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Exponential stability for differential equations with random impulses at random times

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Abstract

Impulsive differential equations with impulses occurring at random times arise in the modeling of real world phenomena in which the state of the investigated process changes instantaneously at uncertain moments. The investigation of these differential equations uses ideas in the qualitative theory of differential equations and probability theory. In this paper differential equations with randomly occurring impulses are considered and the *p*-moment exponential stability of the solutions is studied.

MSC: 34A37; 34E05

Keywords: impulsive differential equations; random moments of impulses; *p*-moment exponential stability

1 Introduction

Impulsive differential equations are studied extensively in the literature. Many authors consider impulsive differential equations with determined impulsive moments (see, for example, the monographs [1–4] and the references cited therein). However, in some real world phenomena the investigated process changes instantaneously at uncertain moments. When modeling such processes, it is necessary to use random variables in jump conditions and impulsive differential equations with random impulses occurring at random moments. The presence of randomness in the jump condition changes the behavior of solutions of differential equations significantly. In the case of impulses occurring at random moments, the solution is a stochastic process.

In the literature a number of results have been obtained for stochastic differential equations with jumps [5–7]. Also, some results on the qualitative properties of equations with random impulses have been obtained [8–10]. In the monograph [11], impulsive differential equations with fixed impulses and random amplitude of jumps were studied.

In this paper we study nonlinear differential equations subject to random impulses occurring at random moments. Randomness is introduced both through the time between impulses, which is distributed exponentially, and through the amount of impulses. The *p*-moment exponential stability of the solution is studied by employing appropriate generalized Lyapunov's functions. In the literature many authors study the stability of impulsive systems with deterministic moments of impulses (see [1, 2, 4] and the references cited therein), the exponential stability of impulsive delay differential equations with deterministic moments of impulses [12–15] and the *p*-moment stability of stochastic differential equations with or without impulses [6, 7, 16, 17]. The behavior of solutions of stochastic equations is totally different from the behavior of ordinary differential equations, so the study of stability properties of differential equations with impulses occurring at random moments is important.

2 Preliminary notes and results

Let the probability space $(\Omega, \mathcal{F}, P)$ be given. Let $\{\tau_k\}_{k=1}^{\infty}$ be a sequence of independent exponentially distributed random variables with a parameter $\lambda > 0$ that are defined on the sample space Ω. We will call the random variables τ_k waiting times since they will define the time between two consecutive impulses of the considered impulsive differential equation.

Define the sequence of random variables $\{\xi_k\}_{k=0}^{\infty}$ such that $\xi_0 = T_0$ and $\xi_k = T_0 + \sum_{i=1}^{k} \tau_i$, $k = 1, 2, \ldots$, where $T_0 \geq 0$ is a fixed point.

We note that $\{\xi_k\}_{k=0}^{\infty}$ is an increasing sequence of random variables.

Assume $\sum_{k=1}^{\infty} \tau_k = \infty$ with probability 1.

Let $t \geq T_0$ be a fixed point. Consider the events

$$S_k(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_{k+1}(\omega)\}, \quad k = 1, 2, \ldots.$$

Let the points t_k be arbitrary values of the random variables τ_k, $k = 1, 2, \ldots$, correspondingly. Define the increasing sequence of points $T_k = \sum_{i=0}^{k} t_i$, $k = 1, 2, 3, \ldots$, that are values of the random variables ξ_k. Consider the initial value problem for the scalar impulsive differential equation with fixed points of impulses

$$\begin{aligned} & x' = f(t, x(t)) \quad \text{for } t \geq T_0, t \neq T_k, \\ & x(T_k + 0) = I_k(t_k, x(T_k - 0)) \quad \text{for } k = 1, 2, \ldots, \\ & x(T_0) = x_0, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$.

The solution of the impulsive differential equation with fixed moments of impulses (1) depends not only on the initial condition (T_0, x_0) but on the moments of impulses T_k, $k = 1, 2, \ldots$, *i.e.*, the solution depends on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. We denote the solution of initial value problem (1) by $x(t; T_0, x_0, \{t_k\})$. We will assume that $x(T_k; T_0, x_0, \{t_k\}) = \lim_{t \to T_k - 0} x(t; T_0, x_0, \{t_k\})$.

Remark 1 We note that the length of the interval (T_k, T_{k+1}) of the continuity of the solutions of the initial value problem for impulsive differential equation with fixed moments of impulses (1) is equal to t_k, that is, a value of the random variable τ_k, called the waiting time.

The set of all solutions $x(t; T_0, x_0, \{t_k\})$ of the initial value problem for the impulsive differential equation (1) for any values t_k of the random variables τ_k, $k = 1, 2, \ldots$, generates a stochastic process with state space $\mathbb{R}^n$. We denote it by $x(t; T_0, x_0, \{\tau_k\})$ and we will say that it is a solution of the following initial value problem for impulsive differential equations with random moments of impulses (RIDE):

$$\begin{aligned} & x' = f(t, x(t)) \quad \text{for } t \geq T_0, \xi_k < t < \xi_{k+1}, \\ & x(\xi_k + 0) = I_k(\tau_k, x(\xi_k - 0)) \quad \text{for } k = 1, 2, \ldots, \\ & x(T_0) = x_0. \end{aligned} \tag{2}$$

For any values t_k of the random variables τ_k, the solution $x(t; T_0, x_0, \{t_k\})$ of the initial value problem for the impulsive equation with fixed points of impulses (1) will be called *a sample path solution* of RIDE (2).

Define the stochastic processes $\Delta_k(t)$, $k = 1, 2, \ldots$, by the equality

$$\Delta_k(t) = \begin{cases} 1 & \text{for } \omega \in S_k(t), \\ 0 & \text{for } \omega \notin S_k(t). \end{cases} \tag{3}$$

We note that for any fixed point t and any element $\omega \in \Omega$, there exists a natural number k such that $\omega \in S_k(t)$ and $\omega \notin S_j(t)$ for $j \neq k$, or for any fixed point t, there exists a natural number k such that $\Delta_k(t) = 1$ and $\Delta_j(t) = 0$ for $j \neq k$.

We will prove the following result for the stochastic processes $\Delta_k(t)$.

Lemma 2.1 *Let $\{\tau_k\}_1^\infty$ be independent exponentially distributed random variables (IED) with a parameter λ and $\xi_k = T_0 + \sum_{i=1}^k \tau_i$.*

Then

$$E(\Delta_k(t)) = \frac{\lambda^k (t - T_0)^k}{k!} e^{-\lambda(t-T_0)} \quad \text{for } t \geq T_0 \text{ and } k = 1, 2, \ldots. \tag{4}$$

Proof Using the distribution of random variables τ_k, the definition of the mean of joint distributed exponentially random variables, and the definition of the sequence of random variables $\{\xi_k\}_1^\infty$, we obtain

$$\begin{aligned} E(\Delta_k(t)) &= \lambda^{k+1} \underbrace{\int \cdots \int}_{\sum_{i=1}^k \theta_i \leq t - T_0 < \sum_{i=1}^{k+1} \theta_i} e^{-\lambda(\theta_1 + \cdots + \theta_{k+1})} \, d\theta_1 \cdots d\theta_{k+1} \\ &= \lambda^k e^{-\lambda(t-T_0)} \underbrace{\int \cdots \int}_{\sum_{i=1}^k \theta_i \leq t - T_0} d\theta_1 \cdots d\theta_k \quad \text{for } t \geq T_0. \end{aligned} \tag{5}$$

From (5) and the equality

$$\underbrace{\int \cdots \int}_{\sum_{i=1}^k \theta_i \leq t} d\theta_1 \cdots d\theta_k = \frac{t^k}{k!},$$

we have (4). □

Corollary 1 *The probability that there will be exactly k impulses until the time t, $t \geq T_0$, is given by the equality*

$$P(S_k(t)) = \frac{\lambda^k (t - T_0)^k}{k!} e^{-\lambda(t-T_0)}.$$

Proof The result follows immediately from the definition of the event $S_k(t)$, Lemma 2.1 and the fact that $P(S_k(t)) = E(\Delta_k(t))$. □

Now we will illustrate some differences between impulsive differential equations with deterministic moments of impulses and differential equations with randomly occurring impulses.

Example 1 (Ordinary differential equation) Consider the IVP

$$x' = 0, \qquad x(0) = x_0 \neq 0. \tag{6}$$

The solution of (6) $x(t) = x_0$ is stable but is not approaching 0.

Example 2 (Impulsive differential equations with fixed points of impulses) Let the increasing sequence of points T_i, $i = 1, 2, \dots$, be given and $\lim_{k\to\infty} T_k = \infty$. Consider the initial value problem (IVP) for the impulsive differential equation

$$\begin{aligned} &x' = 0 \quad \text{for } t \geq 0, t \neq T_k, \\ &x(T_k + 0) = ax(T_k - 0) \quad \text{for } k = 1, 2, \dots, \\ &x(0) = x_0 \neq 0. \end{aligned} \tag{7}$$

The solution of IVP (7) is $x(t) = a^k x_0$ for $t \in (T_k, T_{k+1}]$.

The solution is a piecewise continuous function.

The behavior of $x(t)$ depends significantly on the amplitude of jumps.

If $|a| < 1$, then $|x(t)|$ is approaching 0 (*different* from the corresponding ordinary differential equation considered in Example 1).

Example 3 (Impulsive differential equation with random points of impulses) Let the sequence of independent exponentially distributed random variables τ_i, $i = 1, 2, \dots$ (waiting time) be given. Define $\xi_k = \sum_{j=1}^{k} \tau_k$ (moments of impulses).

Consider the IVP for the impulsive differential equation with random moments of impulses

$$\begin{aligned} &x' = 0 \quad \text{for } t \geq 0, \xi_k < t < \xi_{k+1}, \\ &x(\xi_k + 0) = ax(\xi_k - 0) \quad \text{for } k = 1, 2, \dots, \\ &x(0) = x_0 \neq 0. \end{aligned} \tag{8}$$

Let, for any $k = 1, 2, \dots$, the point t_k be an arbitrary value of the random variable τ_k. Define the increasing sequence of points $T_k = \sum_{i=0}^{k} t_i$, $k = 1, 2, 3 \dots$, that are values of the random variables ξ_k.

Consider the IVP for the corresponding impulsive differential equation with fixed points of impulses

$$\begin{aligned} &x' = 0 \quad \text{for } t \geq 0, t \neq T_k, \\ &x(T_k + 0) = ax(T_k - 0) \quad \text{for } k = 1, 2, \dots, \\ &x(0) = x_0. \end{aligned} \tag{9}$$

The solution of (9) is $x(t) = a^k x_0$ for $T_k < t \leq T_{k+1}$.

It depends not only on x_0 but on the moments of impulses T_k, *i.e.*, on the initially chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$.

The set of all solutions of IVP (9) for any values t_k of the random variables τ_k generates a stochastic process with state space $\mathbb{R}^n$. We will say it is a solution of the impulsive differential equations with random moments of impulses (8) and it is $x(t) = a^k x_0$ for $\xi_k < t \le \xi_{k+1}$.

The solution is a stochastic process.

Now, consider the expected value of the solution,

$$E|x(t)| = |x_0|e^{-\lambda t}\sum_{k=0}^{\infty}|a|^k\frac{(\lambda t)^k}{k!} = |x_0|e^{-\lambda t(1-|a|)}.$$

If $|a| < 1$, then $E|x(t)|$ is approaching 0 (compare with the impulsive differential equation with fixed moments of impulses considered in Example 2).

We will say that conditions (H) are satisfied if

(H1) For any initial values $(t_0, x_0) : t_0 \ge T_0$, $x_0 \in \mathbb{R}^n$, the initial value problem $x' = f(t, x(t))$, $x(t_0) = x_0$ has a unique solution $x(t) = x(t; t_0, x_0)$ defined for $t \ge t_0$.

(H2) $f(t, 0) = 0$ and $I_k(t, 0) = 0$ for $t \ge 0$, $k = 1, 2, \ldots$.

(H3) The random variables $\{\tau_k\}_1^\infty$ are independent exponentially distributed random variables with a parameter λ and $\xi_k = T_0 + \sum_{i=1}^k \tau_i$, $k = 0, 1, 2, \ldots$.

Definition 1 A stochastic process $y(t)$ with an uncountable state space $\mathbb{R}^n$ is said to be a solution of the initial value problem for the system of equations with randomly occurring impulses (2), $t_0 \ge T_0$, if for any sample values t_k of the random variables τ_k, $k = 1, 2, \ldots$, correspondingly, the process $y(t)$ satisfies the initial value problem for the impulsive equation with fixed points of impulses (1), where the moments of impulses are defined by $T_k = T_0 + \sum_{i=1}^k t_i$, $k = 1, 2, \ldots$.

Remark 2 We note that if condition (H1) is satisfied, then the sample path solution of the initial value problem for the impulsive equation with impulses at random moments (2) exists for all $t > t_0$, $t_0 \ge T_0$ provided that the times between two consecutive impulses t_k are such that $\sum t_k = \infty$.

Remark 3 We note that if the values t_k are values of the random variables τ_k, $k = 1, 2, \ldots$, correspondingly, then the value $T_k = t_0 + \sum_{i=1}^k t_i$ is a value of the random variable ξ_k, $k = 1, 2, \ldots$.

Definition 2 We will say that the stochastic processes $y(t)$ and $u(t)$ satisfy the inequality $y(t) \le u(t)$ for $t \in J \subset \mathbb{R}$ if the state space of $v(t) = y(t) - u(t)$ is $[0, \infty)$.

Definition 3 Let $p > 0$. Then the trivial solution of the impulsive differential equation with random impulses (2) is said to be p-moment exponentially stable if for any initial data $t_0 \ge T_0$ and $x_0 \in \mathbb{R}^n$, there exist constants $\alpha, \mu > 0$ such that $E[\|x(t; T_0, x_0, \{\tau_k\})\|^p] < \alpha\|x_0\|^p e^{-\mu(t-t_0)}$ for all $t > t_0$, where $x(t; T_0, x_0, \{\tau_k\})$ is the solution of the initial value problem for the impulsive differential equation with random impulses (2).

Remark 4 We note that the two-moment exponential stability for stochastic equations is known as exponential stability in mean square.

3 Main results

In this section we will use Lyapunov functions to obtain sufficient conditions for the p-moment exponential stability of the trivial solution of the nonlinear impulsive random system at random moments (2).

First we obtain a formula for the solution of the following initial value problem for a scalar linear impulsive differential equation with random moments of impulses and random amplitude of jumps:

$$\begin{aligned} &u' = a(t)u \quad \text{for } t \geq 0, \xi_k < t < \xi_{k+1}, \\ &u(\xi_k + 0) = B_k(\tau_k)u(\xi_k - 0) \quad \text{for } k = 1, 2, \ldots, \\ &u(t_0) = u_0, \end{aligned} \tag{10}$$

where $u, u_0 \in \mathbb{R}$, $a, B_k : \mathbb{R}_+ \to \mathbb{R}$, $k = 1, 2, \ldots$.

Lemma 3.1 *Let the following conditions be fulfilled*:

1. *Condition* (H3) *is satisfied*.
2. *The functions* $a, B_k \in C(\mathbb{R}_+, \mathbb{R})$ $(k = 1, 2, \ldots)$.

Then the solution $u(t; T_0, x_0, \{\tau_k\})$ *of the initial value problem for the linear impulsive differential equation with random moments of impulses* (10) *is given by the formula*

$$u\big(t; T_0, u_0, \{\tau_k\}\big) = u_0 \left(\prod_{i=1}^{k} B_i(\tau_i) \right) e^{\int_{t_0}^{t} a(s)\,ds} \quad \textit{for } \xi_k < t < \xi_{k+1}, k = 0, 1, 2, \ldots \tag{11}$$

and the expected value of the solution satisfies the inequality

$$E\big(\big|u\big(t; T_0, u_0, \{\tau_k\}\big)\big|\big) \leq |u_0| e^{\int_{t_0}^{t} (a(s) - \lambda)\,ds} \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} E\big(\big|B_i(\tau_i)\big|\big) \right) \frac{\lambda^k (t - t_0)^k}{k!}.$$

Proof Choose arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. Define the increasing sequence of points $T_k = \sum_{i=0}^{k} t_i$, $k = 1, 2, 3, \ldots$, that are values of the random variables ξ_k and consider the initial value problem for the linear impulsive differential equation with fixed points of impulses

$$\begin{aligned} &u' = a(t)u \quad \text{for } t \geq 0, t \neq T_k, \\ &u(T_k + 0) = B_k(T_k - T_{k-1})u(T_k - 0) \quad \text{for } k = 1, 2, \ldots, \\ &u(t_0) = u_0. \end{aligned} \tag{12}$$

The solution of initial value problem (12) is given by the formula (see [4])

$$u\big(t; T_0, u_0, \{t_k\}\big) = u_0 \left(\prod_{i=1}^{k} B_i(T_k - T_{k-1}) \right) e^{\int_{t_0}^{t} a(s)\,ds} \quad \text{for } t \in (T_k, T_{k+1}).$$

This solution generates a continuous stochastic process $u(t; T_0, u_0, \{\tau_k\})$ that is defined by (11). It is a solution of the initial value problem for the linear impulsive differential equation with random moments of impulses (10).

According to Corollary 1, formula (11), and the independence of the random variables τ_k, $k = 1, 2, \ldots$, we obtain that the random variables $B_k(\tau_k)$, $k = 1, 2, \ldots$, are independent and the expected value of the solution of the initial value problem for the scalar linear impulsive differential equation with random moments of impulses (10) satisfies

$$\begin{aligned} E\big(\big|u\big(t;T_0,u_0,\{\tau_k\}\big)\big|\big) &= \sum_{k=0}^{\infty} E\big(\big|u\big(t;T_0,u_0,\{\tau_k\}\big)\big| |S_k(t)\big)P\big(S_k(t)\big) \\ &\le \sum_{k=0}^{\infty} |u_0| e^{\int_{t_0}^{t} a(s)\,ds} E\left(\left|\prod_{i=1}^{k} B_i(\tau_i)\right|\right) P\big(S_k(t)\big) \\ &= |u_0| e^{\int_{t_0}^{t} a(s)\,ds} \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} E\big(\big|B_i(\tau_i)\big|\big)\right)\frac{\lambda^k (t-t_0)^k}{k!} e^{-\lambda(t-t_0)} \\ &= |u_0| e^{\int_{t_0}^{t} a(s)\,ds} e^{-\lambda(t-t_0)} \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} E\big(\big|B_i(\tau_i)\big|\big)\right)\frac{\lambda^k (t-t_0)^k}{k!}. \end{aligned} \tag{13}$$

□

In the special case when the expected values of all amplitudes of jumps are the same, we obtain the following result.

Corollary 2 *In the special case when additionally to the conditions of Lemma* 3.1 *the condition* $E(|B_k(\tau_k)|) \le C < \infty$, $k = 1, 2, \ldots$, *holds, then the expected value of the solution* $x(t; T_0, x_0, \{\tau_k\})$ *of initial value problem* (10) *satisfies the inequality*

$$E\big(\big|u\big(t;T_0,u_0,\{\tau_k\}\big)\big|\big) \le |u_0| e^{\int_{t_0}^{t} (a(s)-\lambda(1-C))\,ds}. \tag{14}$$

In the special case when all amplitudes of jumps are deterministic, we obtain the following result.

Corollary 3 *Let* $B_k(u) \equiv b_k$ *be nonnegative constants. Then the expected value of the solution* $u(t; T_0, u_0, \{\tau_k\})$ *of initial value problem* (10) *satisfies the inequality*

$$E\big(\big|u\big(t;T_0,u_0,\{\tau_k\}\big)\big|\big) \le |u_0| e^{\int_{t_0}^{t} a(s)\,ds} e^{-\lambda(t-t_0)} \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k} b_i\right)\frac{\lambda^k (t-t_0)^k}{k!}. \tag{15}$$

Now consider the following scalar linear impulsive differential inequality with random moments of impulses:

$$\begin{aligned} &u' \ge a(t)u \quad \text{for } t \ge 0, \xi_k < t < \xi_{k+1}, \\ &u(\xi_k + 0) \ge B_k(\tau_k)u(\xi_k) \quad \text{for } k = 1, 2, \ldots, \\ &u(t_0) = 0, \end{aligned} \tag{16}$$

where $x \in \mathbb{R}$.

Lemma 3.2 *Let the following conditions be fulfilled*:

1. *Condition* (H3) *is satisfied.*
2. *The functions* $a \in C(\mathbb{R}_+, (0, \infty))$, $B_k \in C(\mathbb{R}_+, \mathbb{R}_+)$ $(k = 1, 2, \ldots)$.

Then the state space of the solution $u(t;T_0,\{\tau_k\})$ *of the linear impulsive differential inequalities with random moments of impulses* (16) *is* $[0,\infty)$.

Proof For any values t_k of the random variables τ_k, $k=1,2,\dots$, we consider the increasing sequence of points $T_k=\sum_{i=0}^{k} t_i$, $k=1,2,3,\dots$, and the linear impulsive differential inequalities with fixed points of impulses

$$\begin{aligned} &u' \geq -a(t)u \quad \text{for } t\geq 0, t\neq T_k,\\ &u(T_k+0)\geq B_k(T_k-T_{k-1})u(T_k) \quad \text{for } k=1,2,\dots,\\ &u(t_0)=0. \end{aligned} \tag{17}$$

We prove that any solution $u(t)$ of inequalities (17) is a piecewise continuous function that is nonnegative. Assume the contrary.

Suppose that there exists a point $\xi\in(t_0,T]$ such that $u(\xi)<0$. Then there exists a point $\xi_1\in(t_0,T]$ such that $u(\xi_1)<0$ and $u'(\xi_1)<0$. That contradicts the first inequality of (17). Therefore $u(t)\geq 0$ on $[t_0,T_1]$.

According to the second inequality of (17), it follows that if $u(T_k)\geq 0$, then $u(T_k+0)\geq 0$.

Using induction, assume that $u(t)\geq 0$ on $[t_0,T_k]$. If there exists a point $\xi_k\in(T_k,T_{k+1}]$ such that $u(\xi_k)<0$, then, as in the proof above, we obtain a contradiction.

Since the above proof does not depend on the points of jumps T_k, it follows that for any points T_k, the solution of (17) will be nonnegative.

Therefore the stochastic process $u(t;T_0,\{\tau_k\})$, generated by all nonnegative functions $u(t)$, will have $[0,\infty)$ as a state space. □

We will use Lyapunov functions in order to obtain sufficient conditions for the p-moment exponential stability of the trivial solution of systems of nonlinear impulsive differential equations with impulses occurring at random moments.

We will say that the function $V:\mathbb{R}^n\to[0,\infty)$ belongs to the class Ξ if $V(x)$ is a continuous differentiable almost everywhere function. We will use the following known definition in the literature for the derivative of the function $V\in\Xi$ along the trajectory of a differential equation:

$$\mathbf{D}V(t,x)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}V(x)f_i(t,x), \tag{18}$$

where $t\geq 0$, $x\in\mathbb{R}^n$.

Remark 5 The function V does not depend on t but its derivative $\mathbf{D}V$ depends on t because of the function f.

Theorem 3.1 *Let the following conditions be fulfilled*:

1. *Conditions* (H) *are satisfied.*
2. *The function* $V\in\Xi$ *and there exist positive constants* a, b *such that*
   (i) $a\|x\|^p\leq V(x)\leq b\|x\|^p$ *for* $x\in\mathbb{R}^n$;

(ii) *for any* $(t,x) \in [0,\infty) \times \mathbb{R}^n$, *the inequality*

$$\mathbf{D}V(t,x) \leq -m(t)V(x)$$

holds, where $m \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and* $\inf_{t\geq 0} m(t) = L \geq 0$;

(iii) *there exist constants* w_k *and* C *such that* $0 \leq w_k \leq C < 1 + \frac{L}{\lambda}$, $k = 1,2,\ldots$, *such that*

$$V\big(I_k(x)\big) \leq w_k V(x) \quad \text{for } x \in \mathbb{R}^n. \tag{19}$$

Then the trivial solution of the impulsive differential equations with random moments of impulses (2) *is p-moment exponentially stable.*

Proof Let $(t_0, x_0) \in [T_0,\infty) \times \mathbb{R}^n$ be arbitrary initial data and the stochastic process $x_\tau(t) = x(t; T_0, x_0, \{\tau_k\})$ be a solution of the initial value problem for impulsive differential equation with random impulses (2). From condition (i) it follows that $a\|x_\tau(t)\|^p \leq V(x_\tau(t))$ for $t \geq T_0$ and

$$E\big(\|x_\tau(t)\|^p\big) \leq \frac{1}{a}E\big(V\big(x_\tau(t)\big)\big), \quad t \geq T_0. \tag{20}$$

Define the stochastic process $v_\tau(t) = V(x_\tau(t))$ with an uncountable state space $\mathbb{R}_+$. From conditions (ii), (iii) we obtain that the stochastic process $v_\tau(t)$ satisfies the inequalities

$$\begin{aligned} &v'_\tau(t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} V\big(x_\tau(t)\big) f_i\big(t, x_\tau(t)\big) \leq -m(t)v_\tau(t) \quad \text{for } \xi_k < t < \xi_{k+1}, \\ &v_\tau(\xi_k+) = V\big(I_k\big(x_\tau(\xi_k)\big)\big) \leq w_k V\big(x_\tau(\xi_k)\big) = w_k v_\tau(\xi_k), \quad k = 1,2,\ldots, \\ &v_\tau(t_0) = V(x_0). \end{aligned} \tag{21}$$

Consider the initial value problem for the scalar linear differential equation with random impulses at random moments

$$\begin{aligned} &u'(t) = -m(t)u(t) \quad \text{for } \xi_k < t \leq \xi_{k+1}, \\ &u(\xi_k+) = w_k u(\xi_k), \quad k = 1,2,\ldots, \\ &u(t_0) = V(x_0). \end{aligned} \tag{22}$$

According to Lemma 3.1, the stochastic process

$$u_\tau(t) = V(x_0)\left(\prod_{i=1}^{k} w_i\right) e^{-\int_{t_0}^{t} m(s)\,ds} \quad \text{for } \xi_k < t < \xi_{k+1}, k = 0,1,2,\ldots,$$

is a solution of initial value problem (22). Since $E(|w_k|) = w_k \leq C$, $k = 1,2,\ldots$, from Corollary 2 we obtain the inequality

$$E\big(\big|u_\tau(t)\big|\big) \leq V(x_0)e^{-\int_{t_0}^{t}(m(s)+\lambda(1-C))\,ds} \leq V(x_0)e^{-(L+\lambda(1-C))(t-t_0)}. \tag{23}$$

Consider $y(t) = u_\tau(t) - v_\tau(t)$, $t \geq t_0$. The stochastic process $y(t)$ satisfies (16) with $a(t) = m(t) > 0$ and $B_k(u) = w_k$. According to Lemma 3.2, the state space of $w(t)$ is $[0, \infty)$, *i.e.*, the inequality $v_\tau(t) \leq u_\tau(t)$, $t \geq t_0$ holds.

From inequality (23) and condition (i) we obtain the inequalities

$$\begin{aligned} E\big(\|x_\tau(t)\|^p\big) &\leq \frac{1}{a}E\big(v_\tau(t)\big) \leq \frac{1}{a}E\big(u_\tau(t)\big) = \frac{1}{a}E\big(|u_\tau(t)|\big) \\ &\leq \frac{1}{a}V(x_0)e^{-(L+\lambda(1-C))(t-t_0)} \\ &\leq \frac{b}{a}\|x_0\|^p e^{-(L+\lambda(1-C))(t-t_0)}. \end{aligned} \tag{24}$$

Inequality (24) proves the p-moment exponential stability. □

Corollary 4 *Let all the conditions of Theorem* 3.1 *be satisfied where inequality* (19) *is replaced by*

$$\mathbf{D}V(t,x) \leq -c\|x\|^p, \quad x \in \mathbb{R}^n.$$

Then the inequality $E(\|x_\tau(t)\|^p) \leq \frac{b}{a}\|x_0\|^p e^{-((1-C)\lambda + \frac{c}{b})(t-t_0)}$ *holds.*

Example 4 (Exponential stability of IDE with random moments of impulses) Let τ_i, $i = 1, 2, \ldots$, be independent exponentially distributed random variables with a parameter λ, *i.e.*, $E(\tau_i) = \frac{1}{\lambda}$, $i = 1, 2, \ldots$. Consider the following initial value problem for the system of impulsive differential equations with random moments of impulses:

$$\begin{aligned} &x' = -tx\big(x^2 + y^2\big), \\ &y' = ty\big(x^2 - y^2\big) \quad \text{for } t \geq 0, \xi_k < t < \xi_{k+1}, \\ &x(\xi_k + 0) = ax(\xi_k - 0), \qquad y(\xi_k + 0) = by(\xi_k - 0) \quad \text{for } k = 1, 2, \ldots, \\ &x(0) = x_0, \qquad y(0) = y_0, \end{aligned} \tag{25}$$

where a, b are constants such that $|a| < 1$, $|b| < 1$.

Consider the Lyapunov function $V(x, y) = 0.5(x^2 + y^2)$.

Since $0.5(x^2 + y^2) = 0.5\|(x, y)\|^2$, condition 2(i) of Theorem 3.1 is satisfied for $a = b = 0.5$.

In this particular case,

$$\mathbf{D}V(t,x,y) = -tx^2\big(x^2 + y^2\big) + ty^2\big(x^2 - y^2\big) = -tx^4 - ty^4 \leq 0, \quad t \in \mathbb{R}_+$$

and

$$V(ax, by) = a^2x^2 + b^2y^2 \leq CV(x, y),$$

where $C = \max\{a^2, b^2\}$.

Therefore the conditions of Theorem 3.1 are satisfied for $m(t) \equiv 0$, $L = 0$, $w_k = C$ and $0 < C < 1 = 1 + \frac{L}{\lambda}$.

Therefore, according to Theorem 3.1, the solution of (25) is exponentially stable in mean square, *i.e.*,

$$E\big[\big\|\big(x(t),y(t)\big)\big\|^2\big] < \alpha\big\|(x_0,y_0)\big\|^2 e^{-\mu(t-t_0)}, \quad t > t_0,$$

where $\mu = \lambda(1-C) > 0$, $\alpha = \frac{b}{a} = 1$ and $\|(x_0,y_0)\|^2 = x_0^2 + y_0^2$.

Now, consider the system without any impulses, *i.e.*, the system of ordinary differential equations

$$\begin{aligned} x' &= -\,tx\big(x^2+y^2\big), \\ y' &= ty\big(x^2-y^2\big) \quad \text{for } t \geq 0. \end{aligned} \tag{26}$$

The solution of (26) is asymptotically stable (see Figures 1 and 2) but it is not exponentially stable, since $\mathbf{D}V(t,x,y) \leq 0$ and $0.5\|(x(t),y(t))\|^2 = V(x(t),y(t)) \leq V(x_0,y_0) =$

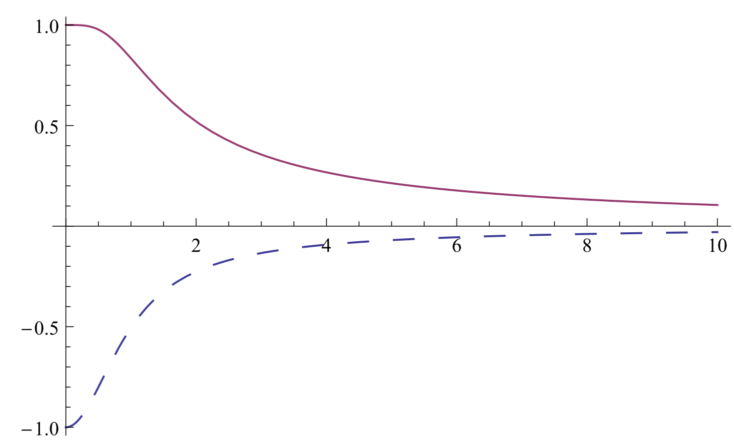

Figure 1 The graph of the solution $x(t), y(t) : x(0) = -1, y(0) = 1$ of (26).

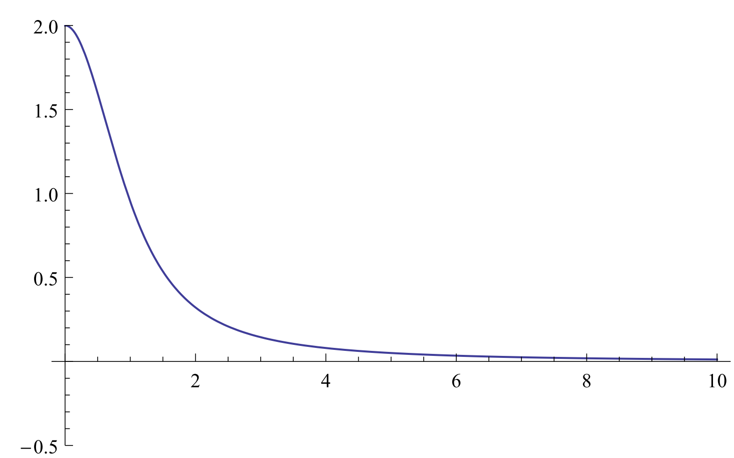

Figure 2 The graph of $x^2(t) + y^2(t)$, where $x(t), y(t) : x(0) = -1, y(0) = 1$ is the solution of (26).

$0.5\|(x_0, y_0)\|^2$, or $(x^2(t) + y^2(t))^2 \leq (x_0^2 + y_0^2)^2$ and there are no positive constants α and μ such that $(x^2(t) + y^2(t))^2 \leq \alpha(x_0^2 + y_0^2)^2 e^{-\mu t}$, $t > 0$, for any initial points x_0, y_0.

Therefore, the presence of impulses at random time changes the behavior of the solution, *i.e.*, the solution of IDE with random impulses is exponentially stable in mean square.

Competing interests
The authors did not provide this information.

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