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Two types of permanence of a stochastic mutualism model

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Abstract

A stochastic mutualism model is proposed and investigated in this paper. We show that there is a unique solution to the model for any positive initial value. Moreover, we show that the solution is stochastically bounded, uniformly continuous and globally attractive. Under some conditions, we conclude that the stochastic model is stochastically permanent and persistent in mean. Finally, we introduce some figures to illustrate our main results.

Keywords: global attractivity; stochastic permanence; persistence in mean; extinction

1 Introduction

Population systems have long been an important theme in mathematical biology due to their universal existence and importance. As far as mutualism system is concerned, lots of proofs have been found in many types of communities. Mutualism occurs when one species provides some benefit in exchange for some benefit. One of the simplest models is the classical Lotka-Volterra two-species mutualism model, which reads

$$\begin{cases} \frac{dN_1(t)}{dt} = N_1(t)[a_1 - b_1N_1(t) + c_1N_2(t)], \\ \frac{dN_2(t)}{dt} = N_2(t)[a_2 - b_2N_2(t) + c_2N_1(t)]. \end{cases} \tag{1}$$

There are many excellent results on the two-species mutualism model (1). It is well known that in nature, with the restriction of resources, it is impossible for one species to survive if its density is too high. Thus the above model is not so good in describing the mutualism of two species (see [1]). Gopalsamy [2] proposed the mutualism model as follows:

$$\begin{cases} \frac{dN_1(t)}{dt} = r_1(t)N_1(t)[\frac{K_1(t)+\alpha_1(t)N_2(t)}{1+N_2(t)} - N_1(t)], \\ \frac{dN_2(t)}{dt} = r_2(t)N_2(t)[\frac{K_2(t)+\alpha_2(t)N_1(t)}{1+N_1(t)} - N_2(t)], \end{cases} \tag{2}$$

where $N_1(t)$ and $N_2(t)$ denote population densities of each species at time t, r_i denotes the intrinsic growth rate of species N_i and $\alpha_i > K_i$, $i = 1, 2$. The carrying capacity of species N_i is K_i in the absence of other species, while with the help of the other species, the carrying capacity becomes $(K_i(t) + \alpha_i(t)N_{3-i}(t))/(1 + N_{3-i}(t))$, $i = 1, 2$. It is assumed that the coefficients of the system are all continuous and bounded. Li and Xu [3] obtained sufficient conditions for the existence of positive periodic solutions. Chen and You [4] gave the sufficient conditions for the permanence of the model. Chen *et al.* [1] considered the permanence of a delayed discrete mutualism model with feedback controls. Here we transform the system (2) into the following form:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)[\frac{a_1(t)+a_2(t)y(t)}{1+y(t)} - c_1(t)x(t)], \\ \frac{dy(t)}{dt} = y(t)[\frac{b_1(t)+b_2(t)x(t)}{1+x(t)} - c_2(t)y(t)]. \end{cases} \tag{3}$$

As a matter of fact, population systems are often subject to environmental noise, *i.e.*, due to environmental fluctuations, parameters involved in population models are not absolute constants, and they may fluctuate around some average values. Based on these factors, more and more people began to be concerned about stochastic population systems (see [5–11]). Especially, Mao *et al.* [5] obtained the interesting and surprising conclusion: even a sufficiently small noise can suppress explosions in population dynamics. Jiang *et al.* [6] considered the global stability and stochastic permanence of a stochastic logistic model. Ji *et al.* [7] discussed the persistence in mean of a predator-prey model with stochastic perturbation. Now, taking into account the effect of randomly fluctuating environment, we incorporate white noise in each equation of the system (3). Therefore, the non-autonomous stochastic system can be described by the Itô equation

$$\begin{cases} dx(t) = x(t)[\frac{a_1(t)+a_2(t)y(t)}{1+y(t)} - c_1(t)x(t)]\,dt + \sigma_1(t)x(t)\,dB_1(t), \\ dy(t) = y(t)[\frac{b_1(t)+b_2(t)x(t)}{1+x(t)} - c_2(t)y(t)]\,dt + \sigma_2(t)y(t)\,dB_2(t), \end{cases} \tag{4}$$

where $a_i(t)$, $b_i(t)$, $c_i(t)$, $\sigma_i(t)$, $i = 1, 2$ are all positive, continuous and bounded functions on $[0, +\infty)$, and $B_1(t)$, $B_2(t)$ are independent Brownian motions, σ_1 and σ_2 represent the intensities of the white noises.

For convenience, if $f(t)$ is a continuous bounded function on $[0, +\infty)$, we define

$$\hat{f} = \inf_{t\in[0,+\infty)} f(t), \qquad \check{f} = \sup_{t\in[0,+\infty)} f(t).$$

For any sequence $\{f_i(t)\}$ $(i = 1, 2)$ define

$$\hat{f} = \inf_{t\in[0,+\infty)} f_i(t), \qquad \check{f} = \sup_{t\in[0,+\infty)} f_i(t).$$

To the best of our knowledge, a very little amount of work has been done on the stochastic system (4). Therefore, we aim to consider dynamical properties of the stochastic model (4) in this paper.

Since stochastic differential equation (4) describes population dynamics, it is necessary for the solution of the system to be positive and not to explode to infinity in a finite time. In this paper, we firstly show that the stochastic system (4) has a unique global (no explosion in a finite time) solution for any positive initial value in Section 2.1. To a population system, the stochastic boundedness is one of most important topics. Section 2.2 tells us that the stochastic model (4) is stochastically ultimately bounded. Furthermore, we will show that the solution of (4) is uniformly continuous and globally attractive in Section 2.3 and Section 2.4 respectively. Moreover, we obtain that the stochastic system is stochastically permanent (*cf.* [6, 8]) in Section 3. Section 4 shows that the stochastic system is persistent in mean (*cf.* [7, 12]). And under some conditions, we discuss the stochastic extinction of the system (4) in Section 5. We work out some figures to illustrate the various theorems obtained before in Section 6. Finally, we close the paper with conclusions in Section 7. The important contributions of this paper are therefore clear.

2 Basic properties of the solution

2.1 Positive and global solution

Throughout this paper, let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. We denote by R_+^2 the positive cone in R^2, $X(t) = (x(t), y(t))$ and $|X(t)| = (x^2(t) + y^2(t))^{\frac{1}{2}}$. And we use K to denote a positive constant whose exact value may be different in different appearances.

Theorem 1 *For any given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is a unique solution $X(t) = (x(t), y(t))$ to stochastic differential equation* (4) *on $t \geq 0$ and the solution will remain in R_+^2 with probability* 1, *that is, $X(t) = (x(t), y(t)) \in R_+^2$ for all $t \geq 0$ almost surely.*

Proof The proof is similar to [5, 8]. Since the coefficients of equation (4) are locally Lipschitz continuous, for any given initial value $X_0 \in R_+^2$, there is a unique local solution $X(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = +\infty$ a.s. Let $m_0 > 0$ be sufficiently large for $x(t)$ and $y(t)$ lying within the interval $[\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : x(t) \text{ or } y(t) \notin \left(\frac{1}{m}, m\right)\right\},$$

where, throughout this paper, we set $\inf \emptyset = \infty$. Obviously, τ_m is increasing as $m \to \infty$. Let $\tau_\infty = \lim_{m\to\infty} \tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. If not, there is $\epsilon \in (0,1)$ and $T > 0$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Hence there is an integer $m_1 \geq m_0$ such that $P\{\tau_m \leq T\} \geq \epsilon$ for all $m \geq m_1$. Define a function $V : R_+^2 \to R_+$ by $V(x) = (x - 1 - \ln x) + (y - 1 - \ln y)$. The non-negativity of this function can be seen from $u - 1 - \ln u \geq 0$ on $u > 0$. If $X(t) \in R_+^2$, we obtain that

$$\begin{aligned} LV &= (x-1)\left[\frac{a_1(t) + a_2(t)y(t)}{1 + y(t)} - c_1(t)x(t)\right] \\ &\quad + (y-1)\left[\frac{b_1(t) + b_2(t)x(t)}{1 + x(t)} - c_2(t)y(t)\right] + \frac{\sigma_1^2 + \sigma_2^2}{2} \\ &\leq (\check{a}_i + \check{c}_i)x(t) - \hat{c}_i(t)x^2(t) + (\check{b}_i + \check{c}_i)y(t) - \hat{c}_i(t)y^2(t) + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right) \leq K. \end{aligned}$$

Therefore

$$EV\big(X(\tau_m \wedge T)\big) = V(X_0) + E\int_0^{\tau_m\wedge T} LV\big(X(t)\big)\,dt \leq V(X_0) + KT. \tag{5}$$

On the other hand, we have

$$V\big(X(\tau_m)\big) \geq [m - 1 - \ln m] \wedge \left[\ln m - 1 + \frac{1}{m}\right].$$

It follows from (5) that

$$V(X_0) + KT \geq \epsilon\left([m - 1 - \ln m] \wedge \left[\ln m - 1 + \frac{1}{m}\right]\right).$$

Letting $m \to \infty$ leads to the contradiction $\infty > V(X_0) + KT = \infty$. Hence, we have $\tau_\infty = \infty$ a.s. The proof is complete. □

2.2 Stochastic boundedness

Stochastic boundedness is one of most important topics because boundedness of a system guarantees its validity in a population system. We first present the definition of stochastically ultimate boundedness.

Definition 1 (see [8]) The solution $X(t) = (x(t), y(t))$ of equation (4) is said to be stochastically ultimately bounded if for any $\epsilon \in (0,1)$, there is a positive constant $\delta = \delta(\epsilon)$ such that for any initial value $X_0 \in R_+^2$, the solution $X(t)$ to (4) has the property that

$$\limsup_{t\to\infty} P\big\{|X(t)| > \delta\big\} < \epsilon.$$

Theorem 2 *The solution of the system* (4) *is stochastically ultimately bounded for any initial value* $X_0 = (x_0, y_0) \in R_+^2$.

Proof By Theorem 1, the solution $X(t)$ will remain in R_+^2 for all $t \geq 0$ with probability 1. Define the function $V = e^t x^p$ for $p > 0$. By the Itô formula, we obtain

$$\begin{aligned} LV &= e^t x^p(t)\left[1 + p\left(\frac{a_1(t) + a_2(t)y(t)}{1 + y(t)} - c_1(t)x(t)\right) + \frac{p(p-1)}{2}\sigma_1^2\right] \\ &\leq e^t\left[\left(1 + p\check{a} + \frac{p(p-1)}{2}\check{\sigma}^2\right)x^p(t) - p\hat{c}x^{p+1}(t)\right] \\ &\leq \frac{1}{(\hat{c})^p}\left[\frac{1 + p\check{a} + \frac{p(p-1)}{2}\check{\sigma}^2}{p+1}\right]^{p+1} e^t := K_1(p)e^t. \end{aligned}$$

Hence we have

$$d\big(e^t x^p\big) = LV\,dt + pe^t x^p \sigma_1\,dB_1(t) \leq K_1(p)e^t\,dt + pe^t x^p \sigma_1(t)\,dB_1(t).$$

Thus $e^t Ex^p - Ex_0^p \leq K_1(p)e^t$. So, we have $\limsup_{t\to\infty} Ex^p \leq K_1(p) < +\infty$.

On the other hand, define the function $V = e^t y^p$ for $p > 0$. We have

$$\begin{aligned} LV &= e^t y^p(t)\left[1 + p\left(\frac{b_1(t) + b_2(t)x(t)}{1 + x(t)} - c_2(t)y(t)\right) + \frac{p(p-1)}{2}\sigma_2^2\right] \\ &\leq e^t\left[\left(1 + p\check{b} + \frac{p(p-1)}{2}\check{\sigma}^2\right)y^p(t) - p\hat{c}y^{p+1}(t)\right] \\ &\leq \frac{1}{(\hat{c})^p}\left[\frac{1 + p\check{b} + \frac{p(p-1)}{2}\check{\sigma}^2}{p+1}\right]^{p+1} e^t := K_2(p)e^t. \end{aligned}$$

This implies $d(e^t y^p(t)) = K_2(p)e^t\,dt + pe^t y^p \sigma_2(t)\,dB_2(t)$. Then $\limsup_{t\to\infty} Ey^p \le K_2(p) < +\infty$. For $X(t) = (x(t), y(t)) \in R_+^2$, note that $|X(t)|^p \le 2^{\frac{p}{2}}(x^p + y^p)$, therefore

$$\limsup_{t\to\infty} E\left|X(t)\right|^p \le K < +\infty.$$

Applying the Chebyshev inequality yields the required assertion. □

2.3 Uniform continuity

In this section, we show the positive solution $X(t) = (x(t), y(t))$ is uniformly Hölder continuous. Main tools are to use appropriate Lyapunov functions and fundamental inequalities. Main methods are motivated by [6, 8].

Lemma 1 ([13, 14]) *Suppose that a stochastic process $X(t)$ on $t \ge 0$ satisfies the condition $E|X(t) - X(s)|^\alpha \le c|t-s|^{1+\beta}$, $0 \le s, t < +\infty$, for some positive constants α, β and c. Then there exists a continuous modification $\tilde{X}(t)$ of $X(t)$, which has the property that for every $\gamma \in (0, \frac{\beta}{\alpha})$, there is a positive random variable $h(w)$ such that*

$$P\left\{\omega : \sup_{0<|t-s|<h(\omega), 0\le s,t<+\infty} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t-s|^\gamma} \le \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder-continuous with exponent γ.

Theorem 3 *For any initial value $(x_0, y_0) \in R_+^2$, almost every sample path of $X(t) = (x(t), y(t))$ to (4) is uniformly continuous on $t \ge 0$.*

Proof Let us consider the stochastic equation as follows:

$$x(t) = x_0 + \int_0^t f_1\bigl(s, x(s), y(s)\bigr)\,ds + \int_0^t g_1\bigl(s, x(s), y(s)\bigr)\,dB_1(t),$$

where

$$f_1\bigl(s, x(s), y(s)\bigr) = x(s)\left[\frac{a_1(s) + a_2(s)y(s)}{1 + y(s)} - c_1(s)x(s)\right],$$
$$g_1\bigl(s, x(s), y(s)\bigr) = \sigma_1(s)x(s).$$

It follows from Theorem 2 that

$$\begin{aligned} E\left|f_1\bigl(s, x(s), y(s)\bigr)\right|^p &= E\left(x^p(s)\left|\frac{a_1(s) + a_2(s)y(s)}{1 + y(s)} - c_1(s)x(s)\right|^p\right) \\ &\le \frac{1}{2}Ex^{2p}(s) + \frac{1}{2}E\left(\frac{a_1(s) + a_2(s)y(s)}{1 + y(s)} - c_1(s)x(s)\right)^{2p} \\ &\le \frac{1}{2}Ex^{2p}(s) + 2^{2p-2}\check{a}^{2p} + 2^{2p-2}\check{c}Ex^{2p}(s) \le K_1(2p) \end{aligned}$$

and

$$E\left|g_1\bigl(s, x(s), y(s)\bigr)\right|^p = E\bigl(\sigma_1^p(s)x_1^p(s)\bigr) \le \check{\sigma}^p Ex^p(s) \le \check{\sigma}^p K_1(p).$$

By the moment inequality (*cf.* [15, 16]) then for $0 \le t_1 < t_2 < \infty$ and $p > 2$,

$$E\left|\int_{t_1}^{t_2} g_1(s)\,dB_1(s)\right|^p \le \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} E\left|g_1(s)\right|^p ds,$$

where dropping $(s, x(s), y(s))$ from $g_1(s, x(s), y(s))$.

Let $0 \le t_1 < t_2 < \infty$, $t_2 - t_1 \le 1$, $1/p + 1/q = 1$, we can compute

$$\begin{aligned}
&E\left|x(t_2) - x(t_1)\right|^p \\
&\quad \le 2^{p-1} E\left(\int_{t_1}^{t_2} |f_1|\,ds\right)^p + 2^{p-1} E\left|\int_{t_1}^{t_2} g_1\,dB_1(s)\right|^p \\
&\quad \le 2^{p-1}\left(\int_{t_1}^{t_2} 1^q\,ds\right)^{\frac{p}{q}} E\left(\int_{t_1}^{t_2} |f_1|^p\,ds\right) \\
&\qquad + 2^{p-1}\left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} (t_2 - t_1)^{\frac{p-2}{2}} \int_{t_1}^{t_2} E|g_1|^p\,ds \\
&\quad \le 2^{p-1}(t_2 - t_1)^{\frac{p}{2}}\left\{(t_2 - t_1)^{\frac{p}{2}} + \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}}\right\} K \\
&\quad \le K(t_2 - t_1)^{\frac{p}{2}},
\end{aligned}$$

where dropping $(s, x(s), y(s))$ from $f_1(s, x(s), y(s))$ and $g_1(s, x(s), y(s))$. Consequently, it follows from Lemma 1 that almost every sample path of $x(t)$ is locally but uniformly Hölder continuous with an exponent $\gamma \in (0, \frac{p-2}{2p})$, and therefore almost every sample path of $x(t)$ is uniformly continuous on $t \ge 0$.

Similarly, by virtue of Lemma 1, almost every sample path of $y(t)$ is uniformly continuous on $t \ge 0$. In a word, almost every sample path of $(x(t), y(t))$ to (4) is uniformly continuous on $t \ge 0$. □

2.4 Global attractivity

Here we show that the solution of (4) is globally attractive.

Lemma 2 (Barbalat [17]) *Let $f(t)$ be a non-negative function defined on $[0, +\infty)$ such that $f(t)$ is integrable on $[0, +\infty)$ and is uniformly continuous on $[0, +\infty)$. Then $\lim_{t\to\infty} f(t) = 0$.*

Definition 2 Let $X_1(t) = (x_1(t), y_1(t))$ and $X_2(t) = (x_2(t), y_2(t))$ be two arbitrary solutions of the system (4) with initial values $(x_1(0), y_1(0)) \in R_+^2$ and $(x_2(0), y_2(0)) \in R_+^2$ respectively. If

$$\lim_{t\to\infty}\left[\left|x_1(t) - x_2(t)\right| + \left|y_1(t) - y_2(t)\right|\right] = 0 \quad \text{a.s.}$$

then we say the system is globally attractive.

Theorem 4 *Let $c_1(t) - |(b_1(t) - b_2(t))| > 0$, $c_2(t) - |(a_1(t) - a_2(t))| > 0$ on $[0, +\infty)$ hold. Then, for any initial value $X_0 = (x_0, y_0) \in R_+^2$, the solution $X(t) = (x(t), y(t))$ is globally attractive.*

Proof The proof is motivated by the arguments of [6]. Define the Lyapunov function $V(t) = |x_1(t) - x_2(t)| + |y_1(t) - y_2(t)|$. By virtue of the Itô formula, we obtain

$$\begin{aligned}
d^+V(t) &= \operatorname{sgn}\bigl(x_1(t) - x_2(t)\bigr)\, d\bigl[\ln x_1(t) - \ln x_2(t)\bigr] \\
&\quad + \operatorname{sgn}\bigl(y_1(t) - y_2(t)\bigr)\, d\bigl[\ln y_1(t) - \ln y_2(t)\bigr] \\
&= \operatorname{sgn}\bigl[x_1(t) - x_2(t)\bigr] \\
&\quad \times \left\{ \frac{a_1(t) + a_2(t)y_1(t)}{1 + y_1(t)} - \frac{a_1(t) + a_2(t)y_2(t)}{1 + y_2(t)} - c_1(t)\bigl[x_1(t) - x_2(t)\bigr] \right\} dt \\
&\quad + \operatorname{sgn}\bigl[y_1(t) - y_2(t)\bigr] \\
&\quad \times \left\{ \frac{b_1(t) + b_2(t)x_1(t)}{1 + x_1(t)} - \frac{b_1(t) + b_2(t)x_2(t)}{1 + x_2(t)} - c_2(t)\bigl[y_1(t) - y_2(t)\bigr] \right\} dt \\
&= \frac{|(a_1(t) - a_2(t))(y_1(t) - y_2(t))|}{(1 + y_1(t))(1 + y_2(t))} + \frac{|(b_1(t) - b_2(t))(x_1(t) - x_2(t))|}{(1 + x_1(t))(1 + x_2(t))} \\
&\quad - c_1\bigl|x_1(t) - x_2(t)\bigr| - c_2\bigl|y_1(t) - y_2(t)\bigr| \\
&\le -\bigl(c_1(t) - \bigl|\bigl(b_1(t) - b_2(t)\bigr)\bigr|\bigr)\bigl|x_1(t) - x_2(t)\bigr| \\
&\quad - \bigl(c_2(t) - \bigl|\bigl(a_1(t) - a_2(t)\bigr)\bigr|\bigr)\bigl|y_1(t) - y_2(t)\bigr|.
\end{aligned}$$

Integrating the above inequality from 0 to t, there exists a positive constant K such that

$$V(t) + K\int_0^t \bigl(\bigl|x_1(s) - x_2(s)\bigr| + \bigl|y_1(s) - y_2(s)\bigr|\bigr)\, ds \le V(0) < +\infty.$$

Therefore, it follows from Theorem 3 and Lemma 2 that

$$\lim_{t\to\infty} \bigl[\bigl|x_1(t) - x_2(t)\bigr| + \bigl|y_1(t) - y_2(t)\bigr|\bigr] = 0 \quad \text{a.s.}$$

So, we complete the proof. □

3 Stochastic permanence

The property of permanence is more desirable since it means the long time survival in a population dynamics. Now, the definition of stochastic permanence will be given below [6, 8].

Definition 3 The solution $X(t) = (x(t), y(t))$ of equation (4) is said to be stochastically permanent if for any $\epsilon \in (0,1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $\chi = \chi(\epsilon)$ such that for any initial value $X_0 = (x_0, y_0) \in R_+^2$, the solution $X(t)$ to (4) has the properties that

$$\liminf_{t\to\infty} P\bigl\{\bigl|X(t)\bigr| \ge \delta\bigr\} \ge 1 - \epsilon, \qquad \liminf_{t\to\infty} P\bigl\{\bigl|X(t)\bigr| \le \chi\bigr\} \ge 1 - \epsilon.$$

Let us now impose a hypothesis.

Assumption 1 $\min\{\hat{a}, \hat{b}\} > \frac{\check{\sigma}^2}{2}$.

Theorem 5 *Under Assumption* 1, *for any initial value* $X_0 = (x_0, y_0) \in R^2_+$, *the solution* $X(t) = (x(t), y(t))$ *satisfies that*

$$\limsup_{t\to\infty} E\biggl(\frac{1}{|X(t)|^\theta}\biggr) \le K, \tag{6}$$

where θ *is an arbitrary positive constant satisfying*

$$\min\{\hat{a}, \hat{b}\} > \frac{\theta+1}{2}\check{\sigma}^2, \tag{7}$$

and k *is an arbitrary positive constant satisfying*

$$\theta\min\{\hat{a}, \hat{b}\} - \frac{\theta(\theta+1)}{2}\check{\sigma}^2 - k > 0. \tag{8}$$

Proof Define $V(X) = (x + y)$ for $(x, y) \in R^2_+$, then

$$\begin{aligned} dV(X) = {}&\biggl[x(t)\biggl(\frac{a_1(t)+a_2(t)y(t)}{1+y(t)} - c_1(t)x(t)\biggr) + y(t)\biggl(\frac{b_1(t)+b_2(t)x(t)}{1+x(t)} - c_1(t)y(t)\biggr)\biggr]\,dt \\ &+ \bigl(\sigma_1 x\,dB_1(t) + \sigma_2 y\,dB_2(t)\bigr). \end{aligned}$$

And also define $U(X(t)) = \frac{1}{V(X(t))}$ on $t \ge 0$. Applying the Itô formula, we get

$$\begin{aligned} dU = {}& -U^2\biggl[\biggl(x\biggl(\frac{a_1(t)+a_2(t)y}{1+y} - c_1(t)x\biggr) + y\biggl(\frac{b_1(t)+b_2(t)x}{1+x} - c_1(t)y\biggr)\biggr)\,dt \\ &+ \bigl(\sigma_1(t)x\,dB_1(t) + \sigma_2(t)y\,dB_2(t)\bigr)\biggr] + U^3\bigl[\bigl(\sigma_1(t)x\bigr)^2 + \bigl(\sigma_2(t)y\bigr)^2\bigr]\,dt \\ = {}& LU\,dt - U^2\bigl[\sigma_1(t)x\,dB_1(t) + \sigma_2(t)y\,dB_2(t)\bigr], \end{aligned}$$

where

$$\begin{aligned} LU = {}& -U^2\biggl[x\biggl(\frac{a_1(t)+a_2(t)y}{1+y} - c_1(t)x\biggr) + y\biggl(\frac{b_1(t)+b_2(t)x}{1+x} - c_1(t)y\biggr)\biggr] \\ &+ U^3\bigl[\bigl(\sigma_1(t)x\bigr)^2 + \bigl(\sigma_2(t)y\bigr)^2\bigr] \end{aligned}$$

dropping $X(t)$ from $U(X(t))$, $V(X(t))$ and t from $x(t)$, $y(t)$. Under Assumption 1, choose a positive constant θ such that it obeys (7). By the Itô formula again, we have

$$L(1+U)^\theta = \theta(1+U)^{\theta-1}LU + \frac{\theta(\theta-1)}{2}U^4(1+U)^{\theta-2}\bigl[\bigl(\sigma_1(t)x\bigr)^2 + \bigl(\sigma_2(t)y\bigr)^2\bigr].$$

Now, choose $k > 0$ sufficiently small such that it satisfies (8). Thus by the Itô formula,

$$L\bigl[e^{kt}(1+U)^\theta\bigr] = ke^{kt}(1+U)^\theta + e^{kt}L(1+U)^\theta = e^{kt}(1+U)^{\theta-2}\bigl(k(1+U)^2 + F\bigr).$$

The following analysis mainly focuses on the upper boundedness of the function

$$(1+U)^{\theta-2}\bigl(k(1+U)^2 + F\bigr).$$

We compute

$$\begin{aligned}F &= -\theta U^2\left[x\left(\frac{a_1(t)+a_2(t)y}{1+y}-c_1(t)x\right)+y\left(\frac{b_1(t)+b_2(t)x}{1+x}-c_1(t)y\right)\right]\\ &\quad -\theta U^3\left[x\left(\frac{a_1(t)+a_2(t)y}{1+y}-c_1(t)x\right)+y\left(\frac{b_1(t)+b_2(t)x}{1+x}-c_1(t)y\right)\right]\\ &\quad +\theta U^3\left[\left(\sigma_1(t)x\right)^2+\left(\sigma_2(t)y\right)^2\right]+\frac{\theta(\theta+1)}{2}U^4\left[\left(\sigma_1(t)x\right)^2+\left(\sigma_2(t)y\right)^2\right]\end{aligned}$$

dropping t from $x(t)$, $y(t)$. Simplifying the inequalities above, we obtain

$$\begin{aligned}L\left[e^{kt}(1+U)^\theta\right] &= e^{kt}(1+U)^{\theta-2}\left[k(1+U)^2+F\right]\\ &\le e^{kt}(1+U)^{\theta-2}\left[K+KU-\left(\theta\min\{\hat{a},\hat{b}\}-\frac{\theta(\theta+1)}{2}\check{\sigma}^2-k\right)U^2\right].\end{aligned}$$

Then (8) implies that there exists a positive constant K such that

$$Le^{kt}(1+U)^\theta\le Ke^{kt}.$$

This implies

$$E\left[e^{kt}\left(1+U(t)\right)^\theta\right]\le\left(1+U(0)\right)^\theta+\frac{K}{k}e^{kt}=\left(1+U(0)\right)^\theta+Ke^{kt}.$$

Thus, $\limsup_{t\to\infty}EU^\theta(t)\le\limsup_{t\to\infty}E(1+U(t))^\theta\le K$. Note that $(x+y)^\theta\le 2^\theta(x^2+y^2)^{\frac{\theta}{2}}=2^\theta|X|^\theta$ for $X=(x,y)\in R_+^2$. Consequently,

$$\limsup_{t\to\infty}E\left(\frac{1}{|X(t)|^\theta}\right)\le 2^\theta\limsup_{t\to\infty}EU^\theta(t)\le K.\qquad\square$$

Theorem 6 *Let Assumption* 1 *hold. Then the system* (4) *is stochastically permanent.*

The proof is an application of the well-known Chebyshev inequality and Theorems 2 and 5. Here it is omitted.

4 Persistence in mean

In view of ecology, a good situation occurs when all species co-exist. In this section, we will consider another stochastic persistence, that is, stochastic persistence in mean. Now, we present the definition of persistence in mean.

Definition 4 (see [7, 12]) The system (4) is said to be persistent in mean if

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds>0,\qquad\lim_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,ds>0\quad\text{a.s.}$$

Firstly, we introduce a fundamental lemma which will be used.

Lemma 3 *Consider the one-dimensional stochastic equation*

$$dx(t)=x(t)\left[a(t)-b(t)x(t)\right]dt+\sigma(t)x(t)\,dB(t),\tag{9}$$

where $a(t)$, $b(t)$, $\sigma(t)$ are positive, continuous and bounded functions, $B(t)$ is a standard Brownian motion. Under the condition $\hat{a} > \frac{\check{\sigma}^2}{2}$, for any initial value $x_0 > 0$, the solution $x(t)$ to (9) *has the property*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0 \quad a.s.$$

Proof We firstly show $\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le 0$ a.s. To define the Lyapunov function $V(t,x) = e^t \ln x$, using the Itô formula, we obtain

$$d\bigl(e^t \ln x(t)\bigr) = e^t\left[\ln x(t) + a(t) - b(t)x(t) - \frac{\sigma^2(t)}{2}\right]dt + e^t\sigma(t)\,dB(t).$$

Thus

$$e^t \ln x(t) - \ln x_0 = \int_0^t e^s\left[\ln x(s) + a(s) - b(s)x(s) - \frac{\sigma^2(s)}{2}\right]ds + M(t),$$

where $M(t) = \int_0^t e^s\sigma(s)\,dB(s)$, whose quadratic variation is

$$\bigl\langle M(t), M(t)\bigr\rangle = \int_0^t e^{2s}\sigma^2(s)\,ds.$$

By virtue of the exponential martingale inequality, for any positive constants T, δ, β, we have

$$P\left\{\sup_{0\le t\le T}\left[M(t) - \frac{\delta}{2}\bigl\langle M(t), M(t)\bigr\rangle\right] > \beta\right\} \le e^{-\delta\beta}.$$

Choose $T = k\gamma$, $\delta = n\epsilon e^{-k\delta}$ and $\beta = \frac{\theta e^{k\delta}\ln k}{\epsilon n}$, where $k \in Z^+$, $0 < \epsilon < 1$, $\theta > 1$ and $\gamma > 0$ above. Hence

$$P\left\{\sup_{0\le t\le T}\left[M(t) - \frac{n\epsilon e^{-k\delta}}{2}\bigl\langle M(t), M(t)\bigr\rangle\right] > \frac{\theta e^{k\delta}\ln k}{\epsilon n}\right\} \le k^{-\theta}.$$

Obviously, we know $\sum_{k=1}^{\infty} k^{-\theta} < \infty$. Applying the Borel-Cantalli lemma, we obtain that there exists some $\Omega_i \subset \Omega$ with $P(\Omega_i) = 1$ such that for any $\omega \in \Omega_i$, an integer $k_i = k_i(\omega)$ such that for any $k > k_i$, we get

$$M(t) \le \frac{n\epsilon e^{-k\delta}}{2}\bigl\langle M(t), M(t)\bigr\rangle + \frac{\theta e^{k\delta}\ln k}{\epsilon n}$$

for all $0 \le t \le k\gamma$. Then

$$e^t \ln x(t) - \ln x_0 \le \int_0^t e^s\left[\ln x(s) + ar(s) - b(s)x(s) - \frac{\sigma^2(s)}{2} + \frac{n\epsilon e^{s-k\delta}}{2}\sigma^2(s)\right]ds + \frac{\theta e^{k\delta}\ln k}{\epsilon}.$$

Note that $t \in [0, k\gamma]$, $s \in [0, t]$, we have

$$\ln x(s) + a(s) - b(s)x(s) - \frac{\sigma^2(s)}{2} + \frac{n\epsilon e^{s-k\delta}}{2}\sigma^2(s) \le K.$$

For all $t \in [0, k\gamma]$ with $k > k_0(\omega)$, we derive

$$e^t \ln x(t) - \ln x_0 \le \int_0^t K e^s \, ds + \frac{\theta e^{k\delta} \ln k}{\epsilon} = K\bigl(e^t - 1\bigr) + \frac{\theta e^{k\delta} \ln k}{\epsilon}.$$

Thus, for $(k-1)\gamma \le t \le k\gamma$, we get $\ln x(t) \le e^{-t} \ln x_0 + K(1 - e^{-t}) + \frac{\theta e^{\delta} \ln k}{\epsilon}$. This implies

$$\frac{\ln x(t)}{\ln t} \le \frac{\ln x_0}{e^t \ln t} + \frac{K(1 - e^{-t})}{\ln t} + \frac{\theta e^{\delta} \ln k}{\epsilon \ln((k-1)\gamma)}.$$

Letting $k \to \infty$, that is, $t \to \infty$, we can imply $\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \le \frac{\theta e^{\gamma}}{\epsilon}$. By making $\gamma \downarrow 0$, $\epsilon \uparrow 1$ and $\theta \downarrow 1$, we get $\limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \le 1$. Consequently,

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} = \limsup_{t\to\infty} \frac{\ln x(t)}{\ln t} \limsup_{t\to\infty} \frac{\ln t}{t} \le \limsup_{t\to\infty} \frac{\ln t}{t} = 0.$$

Thus it remains to show that $\liminf_{t\to\infty} \frac{\ln x(t)}{t} \ge 0$ a.s. The quadratic variation of the stochastic integral $\int_0^t \sigma(s) \, dB(s)$ is $\int_0^t \sigma^2(s) \, ds \le Kt$. So, the strong law of large numbers of local martingales yields that

$$\frac{1}{t} \int_0^t \sigma(s) \, dB(s) \to 0 \quad \text{a.s. } t \to \infty.$$

Hence, for any $\epsilon > 0$, there exists some positive $T < \infty$ such that

$$\left| \int_0^t \sigma(s) \, dB(s) \right| < \epsilon t \quad \text{a.s. for any } t \ge T.$$

For any $t > s \ge T$, we have

$$\left| \int_s^t \sigma(s) \, dB(s) \right| < \epsilon(s + t) \quad \text{a.s.}$$

Then, for any $t > T$,

$$\begin{aligned}
\frac{1}{x(t)} &= \frac{1}{x(T)} e^{[\int_T^t -(a(s) - \frac{\sigma^2(s)}{2}) \, ds - \int_T^t \sigma(s) \, dB(s)]} + \int_T^t b(s) e^{[\int_s^t -(a(\tau) - \frac{\sigma^2(\tau)}{2}) \, d\tau - \int_T^t \sigma_1(r(\tau)) \, dB(\tau)]} \, ds \\
&\le \frac{1}{x(T)} e^{[\int_T^t -(a(s) - \frac{\sigma^2(s)}{2}) \, ds + \epsilon(t+T)]} + \int_T^t b(s) e^{[\int_s^t -(a(\tau) - \frac{\sigma^2(\tau)}{2}) \, d\tau + \epsilon(t+s)]} \, ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
e^{-2\epsilon(t+T)} \frac{1}{x(t)} &\le \frac{1}{x(T)} e^{[\int_T^t -(a(s) - \frac{\sigma^2(s)}{2}) \, ds - \epsilon(t+T)]} \\
&\quad + \int_T^t b(s) e^{[\int_s^t -(a(\tau) - \frac{\sigma^2(\tau)}{2}) \, d\tau - \epsilon(t-s) - 2\epsilon T]} \, ds \\
&\le K < \infty.
\end{aligned}$$

That is, $\frac{1}{x(t)} \le Ke^{2\epsilon(t+T)}$ a.s. Then $\frac{\ln\frac{1}{x(t)}}{t} \le \frac{1}{t}[\ln K + 2\epsilon(t+T)]$ a.s. Thus $\liminf_{t\to\infty} \frac{\ln x(t)}{t} \ge -2\epsilon$ a.s. Since ϵ is arbitrary, we conclude that

$$\liminf_{t\to\infty} \frac{\ln x(t)}{t} \ge 0 \quad \text{a.s.}$$

So, the proof is complete. □

Remark 1 Lemma 3 generalizes the works of [7] and [11].

To continue our analysis, let us impose the following hypothesis.

Assumption 2 $\hat{a} - \frac{\check{\sigma}^2}{2} > 0$, $\hat{b} - \frac{\check{\sigma}^2}{2} > 0$.

Theorem 1 tells us there is a unique global solution (which is positive for any initial value $X_0 = (x_0, y_0) \in R_+^2$) to the stochastic system (4). So, we conclude the following results by the comparison theorem. We can get

$$dx(t) \le x(t)\big[\check{a} - c_1(t)x(t)\big]\,dt + \sigma_1(t)x(t)\,dB_1(t)$$

and

$$dx(t) \ge x(t)\big[\hat{a} - c_1(t)x(t)\big]\,dt + \sigma_1(t)x(t)\,dB_1(t).$$

Denote that X_2 is the solution to the following stochastic equation:

$$dX_2(t) = X_2(t)\big[\check{a} - c_1(t)X_2(t)\big]\,dt + \sigma_1(t)X_2(t)\,dB_1(t) \tag{10}$$

with $X_2(0) = x_0$. And X_1 is the solution to the equation

$$dX_1(t) = X_1(t)\big(\hat{a} - c_1(t)X_1(t)\big)\,dt + \sigma_1(t)X_1(t)\,dB_1(t) \tag{11}$$

with $X_1(0) = x_0$. It is obvious that $X_1(t) \le x(t) \le X_2(t)$, $t \in [0, +\infty)$ a.s. Moreover, we can have

$$dy(t) \le y(t)\big(\check{b} - c_2(t)y(t)\big)\,dt + \sigma_2(t)y(t)\,dB_2(t)$$

and

$$dy(t) \ge y(t)\big(\hat{b} - c_2(t)y(t)\big)\,dt + \sigma_2(t)y(t)\,dB_2(t).$$

We denote $Y_1(t)$ is the solution of the stochastic differential equation

$$dY_1(t) = Y_1(t)\big[\hat{b} - c_2(t)Y_1(t)\big]\,dt + \sigma_2 Y_1(t)\,dB_2(t) \tag{12}$$

with $Y_1(0) = y_0$. And the stochastic equation

$$dY_2(t) = Y_2(t)\big[(\check{b} - c_2(t)Y_2(t)\big]\,dt + \sigma_2 Y_2(t)\,dB_2(t) \tag{13}$$

has the solution $Y_2(t)$ for initial value $Y_2(0) = y_0$. Consequently, $Y_1(t) \le y(t) \le Y_2(t)$, $t \in [0, +\infty)$ a.s. To sum up, we have

$$X_1(t) \le x(t) \le X_2(t), \qquad Y_1(t) \le y(t) \le Y_2(t), \quad t \in [0, +\infty) \text{ a.s.} \tag{14}$$

Lemma 4 *Under Assumption* 2, *for any initial value* $x_0 > 0$, *the solution* $x(t)$ *to* (4) *satisfies*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0 \quad \textit{a.s.}$$

Lemma 3, (10), (11) and (14) can straightforward imply the assertion.

Lemma 5 *Under Assumption* 2, *for any initial value* $y_0 > 0$, *the solution* $y(t)$ *to* (4) *satisfies*

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} = 0 \quad \textit{a.s.}$$

Lemma 3, (12), (13) and (14) prove the result.

Theorem 7 *Let Assumption* 2 *hold. Then, for any initial value* $(x_0, y_0) \in R_+^2$, *the system* (4) *is persistent in mean. That is, the system* (4) *has the properties*

$$\liminf_{t\to\infty} \frac{\int_0^t y(s)\,ds}{t} \ge \frac{g - \frac{\sigma_2^2}{2}}{h} > 0, \qquad \liminf_{t\to\infty} \frac{\int_0^t x(s)\,ds}{t} \ge \frac{\hat{a} - \frac{\check{\sigma}^2}{2}}{\check{c}} > 0 \quad \textit{a.s.}$$

Proof Denote $V(x) = \ln x$, by the Itô formula, we obtain

$$d\bigl(\ln x(t)\bigr) = \left[\frac{a_1(t) + a_2(t)y(t)}{1 + y(t)} - c_1(t)x(t) - \frac{\sigma_1^2(t)}{2}\right] dt + \sigma_1(t)\,dB_1(t).$$

Then

$$\begin{aligned} \ln x(t) = \ln x_0 &+ \int_0^t \left(\frac{a_1(s) + a_2(s)y(s)}{1 + y(s)} - \frac{\sigma_1^2(s)}{2}\right) ds \\ &- \int_0^t c_1(s)x(s)\,ds + \int_0^t \sigma_1(s)\,dB_1(s), \end{aligned}$$

which yields

$$\check{c} \int_0^t x(s)\,ds \ge -\ln x(t) + \ln x_0 + \left(\hat{a} - \frac{\check{\sigma}^2}{2}\right)t + \int_0^t \sigma_1(s)\,dB_1(s).$$

By virtue of the strong law of large numbers and Lemma 4, we get

$$\liminf_{t\to\infty} \frac{\int_0^t x(s)\,ds}{t} \ge \frac{\hat{a} - \frac{\check{\sigma}^2}{2}}{\check{c}} > 0 \quad \text{a.s.}$$

On the other hand, denote $V(y) = \ln y$, by the Itô formula, we obtain

$$d\bigl(\ln y(t)\bigr) = \left[\frac{b_1(t) + b_2(t)x(t)}{1 + x(t)} - c_2(t)y(t) - \frac{\sigma_2^2(t)}{2}\right] dt + \sigma_2(t)\,dB_2(t).$$

Thus

$$\ln y(t) = \ln y_0 + \int_0^t \left(\frac{b_1(s) + b_2(s)x(s)}{1 + x(s)} - \frac{\sigma_2^2(s)}{2} \right) ds - \int_0^t c_2(s)y(s)\,ds + \int_0^t \sigma_2(s)\,dB_2(s).$$

So, we have

$$\check{c} \int_0^t y(s)\,ds \geq -\ln y(t) + \ln y_0 + \left(\hat{b} - \frac{\check{\sigma}^2}{2} \right) t + \int_0^t \sigma_2(s)\,dB_2(s).$$

Dividing t on both sides yields

$$\check{c} \frac{\int_0^t y(s)\,ds}{t} \geq -\frac{\ln y(t)}{t} + \frac{\ln y_0}{t} + \left(\hat{b} - \frac{\check{\sigma}^2}{2} \right) + \frac{\int_0^t \sigma_2(s)\,dB_2(s)}{t}.$$

Letting $t \to \infty$, by virtue of the strong law of large numbers and Lemma 5, we have

$$\liminf_{t\to\infty} \frac{\int_0^t y(s)\,ds}{t} \geq \frac{\hat{b} - \frac{\check{\sigma}^2}{2}}{\check{c}} > 0 \quad \text{a.s.}$$

The proof is complete. □

5 Extinction

In Sections 3 and 4, we showed that under certain conditions, the system was stochastically permanent and persistent in mean respectively. In view of ecology, a bad thing happens when a species disappears. Here, we will show that if the noise is sufficiently large, the solution to the associated stochastic model will become extinct with probability one.

Theorem 8 *Assume* $\check{a} - \frac{\hat{\sigma}^2}{2} < 0$ *and* $\check{b} - \frac{\hat{\sigma}^2}{2} < 0$ *hold. Then, for any initial value* $(x_0, y_0) \in R_+^2$, *the solution* $(x(t), y(t))$ *to* (4) *will be extinct exponentially with probability one, that is,*

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq \check{a} - \frac{\hat{\sigma}^2}{2} < 0, \qquad \limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \check{b} - \frac{\hat{\sigma}^2}{2} < 0 \quad \textit{a.s.}$$

Proof Define Lyapunov functions $\ln x$ and $\ln y$ respectively. Then, by the Itô formula, we have

$$d\bigl(\ln x(t)\bigr) = \left[\frac{a_1(t) + a_2(t)y(t)}{1 + y(t)} - c_1(t)x(t) - \frac{\sigma_1^2(t)}{2} \right] dt + \sigma_1(t)\,dB_1(t)$$

and

$$d\bigl(\ln y(t)\bigr) = \left[\frac{b_1(t) + b_2(t)x(t)}{1 + x(t)} - c_2(t)y(t) - \frac{\sigma_2^2(t)}{2} \right] dt + \sigma_2(t)\,dB_2(t).$$

Hence

$$\ln x(t) \leq \ln x_0 + \left(\check{a} - \frac{\hat{\sigma}^2}{2} \right) t + \int_0^t \sigma_1(s)\,dB_1(s)$$

and

$$\ln y(t) \le \ln y_0 + \left(\check{b} - \frac{\hat{\sigma}^2}{2}\right)t + \int_0^t \sigma_2(s)\,dB_2(s).$$

Dividing t on the both sides, letting $t \to \infty$ and applying the strong law of large numbers for local martingales, we have

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le \check{a} - \frac{\hat{\sigma}^2}{2} < 0, \qquad \limsup_{t\to\infty} \frac{\ln y(t)}{t} \le \check{b} - \frac{\hat{\sigma}^2}{2} < 0 \quad \text{a.s.}$$

So, we complete the proof. □

6 Numerical simulations

In this section we use the Milstein method mentioned in Higham [18] to substantiate the analytical findings.

For the model (4), we consider the discretization equation:

$$\begin{aligned}
x_{k+1} &= x_k + x_k\left[\frac{a_1(k\Delta t) + a_2(k\Delta t)y_k}{y_k + 1} - c_1(k\Delta t)x_k\right]\Delta t + \sigma_1(k\Delta t)x_k\sqrt{\Delta t}\xi_k \\
&\quad + \frac{\sigma_1^2(k\Delta t)}{2}x_k\left(\xi_k^2 - 1\right)\Delta t, \\
y_{k+1} &= y_k + y_k\left[\frac{b_1(k\Delta t) + b_2(k\Delta t)x_k}{x_k + 1} - c_2(k\Delta t)y_k\right]\Delta t + \sigma_2(k\Delta t)y_k\sqrt{\Delta t}\eta_k \\
&\quad + \frac{\sigma_2^2(k\Delta t)}{2}y_k\left(\eta_k^2 - 1\right)\Delta t,
\end{aligned}$$

where ξ_k and η_k are Gaussian random variables that follow $N(0,1)$.

In Figure 1a,b, we choose $a_1 = b_1 = 0.4$, $a_2 = b_2 = 0.5$, $c_1 = c_2 = 0.1$. In Figure 1a,b, we choose $\sigma_1(t)^2/2 = \sigma_2^2(t)/2 = 0.01$. By virtue of Theorem 6, the system will be stochastically permanent. It follows from Theorem 7 that the system will be persistent in mean. What we mentioned above can be seen from Figure 1a,b. The difference between conditions of Figure 1a,b,c is that the values of σ_1 and σ_2 are different. In Figure 1a,b, we choose $\sigma_1^2/2 = \sigma_2^2/2 = 0.01$. In Figure 1c, we choose $\sigma_1^2/2 = \sigma_2^2/2 = 1$. In view of Theorem 8, both species x and y will go to extinction. Figure 1c confirms this.

By comparing Figure 1a,b with Figure 1c, we can observe that small environmental noise can retain the stochastic system permanent; however, sufficiently large environmental noise makes the stochastic system extinct.

7 Conclusions

In this paper, we consider the stochastic mutualism system (4). We show that there is a unique positive solution to the model for any positive initial value. Moreover, we show that the positive solutions are uniformly continuous, globally attractive. Especially, we conclude the following: under Assumption 1, the stochastic model (4) is stochastically permanent; under Assumption 2, the stochastic model (4) is persistent in mean. It is interesting and surprising to obtain the results. It is easy to see that Assumptions 1 and 2 have almost the same meaning. To a great extent, when the intensity of environmental noise is not too big, some nice properties such as non-explosion, boundedness, permanence are

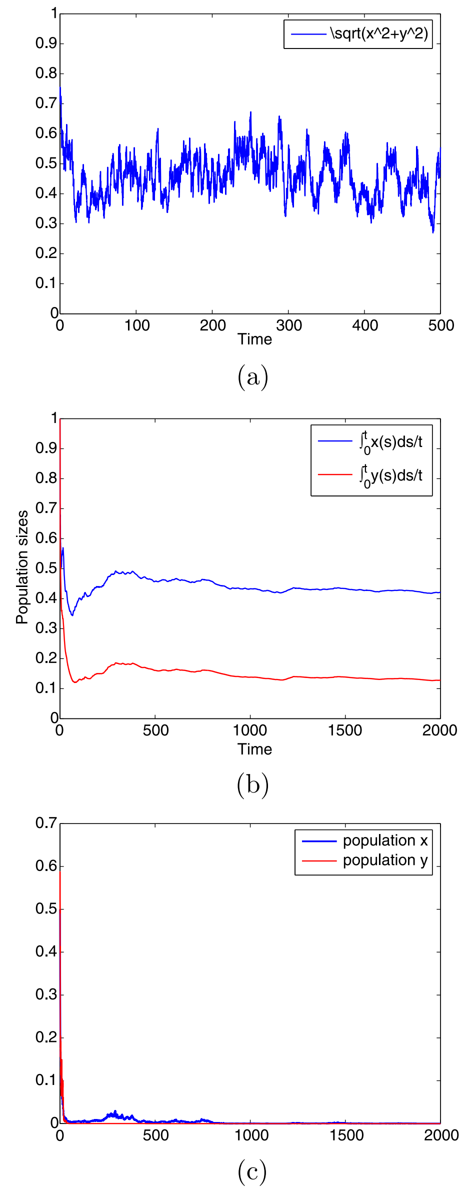

Figure 1 Solution $X(t) = (x(t), y(t))$ of system (4) for $x(0) = 0.5$, $y(0) = 0.5$. The horizontal axis represents the time t. (a) is with $\sigma_1^2/2 = \sigma_2^2/2 = 0.01$ and step size $\Delta t = 0.01$; (b) is with $\sigma_1^2/2 = \sigma_2^2/2 = 0.01$ and step size $\Delta t = 0.002$; (c) is with $\sigma_1^2/2 = \sigma_2^2/2 = 1$ and step size $\Delta t = 0.002$.

desired. However, Theorem 8 reveals that a large white noise will force the population to become extinct.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors carried out the proof of the main part of this article. All authors have read and approved the final manuscript.

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