Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Global dynamics of some systems of higher-order rational difference equations

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Abstract

In the present work, we study the qualitative behavior of two systems of higher-order rational difference equations. More precisely, we study the local asymptotic stability, instability, global asymptotic stability of equilibrium points and rate of convergence of positive solutions of these systems. Our results considerably extend and improve some recent results in the literature. Some numerical examples are given to verify our theoretical results.

MSC: 39A10; 40A05

Keywords: system of difference equations; stability; global character; rate of convergence

1 Introduction

Recently, studying the qualitative behavior of difference equations and systems is a topic of great interest. Applications of discrete dynamical systems and difference equations have appeared recently in many areas such as ecology, population dynamics, queuing problems, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, neural networks, quanta in radiation, genetics in biology, economics, psychology, sociology, physics, engineering, economics, probability theory and resource management. Unfortunately, these are only considered as the discrete analogs of differential equations. It is a well-known fact that difference equations appeared much earlier than differential equations and were instrumental in paving the way for the development of the latter. It is only recently that difference equations have started receiving the attention they deserve. Perhaps this is largely due to the advent of computers where differential equations are solved by using their approximate difference equation formulations. The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. It is very interesting to investigate the behavior of solutions of a system of higher-order rational difference equations and to discuss the local asymptotic stability of their equilibrium points. Systems of rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such systems. For more results on the qualitative behavior of nonlinear difference equations, we refer the interested reader to [1–19].

Zhang *et al.* [20] studied the dynamics of a system of rational third-order difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

Din *et al.* [11] investigated the dynamics of a system of fourth-order rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \qquad y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

To be motivated by the above studies, our aim in this paper is to investigate the qualitative behavior of the following $(k+1)th$-order systems of rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} y_{n-i}}, \qquad y_{n+1} = \frac{\alpha_1 y_{n-k}}{\beta_1 + \gamma_1 \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \dots, \tag{1}$$

where the parameters $\alpha, \beta, \gamma, \alpha_1, \beta_1, \gamma_1$ and initial conditions $x_0, x_{-1}, \dots, x_{-k}, y_0, y_{-1}, \dots, y_{-k}$ are positive real numbers, and

$$x_{n+1} = \frac{a y_{n-k}}{b + c \prod_{i=0}^{k} x_{n-i}}, \qquad y_{n+1} = \frac{a_1 x_{n-k}}{b_1 + c_1 \prod_{i=0}^{k} y_{n-i}}, \quad n = 0, 1, \dots, \tag{2}$$

where the parameters a, b, c, a_1, b_1, c_1 and initial conditions $x_0, x_{-1}, \dots, x_{-k}, y_0, y_{-1}, \dots, y_{-k}$ are positive real numbers. This paper is a natural extension of [11, 20, 21].

Let us consider $(2k+2)$-dimensional discrete dynamical system of the form

$$\begin{aligned} x_{n+1} &= f(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), \\ y_{n+1} &= g(x_n, x_{n-1}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}), \quad n = 0, 1, \dots, \end{aligned} \tag{3}$$

where $f : I^{k+1} \times J^{k+1} \to I$ and $g : I^{k+1} \times J^{k+1} \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=-k}^{\infty}$ of system (3) is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-k, -k+1, \dots, -1, 0\}$. Along with system (3), we consider the corresponding vector map $F = (f, x_n, x_{n-1}, \dots, x_{n-k}, g, y_n, y_{n-1}, \dots, y_{n-k})$. An equilibrium point of (3) is a point $(\bar{x}, \bar{y})$ that satisfies

$$\begin{aligned} \bar{x} &= f(\bar{x}, \bar{x}, \dots, \bar{x}, \bar{y}, \bar{y}, \dots, \bar{y}), \\ \bar{y} &= g(\bar{x}, \bar{x}, \dots, \bar{x}, \bar{y}, \bar{y}, \dots, \bar{y}). \end{aligned}$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map F.

Definition 1 Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (3).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-k, -k+1, \dots, -1, 0\}$, $\| \sum_{i=-k}^{0} (x_i, y_i) - (\bar{x}, \bar{y}) \| < \delta$ implies $\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$ for all $n > 0$, where $\| \cdot \|$ is the usual Euclidian norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\bar{x},\bar{y})$ is said to be unstable if it is not stable.
(iii) An equilibrium point $(\bar{x},\bar{y})$ is said to be asymptotically stable if there exists $\eta > 0$ such that $\| \sum_{i=-k}^{0}(x_i,y_i) - (\bar{x},\bar{y})\| < \eta$ and $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$.
(iv) An equilibrium point $(\bar{x},\bar{y})$ is called a global attractor if $(x_n,y_n) \to (\bar{x},\bar{y})$ as $n \to \infty$.
(v) An equilibrium point $(\bar{x},\bar{y})$ is called an asymptotic global attractor if it is a global attractor and stable.

Definition 2 Let $(\bar{x},\bar{y})$ be an equilibrium point of the map

$$F = (f, x_n, x_{n-1}, \ldots, x_{n-k}, g, y_n, y_{n-1}, \ldots, y_{n-k}),$$

where f and g are continuously differentiable functions at $(\bar{x},\bar{y})$. The linearized system of (3) about the equilibrium point $(\bar{x},\bar{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}$$

and F_J is the Jacobian matrix of system (3) about the equilibrium point $(\bar{x},\bar{y})$.

Lemma 1 [22] *Assume that $X_{n+1} = F(X_n)$, $n = 0,1,\ldots,$ is a system of difference equations and $\bar{X}$ is the fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\bar{X}$ lie inside an open unit disk $|\lambda| < 1$, then $\bar{X}$ is locally asymptotically stable. If one of them has norm greater than one, then $\bar{X}$ is unstable.*

Lemma 2 [23] *Assume that $X_{n+1} = F(X_n)$, $n = 0,1,\ldots,$ is a system of difference equations and $\bar{X}$ is the equilibrium point of this system. The characteristic polynomial of this system about the equilibrium point $\bar{X}$ is $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $P(\lambda)$ lie inside the open unit disk $|\lambda|$ if and only if $\Delta_k > 0$ for $k = 0,1,\ldots,$ where Δ_k is the principal minor of order k of the $n \times n$ matrix*

$$\Delta_n = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}. \tag{4}$$

Let us consider a system of difference equations

$$X_{n+1} = \big(A + B(n)\big)X_n, \tag{5}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{6}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x,y)\| = \sqrt{x^2 + y^2}.$$

Proposition 1 (Perron's theorem)[24] *Suppose that condition* (6) *holds. If X_n is a solution of* (5), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \big(\|X_n\|\big)^{1/n} \tag{7}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 2 [24] *Suppose that condition* (6) *holds. If X_n is a solution of* (5), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{8}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2 On the system $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k} y_{n-i}}$, $y_{n+1} = \frac{\alpha_1 y_{n-k}}{\beta_1+\gamma_1\prod_{i=0}^{k} x_{n-i}}$

In this section, we shall investigate the qualitative behavior of system (1). Let $(\bar{x},\bar{y})$ be an equilibrium point of system (1), then for $\alpha > \beta$ and $\alpha_1 > \beta_1$, system (1) has two positive equilibrium points $P_0 = (0,0)$, $P_1 = (A,B)$, where $A = (\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{k+1}}$ and $B = (\frac{\alpha-\beta}{\gamma})^{\frac{1}{k+1}}$.

To construct the corresponding linearized form of system (1), we consider the following transformation:

$$(x_n, x_{n-1}, x_{n-2}, \dots, x_{n-k}, y_n, y_{n-1}, \dots, y_{n-k}) \mapsto (f, f_1, \dots, f_{n-k}, g, g_1, \dots, g_{n-k}), \tag{9}$$

where $f = \frac{\alpha x_{n-k}}{\beta+\gamma\prod_{i=0}^{k} y_{n-i}}$, $f_1 = x_n$, $f_2 = x_{n-1}$, $\dots$, $f_{n-k} = x_{n-(k-1)}$ and $g = \frac{\alpha_1 y_{n-k}}{\beta_1+\gamma_1\prod_{i=0}^{k} x_{n-i}}$, $g_1 = y_n$, $g_2 = y_{n-1}$, $\dots$, $g_{n-k} = y_{n-(k-1)}$. The Jacobian matrix about the fixed point $(\bar{x},\bar{y})$ under the transformation (9) is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} 0 & 0 & \dots & 0 & A & B & B & \dots & B & B \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ C & C & \dots & C & C & 0 & 0 & \dots & 0 & D \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $A = \frac{\alpha}{\beta+\gamma\bar{y}^{k+1}}$, $B = -\frac{\alpha\gamma\bar{x}\bar{y}^k}{(\beta+\gamma\bar{y}^{k+1})^2}$, $C = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}^k}{(\beta_1+\gamma_1\bar{x}^{k+1})^2}$ and $D = \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^{k+1}}$.

Theorem 1 *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then every solution $\{(x_n, y_n)\}$ of system* (1) *is bounded.*

Proof It is easy to verify that

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-k} \quad \text{if } n = (k+1)m + 1,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_{1-k} \quad \text{if } n = (k+1)m + 2,$$

$$\vdots$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_{-1} \quad \text{if } n = (k+1)m + k,$$

$$0 \le x_n \le \left(\frac{\alpha}{\beta}\right)^{m+1} x_0 \quad \text{if } n = (k+1)m + (k+1),$$

and

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-k} \quad \text{if } n = (k+1)m + 1,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{1-k} \quad \text{if } n = (k+1)m + 2,$$

$$\vdots$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_{-1} \quad \text{if } n = (k+1)m + k,$$

$$0 \le y_n \le \left(\frac{\alpha_1}{\beta_1}\right)^{m+1} y_0 \quad \text{if } n = (k+1)m + k + 1.$$

Take $\delta_1 = \max\{x_{-k}, \dots, x_0\}$ and $\delta_2 = \max\{y_{-k}, \dots, y_0\}$. Then $0 \le x_n < \delta_1$ and $0 \le y_n < \delta_2$ for all $n = 0, 1, 2, \dots$. □

Theorem 2 *The equilibrium point P_0 of system* (1) *is locally asymptotically stable.*

Proof The linearized system of (1) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix},$$

and

$$E = F_J(P_0) = (e_{ij})_{(2k+2)\times(2k+2)} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha}{\beta} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{2k+2}$ denote the $2k+2$ eigenvalues of matrix E. Let $D = \operatorname{diag}(d_1, d_2, \dots, d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_{1+m} = d_{k+2+m} = 1 - m\epsilon$, $1 \leq m \leq k$, and

$$\epsilon = \min\left\{\frac{1}{k}, \frac{1}{k}\left(1 - \frac{\alpha}{\beta}\right), \frac{1}{k}\left(1 - \frac{\alpha_1}{\beta_1}\right)\right\}.$$

Clearly, D is invertible. Computing DED^{-1}, we obtain

$$DED^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha}{\beta}d_1 d_{k+1}^{-1} & 0 & 0 & \dots & 0 & 0 \\ d_2 d_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{k+1}d_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_1}{\beta_1}d_{k+2}d_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & d_{k+3}d_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & d_{2k+2}d_{2k+1}^{-1} & 0 \end{pmatrix}.$$

We obtain the following two inequalities:

$$0 < d_{k+1} < d_k < \cdots < d_2, \qquad 0 < d_{2k+2} < d_{2k+1} < \cdots < d_{k+3},$$

which implies that

$$d_2 d_1^{-1} < 1, \qquad d_3 d_2^{-1} < 1, \qquad \ldots, \qquad d_{k+1} d_k^{-1} < 1,$$

and

$$d_{k+3} d_{k+2}^{-1} < 1, \qquad d_{k+4} d_{k+3}^{-1} < 1, \qquad \ldots, \qquad d_{2k+2} d_{2k+1}^{-1} < 1.$$

Furthermore,

$$\frac{\alpha}{\beta} d_1 d_{k+1}^{-1} = \frac{\alpha}{\beta} d_{k+1}^{-1} = \frac{\alpha}{\beta} \frac{1}{1 - k\epsilon} < 1,$$

and

$$\frac{\alpha_1}{\beta_1} d_{k+2} d_{2k+2}^{-1} = \frac{\alpha_1}{\beta_1} d_{2k+2}^{-1} = \frac{\alpha_1}{\beta_1} \frac{1}{1 - k\epsilon} < 1.$$

It is a well-known fact that E has the same eigenvalues as DED^{-1}. Hence, we obtain

$$\begin{aligned} &\max_{1 \le m \le 2k+2} |\lambda_m| \\ &\quad = \| DED^{-1} \| \\ &\quad = \max \left\{ d_2 d_1^{-1}, \ldots, d_{k+1} d_k^{-1}, d_{k+3} d_{k+2}^{-1}, \ldots, d_{2k+2} d_{2k+1}^{-1}, \frac{\alpha}{\beta} d_1 d_{k+1}^{-1}, \frac{\alpha_1}{\beta_1} d_{k+2} d_{2k+2}^{-1} \right\} < 1. \end{aligned}$$

Hence, the equilibrium point P_0 of system (1) is locally asymptotically stable. □

Theorem 3 *The positive equilibrium point P_1 of system* (1) *is unstable.*

Proof The linearized system of (1) about the equilibrium point P_1 is given by

$$X_{n+1} = F_J(P_1) X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}$$

and

$$F_J(P_1) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & L & L & \dots & L & L \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ M & M & \dots & M & M & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $L = -(\frac{\gamma}{\gamma_1})^{\frac{1}{k+1}} \frac{(\alpha_1-\beta_1)^{\frac{1}{k+1}}(\alpha-\beta)^{\frac{k}{k+1}}}{\alpha}$ and $M = -(\frac{\gamma_1}{\gamma})^{\frac{1}{k+1}} \frac{(\alpha-\beta)^{\frac{1}{k+1}}(\alpha_1-\beta_1)^{\frac{k}{k+1}}}{\alpha_1}$. The characteristic polynomial of $F_J(P_1)$ is given by

$$\begin{aligned} P(\lambda) = \lambda^{2k+2} - LM\big[\lambda^{2k} + 2\lambda^{2k-1} + \cdots + k\lambda^{k+1} + (k+1)\lambda^k \\ + k\lambda^{k-1} + (k-1)\lambda^{k-2} + \cdots + 1\big] + 2\lambda^{k+1} + 1. \end{aligned} \tag{10}$$

From (10), we have

$$\Delta_{(2k+2)\times(2k+2)} = \begin{pmatrix} -LM & -3LM & -5LM & \dots & 0 \\ 1 & -2LM & -4LM & \dots & 0 \\ 0 & -LM & -3LM & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1-LM \end{pmatrix}. \tag{11}$$

It is clear that not all of $\Delta_{(2k+2)\times(2k+2)} > 0$. Therefore, by Lemma 1, the unique positive equilibrium point $(\bar{x},\bar{y}) = ((\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{k+1}}, (\frac{\alpha-\beta}{\gamma})^{\frac{1}{k+1}})$ is unstable. □

Theorem 4 *Let $\alpha > \beta$ and $\alpha_1 > \beta_1$, and let $\{(x_n,y_n)\}$ be a solution of system* (1). *Then, for $i = 0,1,\dots,k$, the following statements are true*:

(i) *If $(x_i,y_i) \in (0,(\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{k+1}}) \times ((\frac{\alpha-\beta}{\gamma})^{\frac{1}{k+1}},\infty)$, then*

$$(x_n,y_n) \in \left(0,\left(\frac{\alpha_1-\beta_1}{\gamma_1}\right)^{\frac{1}{k+1}}\right) \times \left(\left(\frac{\alpha-\beta}{\gamma}\right)^{\frac{1}{k+1}},\infty\right).$$

(ii) *If $(x_i,y_i) \in ((\frac{\alpha_1-\beta_1}{\gamma_1})^{\frac{1}{k+1}},\infty) \times (0,(\frac{\alpha-\beta}{\gamma})^{\frac{1}{k+1}})$, then*

$$(x_n,y_n) \in \left(\left(\frac{\alpha_1-\beta_1}{\gamma_1}\right)^{\frac{1}{k+1}},\infty\right) \times \left(0,\left(\frac{\alpha-\beta}{\gamma}\right)^{\frac{1}{k+1}}\right).$$

Proof It follows from induction. □

Theorem 5 *Let $\alpha < \beta$ and $\alpha_1 < \beta_1$, then the equilibrium point P_0 of system* (1) *is globally asymptotically stable*.

Proof For $\alpha < \beta$ and $\alpha_1 < \beta_1$, from Theorem 2, P_0 is locally asymptotically stable. From Theorem 1, every positive solution (x_n,y_n) is bounded, *i.e.*, $0 \le x_n \le \mu$ and $0 \le y_n \le \nu$ for all $n = 0, 1, 2, \ldots$, where $\mu = \max\{x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0\}$ and $\nu = \max\{y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0\}$. So, it is sufficient to prove that $\{(x_n, y_n)\}$ is decreasing. From system (1), one has

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} y_{n-i}} \le \frac{\alpha x_{n-k}}{\beta} < x_{n-k}.$$

This implies that $x_{(k+1)n+1} < x_{(k+1)n-k}$ and $x_{(k+1)n+(k+2)} < x_{(k+1)n+1}$. Hence, the subsequences

$$\{x_{(k+1)n+1}\}, \qquad \{x_{(k+1)n+2}\}, \qquad \ldots, \qquad \{x_{(k+1)n+k}\}, \qquad \{x_{(k+1)n+(k+1)}\}$$

are decreasing, *i.e.*, the sequence $\{x_n\}$ is decreasing. Also,

$$y_{n+1} = \frac{\alpha y_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}} \le \frac{\alpha y_{n-k}}{\beta} < y_{n-k}.$$

This implies that $y_{(k+1)n+1} < y_{(k+1)n-k}$ and $y_{(k+1)n+(k+2)} < y_{(k+1)n+1}$. Hence, the subsequences

$$\{y_{(k+1)n+1}\}, \qquad \{y_{(k+1)n+2}\}, \qquad \ldots, \qquad \{y_{(k+1)n+k}\}, \qquad \{y_{(k+1)n+(k+1)}\}$$

are decreasing, *i.e.*, the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. $\square$

Theorem 6 *Let $\alpha > \beta$ and $\alpha_1 > \beta_1$. Then, for a solution $\{(x_n, y_n)\}$ of system* (1), *the following statements are true*:

(i) *If $x_n \to 0$, then $y_n \to \infty$.*
(ii) *If $y_n \to 0$, then $x_n \to \infty$.*

2.1 Rate of convergence

We investigate the rate of convergence of a solution that converges to the equilibrium point P_0 of system (1).

Assume that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} x_n = \bar{y}$. First we will find a system of limiting equations for the map F. The error terms are given as

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} A_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{k} B_i (y_{n-i} - \bar{y}),$$

$$y_{n+1} - \bar{y} = \sum_{i=0}^{k} C_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{k} D_i (y_{n-i} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, one has

$$e_{n+1}^1 = \sum_{i=0}^{k} A_i e_{n-i}^1 + \sum_{i=0}^{k} B_i e_{n-i}^2, \qquad e_{n+1}^2 = \sum_{i=0}^{k} C_i e_{n-i}^1 + \sum_{i=0}^{k} D_i e_{n-i}^2,$$

where $A_i = 0$ for $i \in \{0, 1, \ldots, k-1\}$,

$$A_k = \frac{\alpha}{\beta + \gamma \prod_{i=0}^{k} y_{n-i}}, \qquad B_0 = -\frac{\alpha\gamma \bar{x} y_{n-1} y_{n-2} \cdots y_{n-k}}{(\beta + \gamma \prod_{i=0}^{k} y_{n-i})(\beta + \gamma \bar{y}^{k+1})},$$

$$B_1 = -\frac{\alpha\gamma \bar{x}\bar{y} y_{n-2} y_{n-3} \cdots y_{n-k}}{(\beta + \gamma \prod_{i=0}^{k} y_{n-i})(\beta + \gamma \bar{y}^{k+1})}, \qquad B_2 = -\frac{\alpha\gamma \bar{x}\bar{y}^2 y_{n-3} \cdots y_{n-k}}{(\beta + \gamma \prod_{i=0}^{k} y_{n-i})(\beta + \gamma \bar{y}^{k+1})}, \qquad \ldots,$$

$$B_{k-1} = -\frac{\alpha\gamma\bar{x}\bar{y}^{k-1}y_{n-k}}{(\beta+\gamma\prod_{i=0}^{k}y_{n-i})(\beta+\gamma\bar{y}^{k+1})}, \qquad B_k = -\frac{\alpha\gamma\bar{x}\bar{y}^{k}}{(\beta+\gamma\prod_{i=0}^{k}y_{n-i})(\beta+\gamma\bar{y}^{k+1})},$$

$$C_0 = -\frac{\alpha_1\gamma_1\bar{y}x_{n-1}x_{n-2}\cdots x_{n-k}}{(\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i})(\beta_1+\gamma_1\bar{x}^{k+1})},$$

$$C_1 = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}x_{n-2}x_{n-3}\cdots x_{n-k}}{(\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i})(\beta_1+\gamma_1\bar{x}^{k+1})},$$

$$C_2 = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}^2x_{n-3}\cdots x_{n-k}}{(\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i})(\beta_1+\gamma_1\bar{x}^{k+1})}, \qquad \ldots,$$

$$C_{k-1} = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}^{k-1}x_{n-k}}{(\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i})(\beta_1+\gamma_1\bar{x}^{k+1})}, \qquad C_k = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}^{k}}{(\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i})(\beta_1+\gamma_1\bar{x}^{k+1})},$$

$$D_i = 0 \quad \text{for } i \in \{0,1,\ldots,k-1\} \quad \text{and} \quad D_k = \frac{\alpha_1}{\beta_1+\gamma_1\prod_{i=0}^{k}x_{n-i}}.$$

Taking the limits, we obtain $\lim_{n\to\infty} A_i = 0$ for $i \in \{0,1,\ldots,k-1\}$, $\lim_{n\to\infty} A_k = \frac{\alpha}{\beta+\gamma\bar{y}^{k+1}}$, $\lim_{n\to\infty} B_i = -\frac{\alpha\gamma\bar{x}\bar{y}^k}{(\beta+\gamma\bar{y}^{k+1})^2}$ for $i \in \{0,1,\ldots,k\}$, $\lim_{n\to\infty} C_i = -\frac{\alpha_1\gamma_1\bar{y}\bar{x}^k}{(\beta_1+\gamma_1\bar{x}^{k+1})^2}$ for $i \in \{0,1,\ldots,k\}$, $\lim_{n\to\infty} D_i = 0$ for $i \in \{0,1,\ldots,k-1\}$ and $\lim_{n\to\infty} D_k = \frac{\alpha_1}{\beta_1+\gamma_1\bar{x}^{k+1}}$. Hence, the limiting system of error terms at $(\bar{x},\bar{y}) = (0,0)$ can be written as

$$E_{n+1} = KE_n, \tag{12}$$

where

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-k}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-k}^2 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{\alpha}{\beta} & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & \frac{\alpha_1}{\beta_1} \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

which is similar to the linearized system of (1) about the equilibrium point $(\bar{x},\bar{y}) = (0,0)$. Using proposition (1), one has the following result.

Theorem 7 *Assume that* $\{(x_n,y_n)\}$ *is a positive solution of system* (1) *such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $(\bar{x},\bar{y}) = (0,0)$. *Then the error vector* E_n *of every solution of* (1) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty}\left(\|E_n\|\right)^{\frac{1}{n}} = \left|\lambda F_J(\bar{x},\bar{y})\right|, \qquad \lim_{n\to\infty}\frac{\|E_{n+1}\|}{\|E_n\|} = \left|\lambda F_J(\bar{x},\bar{y})\right|,$$

where $\lambda F_J(\bar{x},\bar{y})$ *are the characteristic roots of the Jacobian matrix* $F_J(\bar{x},\bar{y})$ *about* $(0,0)$.

3 On the system $x_{n+1} = \frac{ay_{n-k}}{b+c\prod_{i=0}^{k}x_{n-i}}$, $y_{n+1} = \frac{a_1x_{n-k}}{b_1+c_1\prod_{i=0}^{k}y_{n-i}}$

In this section, we shall investigate the qualitative behavior of system (2). Let $(\bar{x},\bar{y})$ be an equilibrium point of system (2), then system (2) has a unique equilibrium point $(0,0)$. To construct the corresponding linearized form of system (2), we consider the following transformation:

$$(x_n,x_{n-1},x_{n-2},\dots,x_{n-k},y_n,y_{n-1},\dots,y_{n-k}) \mapsto (f,f_1,\dots,f_{n-k},g,g_1,\dots,g_{n-k}), \tag{13}$$

$f = \frac{ay_{n-k}}{b+c\prod_{i=0}^{k}x_{n-i}}$, $f_1 = x_n, f_2 = x_{n-1},\dots,f_{n-k} = x_{n-(k-1)}$ and $g = \frac{a_1x_{n-k}}{b_1+c_1\prod_{i=0}^{k}y_{n-i}}$, $g_1 = y_n, g_2 = y_{n-1},\dots,$ $g_{n-k} = y_{n-(k-1)}$. The Jacobian matrix about the fixed point $(\bar{x},\bar{y})$ under the transformation (13) is given by

$$F_J(\bar{x},\bar{y}) = \begin{pmatrix} A & A & \dots & A & A & 0 & 0 & \dots & 0 & B \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & C & D & D & \dots & D & D \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

where $A = -\frac{ac\bar{y}\bar{x}^k}{(b+c\bar{x}^{k+1})^2}$, $B = \frac{a}{b+c\bar{x}^{k+1}}$, $C = \frac{a_1}{b_1+c_1\bar{y}^{k+1}}$ and $D = -\frac{a_1c_1\bar{x}\bar{y}^k}{(b_1+c_1\bar{y}^{k+1})^2}$.

Theorem 8 *Let* $\{(x_n,y_n)\}$ *be a positive solution of system* (2), *then for every* $m \geq 1$, *the following results hold*.

$$\text{(i)}\quad 0 \leq x_n \leq \begin{cases} (\frac{a}{b})^{m+1}(\frac{a_1}{b_1})^m y_{-k} & \textit{if } n = (2k+2)m+1, \\ (\frac{a}{b})^{m+1}(\frac{a_1}{b_1})^m y_{1-k} & \textit{if } n = (2k+2)m+2, \\ \vdots & \\ (\frac{a}{b})^{m+1}(\frac{a_1}{b_1})^m y_{-1} & \textit{if } n = (2k+2)m+k, \\ (\frac{a}{b})^{m+1}(\frac{a_1}{b_1})^m y_0 & \textit{if } n = (2k+2)m+(k+1), \\ (\frac{aa_1}{bb_1})^{m+1} x_{-k} & \textit{if } n = (2k+2)m+(k+2), \\ (\frac{aa_1}{bb_1})^{m+1} x_{1-k} & \textit{if } n = (2k+2)m+(k+3), \\ \vdots & \\ (\frac{aa_1}{bb_1})^{m+1} x_{-1} & \textit{if } n = (2k+2)m+(2k+1), \\ (\frac{aa_1}{bb_1})^{m+1} x_0 & \textit{if } n = (2k+2)m+(2k+2). \end{cases}$$

$$\text{(ii)}\quad 0 \le y_n \le \begin{cases} (\frac{a}{b})^m(\frac{a_1}{b_1})^{m+1}x_{-k} & \textit{if } n = (2k+2)m+1, \\ (\frac{a}{b})^m(\frac{a_1}{b_1})^{m+1}x_{1-k} & \textit{if } n = (2k+2)m+2, \\ \vdots & \\ (\frac{a}{b})^m(\frac{a_1}{b_1})^{m+1}x_{-1} & \textit{if } n = (2k+2)m+k, \\ (\frac{a}{b})^m(\frac{a_1}{b_1})^{m+1}x_{0} & \textit{if } n = (2k+2)m+(k+1), \\ (\frac{aa_1}{bb_1})^{m+1}y_{-k} & \textit{if } n = (2k+2)m+(k+2), \\ (\frac{aa_1}{bb_1})^{m+1}y_{1-k} & \textit{if } n = (2k+2)m+(k+3), \\ \vdots & \\ (\frac{aa_1}{bb_1})^{m+1}y_{-1} & \textit{if } n = (2k+2)m+(2k+1), \\ (\frac{aa_1}{bb_1})^{m+1}y_{0} & \textit{if } n = (2k+2)m+(2k+2). \end{cases}$$

Lemma 3 *Let* $\frac{aa_1}{bb_1} < 1$, *then every solution* $\{x_n, y_n\}_{n=-k}^{\infty}$ *of system* (2) *is bounded.*

Proof Assume that

$$\lambda_1 = \max\left\{\frac{b_1}{a_1}y_{-k}, \frac{b_1}{a_1}y_{-k+1}, \ldots, \frac{b_1}{a_1}y_0, x_{-k}, x_{-k+1}, \ldots, x_0\right\},$$

and

$$\lambda_2 = \max\left\{\frac{b}{a}x_{-k}, \frac{b}{a}x_{-k+1}, \ldots, \frac{b}{a}x_0, y_{-k}, y_{-k+1}, \ldots, y_0\right\}.$$

Then from Theorem 8 one can easily see that $0 \le x_n < \lambda_1$ and $0 \le y_n < \lambda_2$ for all $n = 0, 1, \ldots$. □

Theorem 9 *The equilibrium point* $(0,0)$ *of equation* (2) *is locally asymptotically stable.*

Proof The linearized system of (2) about the equilibrium point $(0,0)$ is given by

$$X_{n+1} = F_J(0,0)X_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \\ y_n \\ y_{n-1} \\ \vdots \\ y_{n-k} \end{pmatrix}$$

and

$$H = F_J(0,0) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{a}{b} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{a_1}{b_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{2k+2}$ denote the $2k+2$ eigenvalues of matrix E. Let $D = \operatorname{diag}(d_1, d_2, \dots, d_{2k+2})$ be a diagonal matrix, where $d_1 = d_{k+2} = 1$, $d_{1+m} = d_{k+2+m} = 1 - m\epsilon$, $1 \le m \le k$ and

$$\epsilon = \min\left\{\frac{1}{k}, \frac{1}{k}\left(1 - \frac{a}{b}\right), \frac{1}{k}\left(1 - \frac{a_1}{b_1}\right)\right\}.$$

Clearly, D is invertible. Computing DHD^{-1}, we obtain

$$DHD^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{a}{b}d_1d_{2k+2}^{-1} \\ d_2d_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{k+1}d_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{a_1}{b_1}d_{k+2}d_{k+1}^{-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & d_{k+3}d_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & d_{2k+2}d_{2k+1}^{-1} & 0 \end{pmatrix}.$$

Next, we have the following two inequalities:

$$0 < d_{k+1} < d_k < \cdots < d_2, \qquad 0 < d_{2k+2} < d_{2k+1} < \cdots < d_{k+3},$$

which implies that

$$d_2d_1^{-1} < 1, \qquad d_3d_2^{-1} < 1, \qquad \dots, \qquad d_{k+1}d_k^{-1} < 1,$$

and

$$d_{k+3}d_{k+2}^{-1} < 1, \qquad d_{k+4}d_{k+3}^{-1} < 1, \qquad \dots, \qquad d_{2k+2}d_{2k+1}^{-1} < 1.$$

Furthermore,

$$\frac{a}{b}d_1d_{2k+2}^{-1} = \frac{a}{b}d_{2k+2}^{-1} = \frac{a}{b}\frac{1}{1-k\epsilon} < 1,$$

and

$$\frac{a_1}{b_1}d_{k+2}d_{k+1}^{-1} = \frac{a_1}{b_1}d_{k+1}^{-1} = \frac{a_1}{b_1}\frac{1}{1-k\epsilon} < 1.$$

Now H has the same eigenvalues as DHD^{-1}, we obtain that

$$\begin{aligned}
&\max_{1\le m\le 2k+2}|\lambda_m| \\
&\quad = \|DHD^{-1}\| \\
&\quad = \max\left\{d_2d_1^{-1},\dots,d_{k+1}d_k^{-1},d_{k+3}d_{k+2}^{-1},\dots,d_{2k+2}d_{2k+1}^{-1},\frac{a}{b}d_1d_{2k+2}^{-1},\frac{a_1}{b_1}d_{k+2}d_{k+1}^{-1}\right\} < 1.
\end{aligned}$$

Hence, the equilibrium point $(0,0)$ of system (2) is locally asymptotically stable. □

Theorem 10 *Let $a < b$ and $a_1 < b_1$, then the equilibrium point* $(0,0)$ *of system* (2) *is globally asymptotically stable*.

Proof Assume that $a < b$ and $a_1 < b_1$. Then from Theorem 9 the equilibrium point $(0,0)$ of system (2) is locally asymptotically stable. Moreover, from Lemma 3 every positive solution (x_n,y_n) is bounded, *i.e.*, $0 \le x_n \le \mu$ and $0 \le y_n \le \nu$ for all $n = 0,1,2,\dots$, where $\mu = \max\{x_{-k},x_{-k+1},\dots,x_0\}$ and $\nu = \max\{y_{-k},y_{-k+1},\dots,y_0\}$. Now, it is sufficient to prove that (x_n,y_n) is decreasing. From system (2) one has

$$\begin{aligned}
x_{n+1} &= \frac{ay_{n-k}}{b+c\prod_{i=0}^{k}x_{n-i}} \\
&\le \frac{ay_{n-k}}{b} < y_{n-k}.
\end{aligned}$$

This implies that $x_{(2k+2)n+1} < y_{(2k+2)n-k}$ and $x_{(2k+2)n+(2k+3)} < y_{(2k+2)n+(k+2)}$.

$$\begin{aligned}
y_{n+1} &= \frac{a_1x_{n-k}}{b_1+c_1\prod_{i=0}^{k}y_{n-i}} \\
&\le \frac{ax_{n-4}}{b} < x_{n-k}.
\end{aligned}$$

This implies that

$$y_{(2k+2)n+1} < x_{(2k+2)n-k} \quad \text{and} \quad y_{(2k+2)n+(2k+3)} < x_{(2k+2)n+(k+2)}.$$

Hence, $x_{(2k+2)n+(2k+3)} < y_{(2k+2)n+(k+2)} < x_{(2k+2)n+1}$ and $y_{(2k+2)n+(2k+3)} < x_{(2k+2)n+(k+2)} < y_{(2k+2)n+1}$. Hence, the subsequences

$$\{x_{(2k+2)n+1}\}, \qquad \{x_{(2k+2)n+2}\}, \qquad \dots, \qquad \{x_{(2k+2)n+(2k+2)}\}$$

and

$$\{y_{(2k+2)n+1}\}, \qquad \{y_{(2k+2)n+2}\}, \qquad \dots, \qquad \{y_{(2k+2)n+(2k+2)}\}$$

are decreasing. Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. □

3.1 Rate of convergence

Assume that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$. First we will find a system of limiting equations for system (2). The error terms are given as

$$x_{n+1} - \bar{x} = \sum_{i=0}^{k} A_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{k} B_i(y_{n-i} - \bar{y}),$$

$$y_{n+1} - \bar{y} = \sum_{i=0}^{k} C_i(x_{n-i} - \bar{x}) + \sum_{i=0}^{k} D_i(y_{n-i} - \bar{y}).$$

Set $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = \sum_{i=0}^{k} A_i e_{n-i}^1 + \sum_{i=0}^{k} B_i e_{n-i}^2,$$

$$e_{n+1}^2 = \sum_{i=0}^{k} C_i e_{n-i}^1 + \sum_{i=0}^{k} D_i e_{n-i}^2,$$

where

$$A_0 = -\frac{ac\bar{y}x_{n-1}x_{n-2}\cdots x_{n-k}}{(b + c\prod_{i=0}^{k} x_{n-i})(b + c\bar{x}^{k+1})}, \qquad A_1 = -\frac{ac\bar{x}\bar{y}x_{n-2}x_{n-3}\cdots x_{n-k}}{(b + c\prod_{i=0}^{k} x_{n-i})(b + c\bar{x}^{k+1})}, \qquad \ldots,$$

$$A_{k-1} = -\frac{ac\bar{x}^{k-1}\bar{y}x_{n-k}}{(b + c\prod_{i=0}^{k} x_{n-i})(b + c\bar{x}^{k+1})}, \qquad A_k = -\frac{ac\bar{x}^{k}\bar{y}}{(b + c\prod_{i=0}^{k} x_{n-i})(b + c\bar{x}^{k+1})},$$

$B_i = 0$ for $i \in \{0, 1, \ldots, k-1\}$, $B_k = \frac{a}{b+c\prod_{i=0}^{k} x_{n-i}}$, $C_i = 0$ for $i \in \{0, 1, \ldots, k-1\}$,

$$C_k = \frac{a_1}{b_1 + c_1\prod_{i=0}^{k} y_{n-i}}, \qquad D_0 = -\frac{a_1c_1\bar{x}y_{n-1}y_{n-2}\cdots y_{n-k}}{(b_1 + c_1\prod_{i=0}^{k} y_{n-i})(b_1 + c_1\bar{y}^{k+1})},$$

$$D_1 = -\frac{a_1c_1\bar{x}\bar{y}y_{n-2}y_{n-3}\cdots y_{n-k}}{(b_1 + c_1\prod_{i=0}^{k} y_{n-i})(b_1 + c_1\bar{y}^{k+1})}, \qquad \ldots,$$

$$D_{k-1} = -\frac{a_1c_1\bar{x}\bar{y}^{k-1}y_{n-k}}{(b_1 + c_1\prod_{i=0}^{k} y_{n-i})(b_1 + c_1\bar{y}^{k+1})},$$

and $D_k = -\frac{a_1c_1\bar{x}\bar{y}^k}{(b_1+c_1\prod_{i=0}^{k} y_{n-i})(b_1+c_1\bar{y}^{k+1})}$. Taking the limits, we obtain $\lim_{n\to\infty} A_i = -\frac{ac\bar{y}\bar{x}^k}{(b+c\bar{x}^{k+1})^2}$ for $i \in \{0, 1, \ldots, k\}$, $\lim_{n\to\infty} B_i = 0$ for $i \in \{0, 1, \ldots, k-1\}$, $\lim_{n\to\infty} B_k = \frac{a}{b+c\bar{x}^{k+1}}$, $\lim_{n\to\infty} C_i = 0$ for $i \in \{0, 1, \ldots, k-1\}$, $\lim_{n\to\infty} C_k = \frac{a_1}{b_1+c_1\bar{y}^{k+1}}$, $\lim_{n\to\infty} D_i = -\frac{a_1c_1\bar{x}\bar{y}^k}{(b_1+c_1\bar{y}^{k+1})^2}$ for $i \in \{0, 1, \ldots, k\}$, So, the limiting system of error terms can be written as

$$E_{n+1} = KE_n,$$

where

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-k}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-k}^2 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{a}{b} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{a_1}{b_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

which is similar to the linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y}) = (0, 0)$. Using proposition (1), one has the following result.

Theorem 11 *Assume that* $\{(x_n, y_n)\}$ *is a positive solution of system* (2) *such that* $\lim_{n\to\infty} x_n = \bar{x}$, *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where* $(\bar{x}, \bar{y}) = (0, 0)$. *Then the error vector* E_n *of every solution of* (2) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty} \left(\|E_n\|\right)^{\frac{1}{n}} = \left|\lambda F_J(\bar{x}, \bar{y})\right|, \qquad \lim_{n\to\infty} \frac{\|E_{n+1}\|}{\|E_n\|} = \left|\lambda F_J(\bar{x}, \bar{y})\right|,$$

where $\lambda F_J(\bar{x}, \bar{y})$ *are the characteristic roots of the Jacobian matrix* $F_J(\bar{x}, \bar{y})$ *about* $(\bar{x}, \bar{y}) = (0, 0)$.

4 Examples

In order to verify our theoretical results, we consider some interesting numerical examples in this section. These examples show that the equilibrium point $(0, 0)$ of both systems (1) and (2) is globally asymptotically stable.

Example 1 Consider system (1) with initial conditions $x_{-8} = 1.9$, $x_{-7} = 1.7$, $x_{-6} = 2.5$, $x_{-5} = 0.9$, $x_{-4} = 1.5$, $x_{-3} = 10.4$, $x_{-2} = 6.9$, $x_{-1} = 0.6$, $x_0 = 2.9$, $y_{-8} = 2.8$, $y_{-7} = 1.6$, $y_{-6} = 1.8$, $y_{-5} = 2.6$, $y_{-4} = 2.8$, $y_{-3} = 2.8$, $y_{-2} = 3.5$, $y_{-1} = 2.1$, $y_0 = 1.6$. Moreover, choose the parameters $\alpha = 15.5$, $\beta = 17$, $\gamma = 27$, $\alpha_1 = 11.2$, $\beta_1 = 12$, $\gamma_1 = 23$. Then system (1) can be written as

$$x_{n+1} = \frac{15.5 x_{n-8}}{17 + 27 \prod_{i=0}^{8} y_{n-i}}, \qquad y_{n+1} = \frac{11.2 y_{n-8}}{12 + 23 \prod_{i=0}^{8} x_{n-i}}, \quad n = 0, 1, \dots, \tag{14}$$

with initial conditions $x_{-8} = 1.9$, $x_{-7} = 1.7$, $x_{-6} = 2.5$, $x_{-5} = 0.9$, $x_{-4} = 1.5$, $x_{-3} = 10.4$, $x_{-2} = 6.9$, $x_{-1} = 0.6$, $x_0 = 2.9$, $y_{-8} = 2.8$, $y_{-7} = 1.6$, $y_{-6} = 1.8$, $y_{-5} = 2.6$, $y_{-4} = 2.8$, $y_{-3} = 2.8$, $y_{-2} = 3.85$, $y_{-1} = 2.1$, $y_0 = 1.6$. Moreover, in Figure 1, the plot of x_n is shown in Figure 1a, the plot of y_n is shown in Figure 1b, and an attractor of system (14) is shown in Figure 1c.

Example 2 Consider system (1) with initial conditions $x_{-15} = 0.9$, $x_{-14} = 1.9$, $x_{-13} = 0.8$, $x_{-12} = 0.3$, $x_{-11} = 1.7$, $x_{-10} = 2.9$, $x_{-9} = 1.9$, $x_{-8} = 2.9$, $x_{-7} = 0.3$, $x_{-6} = 3.5$, $x_{-5} = 1.4$, $x_{-4} = 1.5$, $x_{-3} = 13.4$, $x_{-2} = 17.9$, $x_{-1} = 11.7$, $x_0 = 9.8$, $y_{-15} = 2.9$, $y_{-14} = 3.8$, $y_{-13} = 0.9$, $y_{-12} = 0.8$, $y_{-11} = 1.1$, $y_{-10} = 3.9$, $y_{-9} = 3.6$, $y_{-8} = 1.9$, $y_{-7} = 3.7$, $y_{-6} = 2.9$, $y_{-5} = 1.5$, $y_{-4} = 0.9$, $y_{-3} = 1.8$, $y_{-2} = 3.9$, $y_{-1} = 6.7$, $y_0 = 2.9$. Moreover, choose the parameters $\alpha = 120$, $\beta = 125$, $\gamma = 11.5$, $\alpha_1 = 140$, $\beta_1 = 145$, $\gamma_1 = 14.5$. Then system (1) can be written as

$$x_{n+1} = \frac{120x_{n-15}}{125 + 11.5\prod_{i=0}^{15} y_{n-i}}, \qquad y_{n+1} = \frac{140y_{n-15}}{145 + 14.5\prod_{i=0}^{15} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{15}$$

with initial conditions $x_{-15} = 0.9$, $x_{-14} = 1.9$, $x_{-13} = 0.8$, $x_{-12} = 0.3$, $x_{-11} = 1.7$, $x_{-10} = 2.9$, $x_{-9} = 1.9$, $x_{-8} = 2.9$, $x_{-7} = 0.3$, $x_{-6} = 3.5$, $x_{-5} = 1.4$, $x_{-4} = 1.5$, $x_{-3} = 13.4$, $x_{-2} = 17.9$, $x_{-1} = 11.7$, $x_0 = 9.8$, $y_{-15} = 2.9$, $y_{-14} = 3.8$, $y_{-13} = 0.9$, $y_{-12} = 0.8$, $y_{-11} = 1.1$, $y_{-10} = 3.9$, $y_{-9} = 3.6$, $y_{-8} = 1.9$, $y_{-7} = 3.7$, $y_{-6} = 2.9$, $y_{-5} = 1.5$, $y_{-4} = 0.9$, $y_{-3} = 1.8$, $y_{-2} = 3.9$, $y_{-1} = 6.7$, $y_0 = 2.9$. Moreover, in Figure 2, the plot of x_n is shown in Figure 2a, the plot of y_n is shown in Figure 2b, and an attractor of system (15) is shown in Figure 2c.

Example 3 Consider system (2) with initial conditions $x_{-6} = 0.9$, $x_{-5} = 2.9$, $x_{-4} = 1.6$, $x_{-3} = 9.4$, $x_{-2} = 6.9$, $x_{-1} = 2.8$, $x_0 = 1.9$, $y_{-6} = 1.9$, $y_{-5} = 2.7$, $y_{-4} = 2.9$, $y_{-3} = 2.8$, $y_{-2} = 3.9$, $y_{-1} = 2.4$, $y_0 = 0.8$. Moreover, choose the parameters $a = 115$, $b = 117$, $c = 27$, $a_1 = 111$, $b_1 = 112$, $c_1 = 23$. Then system (2) can be written as

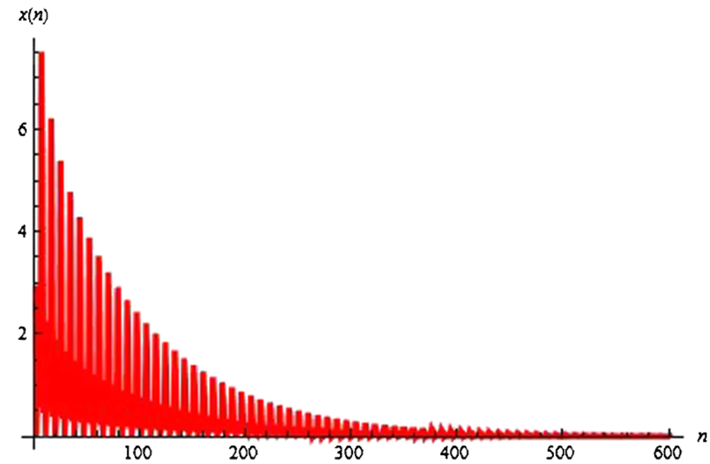
$$x_{n+1} = \frac{115y_{n-6}}{117 + 27\prod_{i=0}^{6} x_{n-i}}, \qquad y_{n+1} = \frac{111x_{n-6}}{112 + 23\prod_{i=0}^{6} y_{n-i}}, \quad n = 0, 1, \ldots, \tag{16}$$

with initial conditions $x_{-6} = 0.9$, $x_{-5} = 2.9$, $x_{-4} = 1.6$, $x_{-3} = 9.4$, $x_{-2} = 6.9$, $x_{-1} = 2.8$, $x_0 = 1.9$, $y_{-6} = 1.9$, $y_{-5} = 2.7$, $y_{-4} = 2.9$, $y_{-3} = 2.8$, $y_{-2} = 3.9$, $y_{-1} = 2.4$, $y_0 = 0.8$. Moreover, in Figure 3, the plot of x_n is shown in Figure 3a, the plot of y_n is shown in Figure 3b, and an attractor of system (16) is shown in Figure 3c.

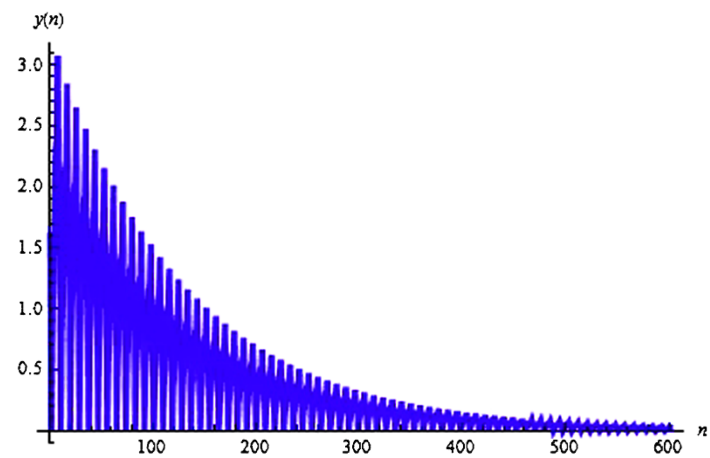
Example 4 Consider system (2) with initial conditions $x_{-12} = 7.1$, $x_{-11} = 2.8$, $x_{-10} = 19.1$, $x_{-9} = 20.9$, $x_{-8} = 9.9$, $x_{-7} = 13.4$, $x_{-6} = 1.9$, $x_{-5} = 10.9$, $x_{-4} = 12.9$, $x_{-3} = 0.9$, $x_{-2} = 1.9$, $x_{-1} = 15.8$, $x_0 = 4.9$, $y_{-12} = 10.9$, $y_{-11} = 1.1$, $y_{-10} = 3.7$, $y_{-9} = 0.6$, $y_{-8} = 1.7$, $y_{-7} = 1.7$, $y_{-6} = 1.9$, $y_{-5} = 1.9$, $y_{-4} = 1.9$, $y_{-3} = 2.8$, $y_{-2} = 17.9$, $y_{-1} = 27.4$, $y_0 = 0.04$. Moreover, choose the parameters $a = 121$, $b = 125$, $c = 9$, $a_1 = 129$, $b_1 = 130$, $c_1 = 7$. Then system (2) can be written as

$$x_{n+1} = \frac{121y_{n-12}}{125 + 9\prod_{i=0}^{12} x_{n-i}}, \qquad y_{n+1} = \frac{129x_{n-12}}{130 + 7\prod_{i=0}^{12} y_{n-i}}, \quad n = 0, 1, \ldots, \tag{17}$$

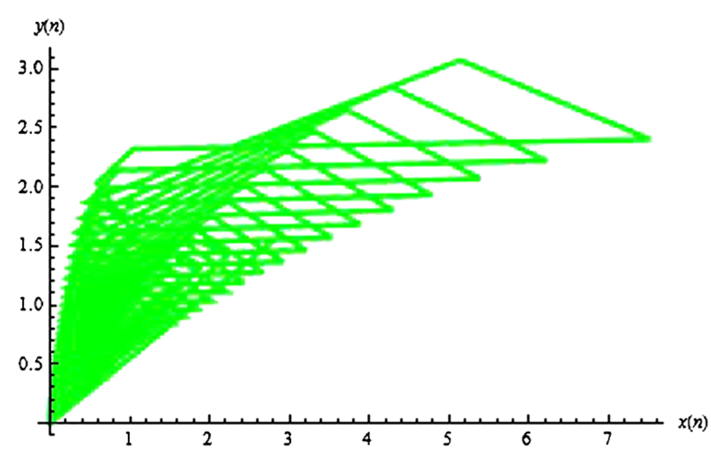
with initial conditions $x_{-12} = 7.1$, $x_{-11} = 2.8$, $x_{-10} = 19.1$, $x_{-9} = 20.9$, $x_{-8} = 9.9$, $x_{-7} = 13.4$, $x_{-6} = 1.9$, $x_{-5} = 10.9$, $x_{-4} = 12.9$, $x_{-3} = 0.9$, $x_{-2} = 1.9$, $x_{-1} = 15.8$, $x_0 = 4.9$, $y_{-12} = 10.9$, $y_{-11} = 1.1$, $y_{-10} = 3.7$, $y_{-9} = 0.6$, $y_{-8} = 1.7$, $y_{-7} = 1.7$, $y_{-6} = 1.9$, $y_{-5} = 1.9$, $y_{-4} = 1.9$, $y_{-3} = 2.8$, $y_{-2} = 17.9$, $y_{-1} = 27.4$, $y_0 = 0.04$. Moreover, in Figure 4, the plot of x_n is shown in Figure 4a, the plot of y_n is shown in Figure 4b, and an attractor of system (17) is shown in Figure 4c.



(a) Plot of x_n for system (14).

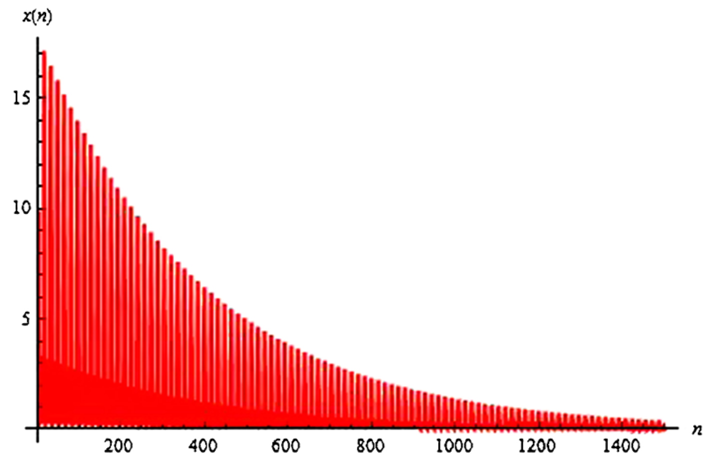


(b) Plot of y_n for system (14).

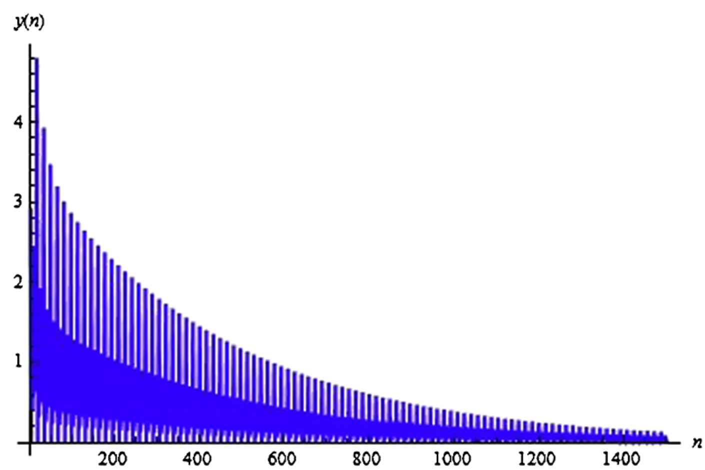


(c) An attractor of system (14).

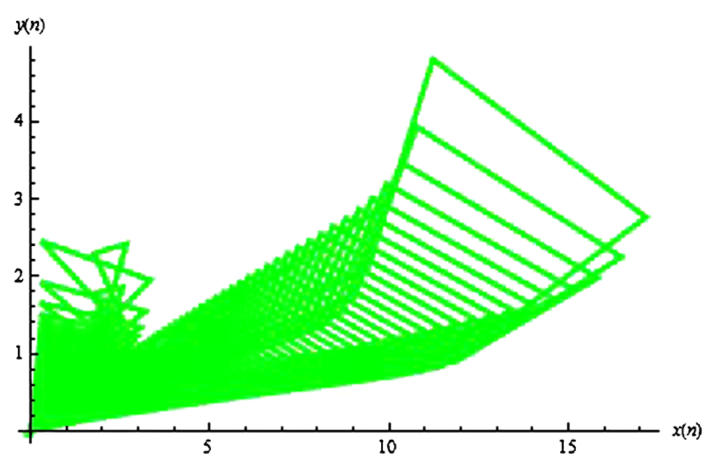
Figure 1 Plots for system (14).



(a) Plot of x_n for system (15).

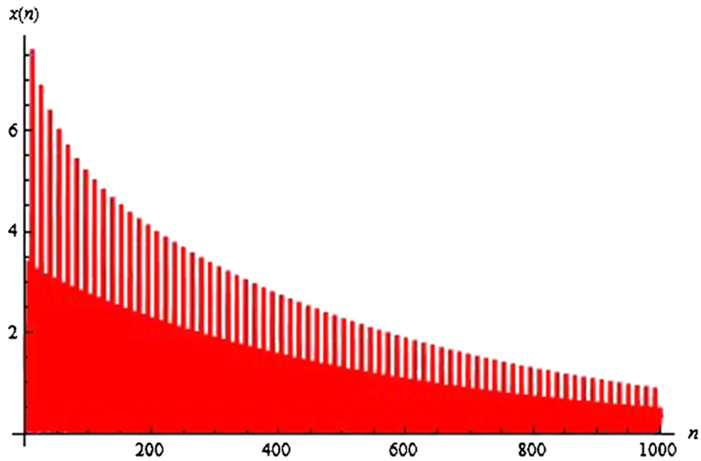


(b) Plot of y_n for system (15).

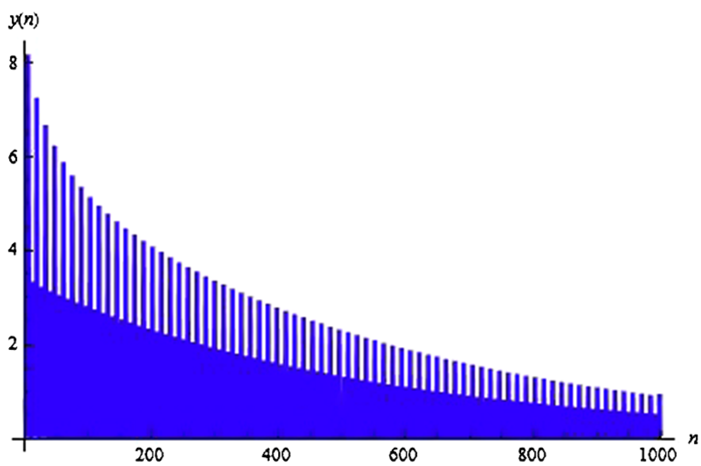


(c) An attractor of system (15).

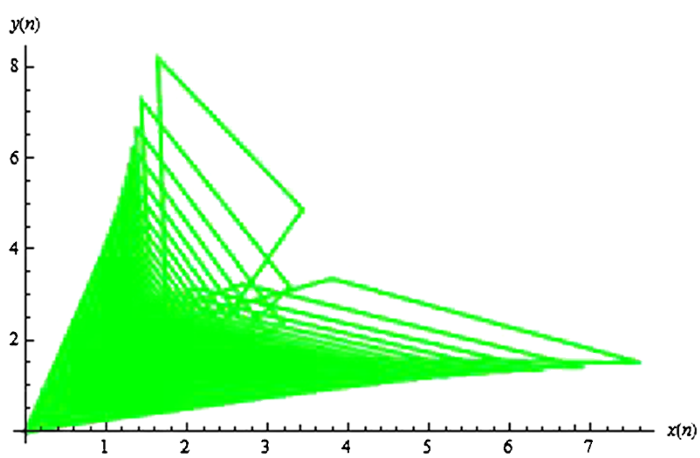
Figure 2 Plots for system (15).



(a) Plot of x_n for system (16).

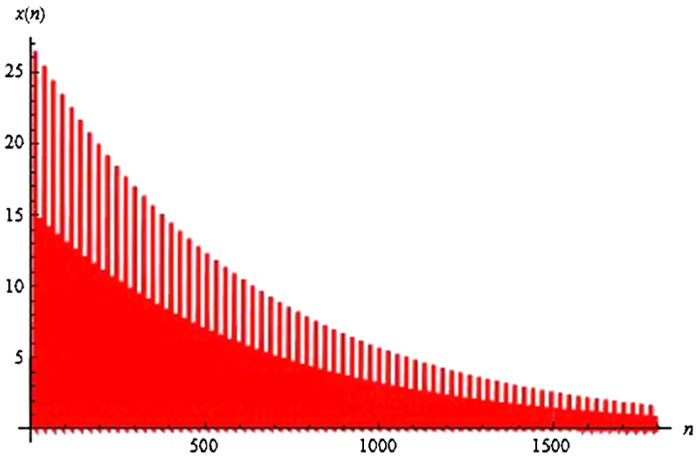


(b) Plot of y_n for system (16).

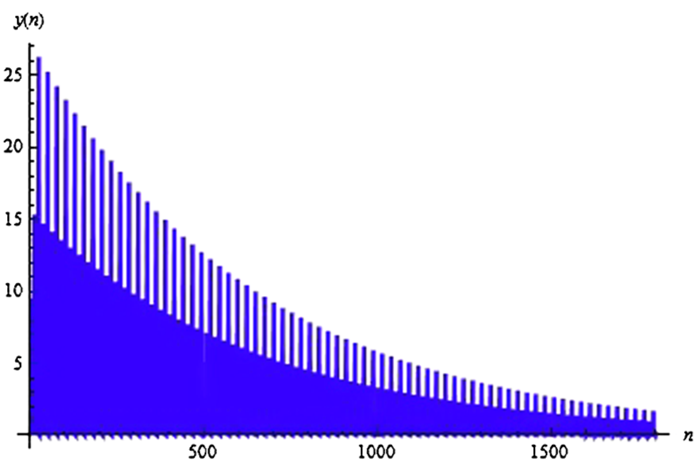


(c) An attractor of system (16).

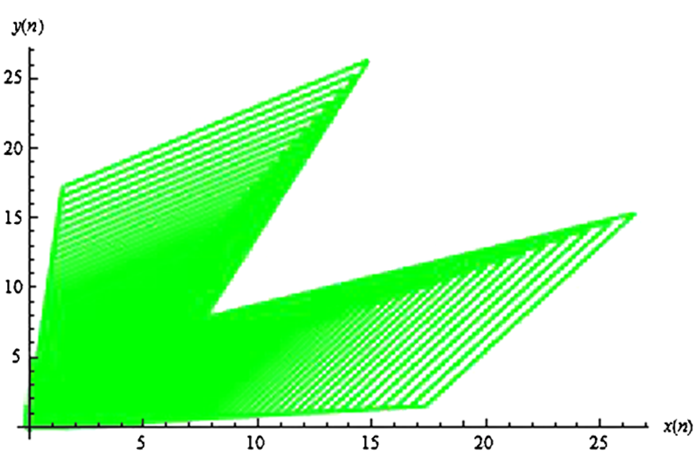
Figure 3 Plots for system (16).



(a) Plot of x_n for system (17).



(b) Plot of y_n for system (17).



(c) An attractor of system (17).

Figure 4 Plots for system (17).

Conclusion

This work is a natural extension of [11, 20, 21]. In the paper, we have investigated the qualitative behavior of $(2k+2)$-dimensional discrete dynamical systems. Each system has only one equilibrium point which is stable under some restriction to parameters. The linearization method is used to show that equilibrium point $(0,0)$ is locally asymptotically stable. The main objective of dynamical systems theory is to predict the global behavior of a system based on the knowledge of its present state. An approach to this problem consists of determining the possible global behaviors of the system and determining which initial conditions lead to these long-term behaviors. In case of higher-order dynamical systems, it is crucial to discuss global behavior of the system. Some powerful tools such as semiconjugacy and weak contraction cannot be used to analyze global behavior of systems (1) and (2). In the paper, we prove the global asymptotic stability of equilibrium point $(0,0)$ by using simple techniques. We have carried out a systematical local and global stability analysis of both systems. The most important finding here is that the unique equilibrium point $(0,0)$ can be a global asymptotic attractor for systems (1) and (2). Moreover, we have determined the rate of convergence of a solution that converges to the equilibrium point $(0,0)$ of systems (1) and (2). Some numerical examples are provided to support our theoretical results. These examples are experimental verifications of theoretical discussions.

Competing interests
The authors have no competing interests.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs.

Acknowledgements
This work was supported by the Higher Education Commission of Pakistan.

Received: 3 October 2013 Accepted: 7 November 2013 Published: 03 Dec 2013

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Cite this article as: Khan et al.: Global dynamics of some systems of higher-order rational difference equations. *Advances in Difference Equations* 2013, 2013:354