keywords: fractional differential equations; nonlocal boundary conditions; fixed point theorems

1 Introduction

We study a boundary value problem of Caputo-type fractional differential equations with new boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 1 < q \le 2, t \in [0,1], \\ x(0) = 0, \qquad x(\xi) = a \int_\eta^1 x(s)\,ds, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and a is a positive real constant.

In (1.1), the second condition may be interpreted as a more general variant of nonlocal integral boundary conditions, which states that the integral contribution due to a sub-strip $(\eta, 1)$ for the unknown function is proportional to the value of the unknown function at a nonlocal point $\xi \in (0,1)$ with $\xi < \eta < 1$. We emphasize that most of the work concerning nonlocal boundary value problems relates the contribution expressed in terms of the integral to the value of the unknown function at a fixed point (left/right end-point of the interval under consideration), for instance, see [1–3] and references therein.

The recent development in the theory, methods and applications of fractional calculus has contributed towards the popularity and importance of the subject. The tools of fractional calculus have been effectively applied in the modeling of many physical and engineering phenomena. Examples include physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, percolation, identification, fitting of experimental data, economics, *etc.* [4–6]. For some recent work on the topic, we refer to [7–22] and the references therein.

2 Preliminaries

Let us recall some basic definitions on fractional calculus.

Definition 2.1 The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.2 For at least nth continuously differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.1 *For any $y \in C[0,1]$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), & 1 < q \le 2, t \in [0,1], \\ x(0) = 0, & x(\xi) = a \int_\eta^1 x(s)\, ds \end{cases} \tag{2.1}$$

is

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds + \frac{t}{A}\left\{ a \int_\eta^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\, du \right) ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} y(s)\, ds \right\}, \tag{2.2}$$

where

$$A = \xi - \frac{a}{2}\left(1-\eta^2\right) \neq 0. \tag{2.3}$$

Proof It is well known that the general solution of the fractional differential equation in (2.1) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds, \tag{2.4}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the given boundary conditions, we find that $c_0 = 0$, and

$$c_1 = \frac{1}{A}\left\{ a \int_\eta^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\, du \right) ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} y(s)\, ds \right\}.$$

Substituting the values of c_0, c_1 in (2.4), we get (2.2). This completes the proof. □

3 Existence results

Let $\mathcal{P} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In relation to the given problem, we define an operator $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ as

$$\begin{aligned}(\mathcal{T}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds \\ &\quad + \frac{t}{A}\left\{a\int_\eta^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\big(u,x(u)\big)\,du\right) ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f\big(s,x(s)\big)\,ds\right\},\end{aligned}$$

where A is given by (2.3). Observe that problem (1.1) has solutions if and only if the operator $\mathcal{T}$ has fixed points.

For the sake of computational convenience, we set

$$\omega = \frac{1}{|A|}\left\{\frac{1}{\Gamma(q+1)}\big(|A| + \xi^q\big) + \frac{a}{\Gamma(q+2)}\big(1-\eta^{q+1}\big)\right\}. \tag{3.1}$$

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the Lipschitz condition*

(H_1) $|f(t,x) - f(t,y)| \le L|x-y|$, $\forall t \in [0,1]$, $x,y \in \mathbb{R}$, $L > 0$.

Then problem (1.1) *has a unique solution if $L\omega < 1$, where ω is given by* (3.1).

Proof Let us denote $\sup_{t\in[0,1]} |f(t,0)| = \gamma$ and show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{x \in \mathcal{P} : \|x\| \le r\}$ with $r > \omega\gamma(1-\omega L)^{-1}$. For $x \in B_r$, $t \in [0,1]$, we have

$$\begin{aligned}\big\|(\mathcal{T}x)\big\| &\le \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f(s,0) + f(s,0)\big|\,ds\right. \\ &\quad + \frac{t}{|A|}\left[a\int_\eta^1\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}\big|f\big(u,x(u)\big) - f(u,0) + f(u,0)\big|\,du\right) ds\right. \\ &\quad \left.\left. + \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f(s,0) + f(s,0)\big|\,ds\right]\right\} \\ &\le (Lr+M)\sup_{t\in[0,1]}\left\{\frac{t^q}{\Gamma(q+1)} + \frac{t}{|A|}\left[\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)}\right]\right\} \\ &\le (Lr+M)\frac{1}{|A|}\left\{\frac{1}{\Gamma(q+1)}\big(|A| + \xi^q\big) + \frac{a}{\Gamma(q+2)}\big(1-\eta^{q+1}\big)\right\} \\ &= (Lr+M)\omega \le r,\end{aligned}$$

which implies that $\mathcal{T}B_r \subset B_r$.

Now, for $x, y \in \mathcal{P}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\| &\le \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds\right. \\ &\quad + \frac{t}{|A|}\left(a\int_\eta^1\left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)}\big|f\big(u,x(u)\big) - f\big(u,y(u)\big)\big|\,du\right) ds\right. \\ &\quad \left.\left. + \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)}\big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|\,ds\right)\right\} \le L\omega|x-y|.\end{aligned}$$

Since $L < 1/\omega$, by the given assumption, therefore the operator $\mathcal{T}$ is a contraction. Thus, by Banach's contraction mapping principle, there exists a unique solution for problem (1.1). This completes the proof. □

The next result is based on Krasnoselskii's fixed point theorem [23].

Theorem 3.2 *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (H_1) *and*

(H_2) $|f(t,x)| \le \mu(t)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$, *and* $\mu \in ([0,1], \mathbb{R}^+)$.

Then problem (1.1) *has at least one solution on* $[0,1]$ *if*

$$\frac{L}{|A|}\left\{\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)}\right\} < 1.$$

Proof Fixing $r \ge \|\mu\|\omega$, we consider $B_r = \{x \in \mathcal{P} : \|x\| \le r\}$ and define the operators Φ and Ψ on B_r as

$$\begin{aligned}
(\Phi x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds, \\
(\Psi x)(t) &= \frac{t}{A}\left\{a\int_\eta^1 \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du\right) ds \right. \\
&\quad \left. - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds\right\}.
\end{aligned}$$

For $x, y \in B_r$, it is easy to show that $\|(\Phi x) + (\Psi y)\| \le \|\mu\|\omega \le r$, which implies that $\Phi x + \Psi y \in B_r$.

In view of the assumption $L\{\frac{1}{|A|}(\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)})\} < 1$, the operator Ψ is a contraction. The continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as $\|\Phi x\| \le \|\mu\|/\Gamma(q+1)$. Moreover, Φ is relatively compact on B_r as

$$\bigl\|(\Phi x)(t_2) - (\Phi x)(t_1)\bigr\| \le \frac{f_m}{\Gamma(q+1)}\bigl(\bigl|t_2{}^q - t_1{}^q\bigr| + 2|t_2 - t_1|^q\bigr),$$

where $\sup_{(t,x)\in[0,1]\times B_r} |f(t,x)| = f_m < \infty$. Hence, by the Arzelá-Ascoli theorem, Φ is compact on B_r. Thus all the assumptions of Krasnoselskii's fixed point theorem are satisfied. So problem (1.1) has at least one solution on $[0,1]$. This completes the proof. □

Our next result is based on the following fixed point theorem [23].

Theorem 3.3 *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \epsilon Tu, 0 < \epsilon < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.4 *Assume that there exists a positive constant L_1 such that $|f(t,x)| \le L_1$ for all $t \in [0,1]$, $x \in \mathcal{P}$. Then there exists at least one solution for problem* (1.1).

Proof As a first step, we show that the operator $\mathcal{T}$ is completely continuous. Clearly, the continuity of $\mathcal{T}$ follows from the continuity of f. Let $D \subset \mathcal{P}$ be bounded. Then, $\forall x \in D$, it is easy to establish that $|(\mathcal{T}x)(t)| \leq L_1\omega = L_2$. Furthermore, we find that

$$\begin{aligned}
\bigl|(\mathcal{T}x)'(t)\bigr| &\leq \biggl|\int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \frac{1}{A}\biggl\{a\int_\eta^1\biggl(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du\biggr)\,ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds\biggr\}\biggr| \\
&\leq L_1\biggl[\frac{t^{q-1}}{\Gamma(q)} + \frac{1}{|A|}\biggl\{\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)}\biggr\}\biggr] \\
&\leq L_1\biggl[\frac{1}{\Gamma(q)} + \frac{1}{|A|}\biggl\{\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)}\biggr\}\biggr] \\
&= L_3.
\end{aligned}$$

Hence, for $t_1, t_2 \in [0,1]$, it follows that

$$\bigl|(\mathcal{T}x)(t_1) - (\mathcal{T}x)(t_2)\bigr| \leq \int_{t_1}^{t_2} \bigl|(\mathcal{T}x)'(s)\bigr|\,ds \leq L_3(t_2 - t_1).$$

Therefore, $\mathcal{T}$ is equicontinuous on $[0,1]$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}$ is completely continuous.

Next, we consider the set $V = \{x \in \mathcal{P} : x = \epsilon\mathcal{T}x, 0 < \epsilon < 1\}$. To show that V is bounded, let $x \in V$, $t \in [0,1]$. Then

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds \\
&\quad + \frac{t}{A}\biggl\{a\int_\eta^1\biggl(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du\biggr)\,ds - \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f\bigl(s,x(s)\bigr)\,ds\biggr\}
\end{aligned}$$

and $|x(t)| = \epsilon|(\mathcal{T}x)(t)| \leq L_1\omega = L_2$. Hence, $\|x\| \leq L_2$, $\forall t \in [0,1]$. So V is bounded. Thus, Theorem 3.3 applies and, in consequence, problem (1.1) has at least one solution. This completes the proof. □

Our final result is based on Leray-Schauder nonlinear alternative.

Lemma 3.1 (Nonlinear alternative for single-valued maps [24]) *Let E be a Banach space, E_1 be a closed, convex subset of E, V be an open subset of E_1 and $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \to E_1$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset of E_1) map. Then either*

(i) *$\mathcal{U}$ has a fixed point in $\overline{V}$, or*
(ii) *there are $x \in \partial V$ (the boundary of V in E_1) and $\kappa \in (0,1)$ with $x = \kappa\mathcal{U}(x)$.*

Theorem 3.5 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_3) *there exist a function $p \in \mathcal{C}([0,1], \mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,x)| \leq p(t)\psi(\|x\|)$, $\forall (t,x) \in [0,1] \times \mathbb{R}$;*
(H_4) *there exists a constant $M > 0$ such that*

$$M\biggl[\frac{\psi(M)\|p\|}{|A|}\biggl\{\frac{1}{\Gamma(q+1)}\bigl(|A| + \xi^q\bigr) + \frac{a}{\Gamma(q+2)}\bigl(1 - \eta^{q+1}\bigr)\biggr\}\biggr]^{-1} > 1.$$

Then problem (1.1) has at least one solution on $[0,1]$.

Proof Let us consider the operator $\mathcal{T}:\mathcal{P}\to\mathcal{P}$ defined by (3.1) and show that $\mathcal{T}$ maps bounded sets into bounded sets in $\mathcal{P}$. For a positive number r, let $B_r=\{x\in\mathcal{P}:\|x\|\le r\}$ be a bounded set in $\mathcal{P}$. Then, for $x\in B_r$ together with (H_3), we obtain

$$\begin{aligned}\big|(\mathcal{T}x)(t)\big| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}p(s)\psi\big(\|x\|\big)\,ds+\frac{t}{|A|}\left\{a\int_\eta^1\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}p(u)\psi\big(\|x\|\big)\,du\right)ds\right.\\&\quad\left.+\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}p(s)\psi\big(\|x\|\big)\,ds\right\}\\&\le\|p\|\psi\big(\|x\|\big)\left[\frac{1}{|A|}\left\{\frac{1}{\Gamma(q+1)}\big(|A|+\xi^q\big)+\frac{a(1-\eta^{q+1})}{\Gamma(q+2)}\right\}\right]\\&\le\|p\|\psi(r)\omega.\end{aligned}$$

Next, it will be shown that $\mathcal{T}$ maps bounded sets into equicontinuous sets of $\mathcal{P}$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $x\in B_r$. Then

$$\begin{aligned}\big|(\mathcal{T}x)(t_2)-(\mathcal{T}x)(t_1)\big| &\le\left|\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}p(s)\psi(r)\,ds-\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}p(s)\psi(r)\,ds\right.\\&\quad+\frac{(t_2-t_1)}{|A|}\left\{a\int_\eta^1\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}p(u)\psi(r)\,du\right)ds\right.\\&\quad\left.\left.-\int_0^\xi\frac{(\xi-s)^{q-1}}{\Gamma(q)}p(s)\psi(r)\,ds\right\}\right|\\&\le\psi(r)\|p\|\left[\frac{|t_2^q-t_1^q|}{\Gamma(q+1)}+\frac{|t_2-t_1|}{|A|}\left\{\frac{a(1-\eta^{q+1})}{\Gamma(q+2)}+\frac{\xi^q}{\Gamma(q+1)}\right\}\right].\end{aligned}$$

Clearly, the right-hand side tends to zero independently of $x\in B_r$ as $t_2\to t_1$. Thus, by the Arzelá-Ascoli theorem, the operator $\mathcal{T}$ is completely continuous.

Let x be a solution for the given problem. Then, for $\lambda\in(0,1)$, following the method of computation used in proving that $\mathcal{T}$ is bounded, we have

$$\big\|x(t)\big\|=\big\|\lambda(\mathcal{T}x)(t)\big\|\le\left[\frac{\psi(\|x\|)\|p\|}{|A|}\left\{\frac{1}{\Gamma(q+1)}\big(|A|+\xi^q\big)+\frac{a}{\Gamma(q+2)}\big(1-\eta^{q+1}\big)\right\}\right],$$

which implies that

$$\|x\|\left[\frac{\psi(\|x\|)\|p\|}{|A|}\left\{\frac{1}{\Gamma(q+1)}\big(|A|+\xi^q\big)+\frac{a}{\Gamma(q+2)}\big(1-\eta^{q+1}\big)\right\}\right]^{-1}\le 1.$$

In view of (H_4), there exists M such that $\|x\|\ne M$. Let us choose $\mathcal{N}=\{x\in\mathcal{P}:\|x\|<M+1\}$.

Observe that the operator $\mathcal{T}:\overline{\mathcal{N}}\to\mathcal{P}$ is continuous and completely continuous. From the choice of $\mathcal{N}$, there is no $x\in\partial\mathcal{N}$ such that $x=\lambda\mathcal{T}(x)$ for some $\lambda\in(0,1)$. Consequently, by Lemma 3.1, we deduce that the operator $\mathcal{T}$ has a fixed point $x\in\overline{\mathcal{N}}$ which is a solution of problem (1.1). This completes the proof. □

4 Examples

Example 4.1 Consider a fractional boundary value problem given by

$$\begin{cases}{}^cD^{\frac{3}{2}}x(t)=\frac{1}{(t^2+2)^2}\left(\frac{|x|}{1+|x|}\right)-\sin^2 t, & t\in[0,1],\\ x(0)=0, \quad x(1/4)=2\int_{1/2}^1 x(s)\,ds.\end{cases}\tag{4.1}$$

Here, $q = 3/2$, $a = 2$, $\eta = 1/2$, $\xi = 1/4$ and $f(t,x) = \frac{1}{(t^2+2)^2}(\frac{|x|}{1+|x|}) - \sin^2 t$. With the given values, $\omega \simeq 1.931151$ and $L = 1/4$ as $|f(t,x) - f(t,y)| \leq 1/4|x - y|$. Clearly, $L\omega \simeq 0.482788 < 1$. Therefore, by Theorem 3.1, there exists a unique solution for problem (4.1).

Example 4.2 Consider a fractional boundary value problem given by

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{1}{\pi\sqrt{1+t}}(\tan^{-1}x + \frac{\pi}{2}), & t \in [0,1], \\ x(0) = 0, \qquad x(1/2) = \int_{3/4}^1 x(s)\,ds. \end{cases} \tag{4.2}$$

Observe that $|f(t,x) - f(t,y)| \leq \frac{1}{\pi}|x - y|$ implies that $L = 1/\pi$ and $|f(t,x)| \leq \frac{1}{\sqrt{1+t}} = \mu(t)$. With the given data, it is found that $A = 9/32$ and

$$\frac{L}{|A|}\left\{\frac{a(1-\eta^{q+1})}{\Gamma(q+2)} + \frac{\xi^q}{\Gamma(q+1)}\right\} = \frac{128}{27\pi\sqrt{\pi}}\left[\frac{2}{5}\left(1 - \left(\frac{3}{4}\right)^{5/2}\right) + \frac{1}{2\sqrt{2}}\right] \simeq 0.475662 < 1.$$

Clearly, all the conditions of Theorem 3.2 are satisfied. Hence there exists a solution for problem (4.2).

Example 4.3 Consider the problem

$$\begin{cases} {}^cD^{3/2}x(t) = \frac{1}{(t+3)^2}(|x| + 1), & t \in [0,1], \\ x(0) = 0, \qquad x(1/2) = \int_{3/4}^1 x(s)\,ds. \end{cases} \tag{4.3}$$

Here, $q = 3/2$, $\xi = 1/2$, $a = 1$, $\eta = 3/4$, $|f(t,x)| \leq \frac{1}{9}(|x| + 1)$ and $\omega \simeq 2.246588$. Let us fix $p(t) = 1$, $\psi(|x|) = \frac{1}{9}(|x| + 1)$. Further, by the condition

$$M\left\{\psi(M)\|p\|\left(\frac{1}{|A|}\left\{\frac{1}{\Gamma(q+1)}\left(|A| + \xi^q\right) + \frac{a}{\Gamma(q+2)}\left(1 - \eta^{q+1}\right)\right\}\right)\right\}^{-1} > 1,$$

it is found that $M > M_1$ with $M_1 \simeq 0.33266$. Thus, Theorem 3.5 applies and there exists a solution for problem (4.3) on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, AA, AAS, and RPA, contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.
[2]Department of Mathematics, Texas A&M University, Kingsville, 78363-8202, USA.

Acknowledgements
This research was partially supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

Received: 29 July 2013 Accepted: 1 December 2013 Published: 20 Dec 2013

References
1. Zhong, W, Lin, W: Nonlocal and multiple-point boundary value problem for fractional differential equations. Comput. Math. Appl. **39**, 1345-1351 (2010)
2. Ahmad, B, Nieto, JJ: Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **2011**, Article ID 36 (2011)

3. Alsaedi, A, Ntouyas, SK, Ahmad, B: Existence results for Langevin fractional differential inclusions involving two fractional orders with four-point multiterm fractional integral boundary conditions. Abstr. Appl. Anal. **2013**, Article ID 869837 (2013)
4. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
5. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
6. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
7. Baleanu, D, Mustafa, OG: On the global existence of solutions to a class of fractional differential equations. Comput. Math. Appl. **59**, 1835-1841 (2010)
8. Baleanu, D, Mustafa, OG, Agarwal, RP: On L^p-solutions for a class of sequential fractional differential equations. Appl. Math. Comput. **218**, 2074-2081 (2011)
9. Agarwal, RP, Ahmad, B: Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions. Comput. Math. Appl. **62**, 1200-1214 (2011)
10. Bhalekar, S, Daftardar-Gejji, V, Baleanu, D, et al.: Fractional Bloch equation with delay. Comput. Math. Appl. **61**, 1355-1365 (2011)
11. Graef, JR, Kong, L, Kong, Q: Application of the mixed monotone operator method to fractional boundary value problems. Fract. Differ. Calc. **2**, 554-567 (2011)
12. Akyildiz, FT, Bellout, H, Vajravelu, K, Van Gorder, RA: Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces. Nonlinear Anal., Real World Appl. **12**, 2919-2930 (2011)
13. Ahmad, B, Nieto, JJ: Sequential fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **64**, 3046-3052 (2012)
14. Bai, ZB, Sun, W: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369-1381 (2012)
15. Sakthivel, R, Mahmudov, NI, Nieto, JJ: Controllability for a class of fractional-order neutral evolution control systems. Appl. Math. Comput. **218**, 10334-10340 (2012)
16. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. **285**, 27-41 (2012)
17. Baleanu, D, Rezapour, S, Mohammadi, H: Some existence results on nonlinear fractional differential equations. Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. **371**(1990), 20120144 (2013)
18. Wang, G, Ahmad, B, Zhang, L, Agarwal, RP: Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. **249**, 51-56 (2013)
19. Zhou, WX, Chu, YD, Baleanu, D: Uniqueness and existence of positive solutions for a multi-point boundary value problem of singular fractional differential equations. Adv. Differ. Equ. **2013**, Article ID 114 (2013)
20. Ahmad, B, Ntouyas, SK, Alsaedi, A: A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. Math. Probl. Eng. **2013**, Article ID 320415 (2013)
21. Ahmad, B, Ntouyas, SK: Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2013**, 20 (2013)
22. Bragdi, M, Debbouche, A, Baleanu, D: Existence of solutions for fractional differential inclusions with separated boundary conditions in Banach space. Adv. Math. Phys. **2013**, Article ID 426061 (2013)
23. Smart, DR: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
24. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)

10.1186/1687-1847-2013-373
Cite this article as: Ahmad et al.: **A new class of fractional boundary value problems.** *Advances in Difference Equations* 2013, 2013:373