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Existence of solutions of multi-point boundary value problems on time scales at resonance

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Abstract

By applying the coincidence degree theorem due to Mawhin, we show the existence of at least one solution to the nonlinear second-order differential equation

$$u^{\Delta\nabla}(t) = f\big(t, u(t), u^{\Delta}(t)\big), \quad t \in [0,1]_{\mathbb{T}},$$

subject to one of the following multi-point boundary conditions:

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u^{\Delta}(1) = 0,$$

and

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u(1) = 0,$$

where $\mathbb{T}$ is a time scale such that $0 \in \mathbb{T}$, $1 \in \mathbb{T}^k$, $\xi_i \in (0,1) \cap \mathbb{T}$, $i = 1, 2, \ldots, m$, $f : [0,1]_{\mathbb{T}} \times \mathrm{R}^2 \to \mathrm{R}$ is continuous and satisfies the Carathéodory-type growth conditions.

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1 Introduction

We assume that the reader is familiar with some notations and basic results for dynamic equations on time scales. Otherwise, the reader is referred to the introductory book on time scales by Bohner and Peterson [1, 2].

There is much current activity focused on dynamic equations on time scales, and a good deal of this activity is devoted to boundary value problems. We refer the readers to Agarwal [3], Morelli [4], Amster [5] and the references therein.

In [6], Anderson studied

$$-u^{\Delta\nabla}(t) = \eta a(t) f\big(u(t)\big), \quad t \in (t_1, t_n)_{\mathbb{T}}, \tag{1.1}$$

subject to one of the following boundary conditions:

$$u(t_1) = \sum_{i=2}^{n-1} \alpha_i u(t_i), \qquad u^{\Delta}(t_n) = 0, \tag{1.2}$$

$$u^{\Delta}(t_1) = 0, \qquad u(t_n) = \sum_{i=2}^{n-1} \alpha_i u(t_i). \tag{1.3}$$

By using a functional-type cone expansion-compression fixed point theorem, the authors get the existence of at least one positive solution to BVP (1.1), (1.2) and BVP (1.1), (1.3) when $\sum_{i=2}^{n-1} \alpha_i \neq 1$.

When $\sum_{i=2}^{n-1} \alpha_i = 1$, the operator $L = u^{\Delta\nabla}$ is non-invertible, this is the so-called resonance case, and the theory used in [6] cannot be used. And to the best knowledge of the authors, the resonant case on time scales has rarely been considered. So, motivated by the papers mentioned above, in this paper, by making use of the coincidence degree theory due to Mawhin [7], we study

$$u^{\Delta\nabla}(t) = f\big(t, u(t), u^{\Delta}(t)\big), \quad t \in [0, a]_{\mathbb{T}}, \tag{1.4}$$

subject to the following two sets of nonlocal boundary conditions:

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u^{\Delta}(1) = 0, \tag{1.5}$$

$$u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i), \qquad u(1) = 0, \tag{1.6}$$

where $\mathbb{T}$ is a time scale such that $0 \in \mathbb{T}$, $1 \in \mathbb{T}^k$, $\xi_i \in (0,1) \cap \mathbb{T}$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} \alpha_i = 1$ holds when (1.4), (1.5) are studied. While $\sum_{i=1}^{m} \alpha_i (1 - \xi_i) = 1$ when (1.4), (1.6) are studied. $f : [0,1]_{\mathbb{T}} \times \mathrm{R}^2 \to \mathrm{R}$ is continuous. We impose Carathéodory-type growth assumptions on f. It is possible, by other methods, to allow nonlinear growth on f, we refer to [8, 9] and the references therein when a time scale is R. A different m-point boundary value problem at resonance is studied in [10].

The main features in this paper are as follows. First, we study two new multi-point BVPs on time scales at resonance, which have rarely been considered, and thus we need to overcome some new difficulties. Second, we give reasons for every important step, which in turn makes this paper easier to be understood. Last but not the least, at the end of this paper, we give examples to illustrate our main results.

We will adopt the following notations throughout:

(i) by $[a, b]_{\mathbb{T}}$ we mean that $[a, b] \cap \mathbb{T}$, where $a, b \in \mathrm{R}$, and $(a, b)_{\mathbb{T}}$ is similarly defined.

(ii) by $u \in L^1[a, b]$ we mean $\int_a^b |u| \nabla t < \infty$.

2 Some definitions and some important theorems

For the convenience of the readers, we provide some background definitions and theorems.

Theorem 2.1 ([1, p.137]) *If $f, g : \mathbb{T} \to \mathrm{R}$ are rd-continuous, then*

$$\int_a^b f(t)g^{\Delta}(t)\Delta t + \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t = (fg)(b) - (fg)(a). \tag{2.1}$$

Theorem 2.2 ([2, p.332]) *If $f, g : \mathbb{T} \to \mathrm{R}$ are ld-continuous, then*

$$\int_a^b f(t)g^{\nabla}(t)\nabla t + \int_a^b f^{\nabla}(t)g(\rho(t))\nabla t = (fg)(b) - (fg)(a). \tag{2.2}$$

Theorem 2.3 ([1, p.139]) *The following formulas hold*:

(i) $(\int_a^t f(t,s)\Delta s)^{\Delta} = f(\sigma(t),t) + \int_a^t f^{\Delta}(t,s)\Delta s$;
(ii) $(\int_a^t f(t,s)\Delta s)^{\nabla} = f(\rho(t),\rho(t)) + \int_a^t f^{\nabla}(t,s)\Delta s$;
(iii) $(\int_a^t f(t,s)\nabla s)^{\Delta} = f(\sigma(t),\sigma(t)) + \int_a^t f^{\Delta}(t,s)\nabla s$;
(iv) $(\int_a^t f(t,s)\nabla s)^{\nabla} = f(\rho(t),t) + \int_a^t f^{\nabla}(t,s)\nabla s$.

Theorem 2.4 ([1, p.89]) *If $f : \mathbb{T} \to \mathrm{R}$ is Δ-differentiable on $\mathbb{T}^k$ and if f^{Δ} is continuous on $\mathbb{T}^k$, then f is ∇-differentiable on $\mathbb{T}_k$ and*

$$f^{\nabla}(t) = f^{\Delta\rho}(t) \quad \text{for } t \in \mathbb{T}_k.$$

If $g : \mathbb{T} \to \mathrm{R}$ is ∇-differentiable on $\mathbb{T}_k$ and if g^{∇} is continuous on $\mathbb{T}_k$, then g is Δ-differentiable on $\mathbb{T}^k$ and

$$g^{\Delta}(t) = g^{\nabla\sigma}(t) \quad \text{for } t \in \mathbb{T}^k.$$

Theorem 2.5 [11] *If f is ∇-integral on $[a,b]$, then so is $|f|$, and*

$$\left|\int_0^t f(t)\nabla t\right| \le \int_0^t |f(t)|\nabla t. \tag{2.3}$$

Definition 2.1 Let X and Y be normed spaces. A linear mapping $L : \operatorname{dom} L \subset X \to Y$ is called a Fredholm operator with index 0, if the following two conditions hold:

(i) $\operatorname{Im} L$ is closed in Y;
(ii) $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

Consider the supplementary subspaces $X1$ and $Y1$ such that $X = \operatorname{Ker} L \oplus X1$ and $Y = \operatorname{Im} L \oplus Y1$, and let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y1$ be the natural projections. Clearly, $\operatorname{Ker} L \cap (\operatorname{dom} L \cap X1) = \{0\}$; thus the restriction $L_P := L|_{\operatorname{dom} L \cap X1}$ is invertible. The inverse of $L_P := L|_{\operatorname{dom} L \cap X1}$ we denote by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$.

If L is a Fredholm operator with index zero, then, for every isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$, the mapping $JQ + K_P(I - Q) : Y \to \operatorname{dom} L$ is an isomorphism and, for every $u \in \operatorname{dom} L$,

$$(JQ + K_p(I - Q))^{-1}u = (L + J^{-1}P)u.$$

Definition 2.2 Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm mapping, E be a metric space, and $N : E \to Y$ be a mapping. We say that N is L-compact on E if $QN : E \to Z$ and $K_P(I - Q)N : E \to X$ are compact on E.

Theorem 2.6 [7] *Suppose that X and Y are two Banach spaces, and $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator with index 0. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$. If*:

(i) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap (\operatorname{dom} L \setminus \operatorname{Ker} L)$, $\lambda \in (0,1)$;

(ii) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(iii) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

then $Lx = Nx$ has a solution in $\bar{\Omega} \cap \operatorname{dom} L$.

3 Related lemmas

Let

$$X = \{u : [0,1] \to \mathrm{R} : u^{\Delta} \in AC[0,1], u^{\Delta\nabla} \in L^1[0,1]\}$$

with the norm $\|u\| = \sup\{\|u\|_0, \|u^{\Delta}\|_0\}$, where $\|u\|_0 = \sup_{t\in[0,1]_{\mathbb{T}}} |u(t)|$.

Let $Y = L^1[0,1]$ with the norm $\|u\|_1 = \int_0^1 |u(t)|\nabla t$.

Define the linear operator $L_1 : \operatorname{dom} L_1 \cap X \to Y$ by $L_1 u = u^{\Delta\nabla}$, with $\operatorname{dom} L_1 = \{u \in X, u \text{ satisfies (1.5)}\}$, and the linear operator $L_2 : \operatorname{dom} L_2 \cap X \to Y$ by $L_2 u = u^{\Delta\nabla}$, with $\operatorname{dom} L_2 = \{u \in X, u \text{ satisfies (1.6)}\}$.

For any open and bounded $\Omega \subset X$, we define $N : \bar{\Omega} \to Y$ by

$$N = f(t, u(t), u^{\Delta}(t)), \quad t \in [0,1]_{\mathbb{T}}. \tag{3.1}$$

Then (1.4), (1.5) (respectively (1.4), (1.6)) can be written as

$$L_1 x = Nx \quad (\text{respectively } L_2 x = Nx).$$

Lemma 3.1 *The mappings $L_1 : \operatorname{dom} L_1 \subset X \to Y$ and $L_2 : \operatorname{dom} L_2 \subset X \to Z$ are Fredholm operators with index zero.*

Proof We first show that L_1 is a Fredholm operator with index zero. We divide this process into two steps.

Step 1: Determine the image of L_1.

Let $y \in Y$ and for $t \in [0,1]_{\mathbb{T}}$,

$$u(t) = \int_t^1 (s-t)y(s)\nabla s + c,$$

then by Theorem 2.3,

$$\begin{aligned} u^{\Delta}(t) &= \left(\int_t^1 (s-t)y(s)\nabla s + c\right)^{\Delta} = \left(\int_1^t (t-s)y(s)\nabla s\right)^{\Delta} \\ &= \big(\sigma(t) - \sigma(t)\big)y\big(\sigma(t)\big) + \int_1^t (t-s)^{\Delta} y(s)\nabla s \\ &= \int_1^t y(s)\nabla s, \end{aligned}$$

and consequently, $u^{\Delta}(1) = 0$ and $u^{\Delta\nabla}(t) = y(t)$. If, in addition, $y(s)$ satisfies

$$\int_0^1 sy(s)\nabla s = \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 (s - \xi_i)y(s)\nabla s, \tag{3.2}$$

then $u(t)$ satisfies the multi-point boundary condition (1.5). That is, $u \in \operatorname{dom} L_1$, and we conclude that

$$\{y \in Y, y \text{ satisfies (3.2)}\} \subseteq \operatorname{Im} L_1.$$

Let $u \in X$, then, by Theorem 2.2,

$$\begin{aligned} \int_t^1 (s - t)u^{\Delta\nabla}(s)\nabla s &= \int_1^t (t - s)u^{\Delta\nabla}(s)\nabla s \\ &= (t - s)u^{\Delta}(s)|_1^t + \int_1^t u^{\Delta}(\rho(s))\nabla s \\ &= -(t - 1)u^{\Delta}(1) + \int_1^t u^{\nabla}(s)\nabla s \\ &= (1 - t)u^{\Delta}(1) + u(t) - u(1), \end{aligned} \tag{3.3}$$

that is, $u(t) = u(1) - (1 - t)u^{\Delta}(1) + \int_t^1 (s - t)u^{\Delta\nabla}(s)\nabla s$. If $y \in \operatorname{Im} L_1$, there exists $u \in \operatorname{dom} L_1 \subset X$ such that

$$u^{\Delta\nabla}(t) = y(t),$$

and the boundary conditions (1.5) are satisfied. Then the expression above becomes

$$u(t) = u(1) + \int_t^1 (s - t)y(s)\nabla s.$$

Since $\sum_{i=1}^{m} \alpha_i = 1$ and $u(0) = \sum_{i=1}^{m} \alpha_i u(\xi_i)$, it follows that (3.2) holds. Hence,

$$\operatorname{Im} L_1 = \{y \in Y : y \text{ satisfies (3.2)}\}.$$

Step 2: Determine the index of L_1.

Let a continuous linear operator $Q_1 : Y \to Y$ be defined by

$$Q_1 y = \frac{1}{C_1}\left(\int_0^1 sy(s)\nabla s - \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 (s - \xi_i)y(s)\nabla s\right), \tag{3.4}$$

where $C_1 = \int_0^1 s\nabla s - \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 (s - \xi_i)\nabla s \neq 0$.

It is clear that $Q_1^2 y = Q_1 y$, that is, $Q_1 : Y \to Y$ is a continuous linear projector. Furthermore, $\operatorname{Im} L_1 = \operatorname{Ker} Q_1$. Let $y = (y - Q_1 y) + Q_1 y \in Y$. It is easy to see that $Q_1(y - Q_1 y) = 0$, thus $y - Q_1 y \in \operatorname{Ker} Q_1 = \operatorname{Im} L_1$ and $Q_1 y \in \operatorname{Im} Q_1$ and so $Y = \operatorname{Im} L_1 + \operatorname{Im} Q_1$. If $y \in \operatorname{Im} L_1 \cap \operatorname{Im} Q_1$, then $y(t) \equiv 0$. Hence, we have $Y = \operatorname{Im} L_1 \oplus \operatorname{Im} Q_1$.

It is clear that $\operatorname{Ker} L_1 = \{u = a, a \in \mathrm{R}\}$. Now, $\operatorname{Ind} L_1 = \dim \operatorname{Ker} L_1 - \operatorname{codim} \operatorname{Im} L_1 = \dim \operatorname{Ker} L_1 - \dim \operatorname{Im} Q_1 = 0$, and so L_1 is a Fredholm operator with index zero.

Next, we show that L_2 is also a Fredholm operator with index zero. We also divide it into two steps.

Step 1: Determine the image of L_2.

Let $y \in Y$ and for $t \in [0,1]_{\mathbb{T}}$,

$$u(t) = \int_t^1 (s-t)y(s)\nabla s + c(1-t),$$

it is obvious that $u(1) = 0$, and then

$$\begin{aligned} u^{\Delta}(t) &= \left(\int_t^1 (s-t)y(s)\nabla s + c(1-t)\right)^{\Delta} = \left(\int_1^t (t-s)y(s)\nabla s - c\right)^{\Delta} \\ &= \big(\sigma(t) - \sigma(t)\big)y\big(\sigma(t)\big) + \int_1^t (t-s)^{\Delta} y(s)\nabla s - c \\ &= \int_1^t y(s)\nabla s - c, \end{aligned}$$

consequently, $u^{\Delta\nabla}(t) = y(t)$. If, in addition, $y(s)$ satisfies (3.2), then $u(t)$ satisfies the multi-point boundary conditions (1.6). That is, $u \in \operatorname{dom} L_2$, and we conclude that

$$\{y \in Y, y \text{ satisfies (3.2)}\} \subseteq \operatorname{Im} L_2.$$

Let $u \in X$, by (3.3), we have $u(t) = u(1) - (1-t)u^{\Delta}(1) + \int_t^1 (s-t)u^{\Delta\nabla}(s)\nabla s$. If $y \in \operatorname{Im} L_2$, there exists $u \in \operatorname{dom} L_2 \subset X$ such that $u^{\Delta\nabla}(t) = y(t)$ and the boundary conditions (1.6) are satisfied. The expression above becomes

$$u(t) = \int_t^1 (s-t)y(s)\nabla s - (1-t)u^{\Delta}(1).$$

Since $\sum_{i=1}^m \alpha_i(1-\xi_i) = 1$ and $u(0) = \sum_{i=1}^m \alpha_i u(\xi_i)$, it follows that (3.2) holds. Hence,

$$\operatorname{Im} L_2 = \{y \in Y : y \text{ satisfies (3.2)}\}.$$

Step 2: Determine the index of L_2.

Let a continuous linear operator $Q_2 : Y \to Y$ be defined by

$$Q_2 y = \frac{1}{C_2}\left(\int_0^1 s y(s)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)y(s)\nabla s\right)(t-1), \tag{3.5}$$

where $C_2 = (\int_0^1 s(s-1)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)(s-1)\nabla s) \neq 0$.

It is clear that $Q_2^2 y = Q_2 y$, that is, $Q_2 : Y \to Y$ is a continuous linear projector. Furthermore, $\operatorname{Im} L_2 = \operatorname{Ker} Q_2$. The remainder of the argument is identical to that concerning L_1 and the proof is completed. □

Lemma 3.2 *N is L_1-compact and L_2-compact.*

Proof Let $P_1 : X \to X$ and $P_2 : X \to X$ be continuous linear operators defined by $P_1 u(t) = u(1)$, $t \in [0,1]_{\mathbb{T}}$, and $P_2 u(t) = -u^{\Delta}(1)(1-t)$, $t \in [0,1]_{\mathbb{T}}$, respectively.

By taking $u \in X$ in the form $u(t) = u(1) + (u(t) - u(1))$, it is clear that $X = \operatorname{Ker} L_1 \oplus \operatorname{Ker} P_1$. Letting $u(t) = -u^{\Delta}(1)(1-t) + (u(t) + u^{\Delta}(1)(1-t))$, we derive $X = \operatorname{Ker} L_2 \oplus \operatorname{Ker} P_2$. Note that the two pairs of projectors P_1, Q_1 and P_2, Q_2 are exact, that is, satisfy the relationships as desired.

Define $K_{P_1} : \operatorname{Im} L_1 \to \operatorname{dom} L_1 \cap \operatorname{Ker} P_1$ by

$$K_{P_1} y(t) = \int_t^1 (s-t) y(s) \nabla s. \tag{3.6}$$

And $K_{P_2} : \operatorname{Im} L_2 \to \operatorname{dom} L_1 \cap \operatorname{Ker} P_2$ is defined by

$$K_{P_2} y(t) = \int_t^1 (s-t) y(s) \nabla s. \tag{3.7}$$

Then, by Theorem 2.5,

$$\sup_{t\in[0,1]_{\mathbb{T}}} \left| K_{P_1} y(t) \right| = \sup_{t\in[0,1]_{\mathbb{T}}} \left| \int_t^1 (s-t) y(s) \nabla s \right| \leq \sup_{t\in[0,1]_{\mathbb{T}}} \int_t^1 \left| (s-t) y(s) \right| \nabla s \leq \|y\|_1,$$

so that

$$\sup_{t\in[0,1]_{\mathbb{T}}} \left| \left(K_{P_1} y(t)\right)^{\Delta} \right| = \sup_{t\in[0,1]_{\mathbb{T}}} \left| \int_t^1 y(s) \nabla s \right| \leq \sup_{t\in[0,1]_{\mathbb{T}}} \int_t^1 \left| y(s) \right| \nabla s \leq \|y\|_1.$$

Therefore,

$$\|K_{P_1} u\| \leq \|y\|_1. \tag{3.8}$$

And similarly,

$$\|K_{P_2} u\| \leq \|y\|_1. \tag{3.9}$$

It is clear that $K_{P_1} = (L_1|_{\operatorname{dom} L_1 \cap \operatorname{Ker} P_1})^{-1}$ and $K_{P_2} = (L_2|_{\operatorname{dom} L_2 \cap \operatorname{Ker} P_2})^{-1}$.

Now, by using (3.4) and (3.5), we have

$$Q_1 N u = \frac{1}{C_1} \left(\int_0^1 s f\left(s, u(s), u^{\Delta}(s)\right) \nabla s - \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 (s - \xi_i) f\left(s, u(s), u^{\Delta}(s)\right) \nabla s \right), \tag{3.10}$$

$$\begin{aligned} Q_2 N u = \frac{1}{C_2} \Bigg(& \int_0^1 s f\left(s, u(s), u^{\Delta}(s)\right) \nabla s \\ & - \sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 (s - \xi_i) f\left(s, u(s), u^{\Delta}(s)\right) \nabla s \Bigg) (t-1). \end{aligned} \tag{3.11}$$

And consequently,

$$K_{P_1}(I - Q_1) N u(t) = \int_t^1 (s-t)(N - Q_1 N) u(s) \nabla s, \tag{3.12}$$

$$K_{P_2}(I - Q_2) N u(t) = \int_t^1 (s-t)(N - Q_2 N) u(s) \nabla s. \tag{3.13}$$

Obviously, both QN and $K_P(I-Q)N$ are compact, thus, N is L_1-compact and L_2-compact. The proof is complete. □

4 Existence of solution to BVP (1.4), (1.5)

For the existence result concerning (1.4), (1.5), we have the following assumptions.

(H_1) There exists a constant $A > 0$ such that for any $u \in \operatorname{dom} L_1 \setminus \operatorname{Ker} L_1$ satisfying $|u(t)| > A$ for all $t \in [0,1]_{\mathbb{T}}$, $Q_1 Nu \neq 0$ holds;

(H_2) There exist functions $p, q, r, \delta \in L^1[0,1]$ and a constant $\varepsilon \in (0,1)$ such that for $(u,v) \in \mathrm{R}^2$ and all $t \in [0,1]_{\mathbb{T}}$, we have

$$\left|f(t,u,v)\right| \leq \delta(t) + p(t)|u| + q(t)|v| + r(t)|v|^{\varepsilon}, \tag{4.1a}$$

or

$$\left|f(t,u,v)\right| \leq \delta(t) + p(t)|u| + q(t)|v| + r(t)|u|^{\varepsilon}; \tag{4.1b}$$

(H_3) There exists a constant $B > 0$ such that, for every $b \in \mathrm{R}$ with $|b| > B$, we have either

$$b\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right) < 0, \tag{4.2a}$$

or

$$b\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right) > 0. \tag{4.2b}$$

Theorem 4.1 *If* (H_1)-(H_3) *hold, then the boundary value problem* (1.4), (1.5) *has at least one solution provided*

$$\|p\|_1 + \|q\|_1 < \frac{1}{2}. \tag{4.3}$$

Proof Firstly, we define an open bounded subset Ω of X. It is based upon four steps to obtain Ω.

Step 1: Let

$$\Omega_1 = \left\{u \in \operatorname{dom} L_1 \setminus \operatorname{Ker} L_1 : L_1 u = \lambda Nu, \lambda \in (0,1)\right\},$$

then for $u \in \Omega_1$, $L_1 u = \lambda Nu$. Thus, we have $Nu \in \operatorname{Im} L_1 = \operatorname{Ker} Q_1$ and

$$\int_0^1 sf\left(s,u(s),u^{\Delta}(s)\right)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f\left(s,u(s),u^{\Delta}(s)\right)\nabla s = 0.$$

It follows from (H_2) that there exists $t_0 \in [0,1]_{\mathbb{T}}$ such that $|u(t_0)| \leq A$. Hence, by Theorems 2.4 and 2.5, we have

$$\left|u(1)\right| = \left|u(t_0) + \int_{t_0}^1 u^{\nabla}(s)\nabla s\right| = \left|u(t_0) + \int_{t_0}^1 u^{\Delta}\left(\rho(s)\right)\nabla s\right| \leq A + \left\|u^{\Delta}\right\|_0. \tag{4.4}$$

Also, $u^{\Delta}(t) = -\int_t^1 u^{\Delta\nabla}(s)\nabla s$ implies

$$\|u^{\Delta}\|_0 \le \|u^{\Delta\nabla}\|_1 = \|L_1 u\|_1 < \|Nu\|_1. \tag{4.5}$$

Combining (4.4), (4.5), one gets

$$|u(1)| \le A + \|Nu\|_1. \tag{4.6}$$

Observe that $(I - P_1)u \in \operatorname{Im} K_{P_1} = \operatorname{dom} L_1 \cap \operatorname{Ker} P_1$ for $u \in \Omega_1$, then we obtain

$$\|(I-P_1)u\| = \|K_{P_1}L_1(I-P_1)u\| \le \|L_1(I-P_1)u\|_1 = \|L_1 u\|_1 < \|Nu\|_1. \tag{4.7}$$

Using (4.6), (4.7), we get

$$\|u\| = \|P_1 u + (I-P_1)u\| \le \|P_1 u\| + \|(I-P_1)u\| < |u(1)| + \|Nu\|_1 < A + 2\|Nu\|_1, \tag{4.8}$$

that is, for all $u \in \Omega_1$,

$$\|u\| < A + 2\|Nu\|_1. \tag{4.9}$$

If (4.1a) holds, then

$$\|u\|_0, \|u^{\Delta}\|_0 \le \|u\| \le A + 2\bigl(\|\delta\|_1 + \|p\|_1\|u\|_0 + \|q\|_1\|u^{\Delta}\|_0 + \|r\|_1\|u^{\Delta}\|_0^{\varepsilon}\bigr), \tag{4.10}$$

and consequently,

$$\|u\|_0 \le \frac{2}{1-2\|p\|_1}\left(\|\delta\|_1 + \|q\|_1\|u^{\Delta}\|_0 + \|r\|_1\|u^{\Delta}\|_0^{\varepsilon} + \frac{A}{2}\right). \tag{4.11}$$

Further, by (4.10) and (4.11),

$$\begin{aligned}\|u^{\Delta}\|_0 &\le 2\|p\|_1\|u\|_0 + 2\left(\|\delta\|_1 + \|q\|_1\|u^{\Delta}\|_0 + \|r\|_1\|u^{\Delta}\|_0^{\varepsilon} + \frac{A}{2}\right)\\ &\le 2\left(\|\delta\|_1 + \|q\|_1\|u^{\Delta}\|_0 + \|r\|_1\|u^{\Delta}\|_0^{\varepsilon} + \frac{A}{2}\right)\left(\frac{2\|p\|_1}{1-2\|p\|_1} + 1\right)\\ &= \frac{2\|q\|_1}{1-2\|p\|_1}\|u^{\Delta}\|_0 + \frac{2\|r\|_1}{1-2\|p\|_1}\|u^{\Delta}\|_0^{\varepsilon} + \frac{2\|\delta\|_1 + A}{1-2\|p\|_1},\end{aligned} \tag{4.12}$$

that is,

$$\|u^{\Delta}\|_0 \le \frac{2\|r\|_1}{1-2(\|p\|_1 + \|q\|_1)}\|u^{\Delta}\|_0^{\varepsilon} + \frac{2\|\delta\|_1 + A}{1-2(\|p\|_1 + \|q\|_1)}. \tag{4.13}$$

Since $\varepsilon \in (0,1)$ and (4.13) holds, we know that there exists $R_1 > 0$ such that $\|u^{\Delta}\|_0 \le R_1$ for all $u \in \Omega_1$. Inequality (4.11) then shows that there exists $R_2 > 0$ such that $\|u\|_0 \le R_2$ for all $u \in \Omega_1$. Therefore, Ω_1 is bounded given (4.1a) holds. If, otherwise, (4.1b) holds, then with minor adjustments to the arguments above we derive the same conclusion.

Step 2: Define

$$\Omega_2 = \{u \in \operatorname{Ker} L_1 : Nu \in \operatorname{Im} L_1\}.$$

Then $u = b \in \mathrm{R}$ and $Nu \in \operatorname{Im} L_1 = \operatorname{Ker} Q_1$ imply that

$$\frac{1}{C_1}\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right) = 0.$$

Hence, by (H_3), $\|u\| = b \leq B$, that is, Ω_2 is bounded.

Step 3: Let

$$\Omega_3 = \bigl\{u \in \operatorname{Ker} L_1 : H(u,\lambda) = 0\bigr\},$$

where

$$H(u,\lambda) = \begin{cases} -\lambda Iu + (1-\lambda)JQ_1 Nu & \text{if (4.2a) holds,} \\ \lambda Iu + (1-\lambda)JQ_1 Nu & \text{if (4.2b) holds,} \end{cases} \tag{4.14}$$

and $J : \operatorname{Im} Q_1 \to \operatorname{Ker} L_1$ is a homomorphism such that $J(b) = b$ for all $b \in \mathrm{R}$.

Without loss of generality, we suppose that (4.2a) holds, then for every $b \in \Omega_3$,

$$\lambda b = (1-\lambda)\frac{1}{C_1}\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right).$$

If $\lambda = 1$, then $b = 0$. And in the case $\lambda \in [0,1)$, if $|b| > B$, then by (4.2a),

$$0 \leq \lambda b^2 = b(1-\lambda)\frac{1}{C_1}\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right) < 0,$$

which is a contradiction.

When (4.2b) holds, by a similar argument, again, we can obtain a contradiction. Thus, for any $u \in \Omega_3$, $\|u\| \leq B$, that is, Ω_3 is bounded.

Step 4: In what follows, we shall prove that all the conditions of Theorem 2.6 are satisfied. Let Ω be an open bounded subset of X such that $\bigcup_{i=1}^3 \overline{\Omega}_i \subset \Omega$, clearly, we have

$$Lu \neq Nx, \quad \lambda \in (0,1), u \in \partial\Omega,$$

and

$$Nu \neq \operatorname{Im} L_1, \quad \forall u \in \partial\Omega \cap \operatorname{Ker} L_1.$$

It can be seen easily that

$$H(u,\lambda) \neq 0, \quad \lambda \in [0,1], u \in \partial\Omega \cap \operatorname{Ker} L_1.$$

Then assumptions (i) and (ii) of Theorem 2.6 are fulfilled. It only remains to verify that the third assumption of Theorem 2.6 applies.

We apply the degree property of invariance under homotopy. To this end, we define the homotopy

$$H(u,\lambda) = \mp\lambda Iu + (1-\lambda)JQ_1Nu.$$

If $u \in \Omega \cap \operatorname{Ker} L_1$, then

$$\begin{aligned}\deg\{JQ_1N, \Omega \cap \operatorname{Ker} L_1, 0\} &= \deg\bigl\{H(\cdot,0), \Omega \cap \operatorname{Ker} L_1, 0\bigr\} \\ &= \deg\bigl\{H(\cdot,1), \Omega \cap \operatorname{Ker} L_1, 0\bigr\} \\ &= \deg\{\mp I, \Omega \cap \operatorname{Ker} L_1, 0\} \neq 0.\end{aligned} \tag{4.15}$$

So, the third assumption of Theorem 2.6 is fulfilled.

Therefore, Theorem 2.6 can be applied to obtain the existence of at least one solution to BVP (1.4) and (1.5). The proof is complete. □

5 Existence of solution to BVP (1.4), (1.6)

In this section, we give the existence result for BVP (1.4), (1.6). We first state the following assumptions:

(H_4) There exists a constant $C > 0$ such that for any $u \in \operatorname{dom} L_2 \setminus \operatorname{Ker} L_2$ satisfying $|u^\Delta(t)| > C$ for all $t \in [0,1]_{\mathbb{T}}$, $Q_2Nu \neq 0$ holds;

(H_5) There exists a constant $D > 0$ such that, for every $d \in \mathrm{R}$ with $|d| > D$, we have either

$$d\left(\int_0^1 sf\bigl(s, d(s-1), d\bigr)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i) f\bigl(s, d(s-1), d\bigr)\nabla s\right) < 0, \tag{5.1a}$$

or

$$d\left(\int_0^1 sf\bigl(s, d(s-1), d\bigr)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i) f\bigl(s, d(s-1), d\bigr)\nabla s\right) > 0. \tag{5.1b}$$

Theorem 5.1 *Assume that* (H_2), (H_4) *and* (H_5) *hold, then BVP* (1.4), (1.6) *has at least one solution provided*

$$\|p\|_1 + \|q\|_1 < \frac{1}{2}. \tag{5.2}$$

Proof Let

$$\Omega_1 = \bigl\{u \in \operatorname{dom} L_2 \setminus \operatorname{Ker} L_2 : L_2u = \lambda Nu, \lambda \in (0,1)\bigr\},$$

then for $u \in \Omega_1$, $L_2u = \lambda Nu$. Thus, we have $Nu \in \operatorname{Im} L_2 = \operatorname{Ker} Q_2$, and thus

$$\int_0^1 sf\bigl(s, u(s), u^\Delta(s)\bigr)\nabla s - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 (s-\xi_i) f\bigl(s, u(s), u^\Delta(s)\bigr)\nabla s = 0.$$

It follows from (H_4) that there exists $t_0 \in [0,1]_{\mathbb{T}}$ such that $|u^{\Delta}(t_0)| \leq C$. Hence, by Theorem 2.5, we have

$$\left|u^{\Delta}(1)\right| = \left|u^{\Delta}(t_0) + \int_{t_0}^{1} u^{\Delta\nabla}(s)\nabla s\right| \leq A + \|Nu\|_1. \tag{5.3}$$

Observe that $(I - P_2)u \in \operatorname{Im} K_{P_2} = \operatorname{dom} L_2 \cap \operatorname{Ker} P_2$ for $u \in \Omega_1$, then we obtain

$$\left\|(I - P_2)u\right\| = \left\|K_{P_2}L_2(I - P_2)u\right\| \leq \left\|L_2(I - P_2)u\right\|_1 = \|L_2u\|_1 < \|Nu\|_1. \tag{5.4}$$

Using (5.3), (5.4), one gets

$$\begin{aligned} \|u\| &= \left\|P_2u + (I - P_2)u\right\| \leq \|P_2u\| + \left\|(I - P_2)u\right\| < \left|u^{\Delta}(1)\right| + \|Nu\|_1 \\ &< C + 2\|Nu\|_1, \end{aligned} \tag{5.5}$$

that is, for all $u \in \Omega_1$,

$$\|u\| < C + 2\|Nu\|_1. \tag{5.6}$$

As in the proof of Theorem 4.1, by applying (H_2) we can show that Ω_1 is bounded.

Step 2: Define

$$\Omega_2 = \{u \in \operatorname{Ker} L_2 : Nu \in \operatorname{Im} L_2\}.$$

Then $u = d(t-1)$, where $d \in \mathrm{R}$ and $Nu \in \operatorname{Im} L_2 = \operatorname{Ker} Q_2$ imply that

$$\frac{1}{C_2}\left(\int_0^1 sf\big(s, d(s-1), s\big)\Delta s - \sum_{i=1}^{m}\alpha_i \int_{\xi_i}^{1}(s - \xi_i)f\big(s, d(s-1), s\big)\Delta s\right) = 0.$$

Hence, by (H_4), $\|u\| = d \leq D$, which means Ω_2 is bounded.

Step 3: Let

$$\Omega_3 = \big\{u \in \operatorname{Ker} L_2 : H(u,\lambda)\big\} = 0,$$

where

$$H(u,\lambda) = \begin{cases} -\lambda Iu + (1-\lambda)JQ_2Nu & \text{if (5.1a) holds,} \\ \lambda Iu + (1-\lambda)JQ_2Nu & \text{if (5.1b) holds,} \end{cases} \tag{5.7}$$

and $J : \operatorname{Im} Q_2 \to \operatorname{Ker} L_2$ is a homomorphism such that $J(d(t-1)) = d(t-1)$ for all $d \in \mathrm{R}$.

Without loss of generality, we suppose that (5.1a) holds, then for every $d \in \Omega_3$,

$$\lambda d = (1-\lambda)\frac{1}{C_2}\left(\int_0^1 sf\big(s, d(s-1), d\big)\nabla s - \sum_{i=1}^{m}\alpha_i \int_{\xi_i}^{1}(s - \xi_i)f\big(s, d(s-1), d\big)\nabla s\right).$$

If $\lambda = 1$, then $d = 0$. And in the case $\lambda \in [0,1)$, if $|d| > D$, then by (5.1a),

$$0 \le \lambda d^2 = d(1-\lambda)\frac{1}{C_2}\left(\int_0^1 sf\big(s,d(s-1),d\big)\nabla s - \sum_{i=1}^{m}\alpha_i\int_{\xi_i}^1 (s-\xi_i)f\big(s,d(s-1),d\big)\nabla s\right) < 0,$$

which is a contradiction.

When (5.1b) holds, by a similar argument, again, we can obtain a contradiction. Thus, for any $u \in \Omega_3$, $\|u\| \le D$, that is, Ω_3 is bounded.

Step 4 is essentially the same as that of Theorem 4.1. Applying Theorem 2.6, we obtain the existence of at least one solution to BVP (1.4), (1.6). The proof is complete. □

6 Examples

In this section, we give an example to illustrate our main results.

Example 6.1 Let $\mathbb{T} = [\frac{k}{2}, \frac{2k+1}{4}]$, where $k \in Z$. We consider the following BVP on $\mathbb{T}$.

$$\begin{cases} u^{\Delta\nabla}(t) = \frac{1}{6}(50 + 2t^2 + u\sin t + 20t^2(u^{\Delta}(t))^{1/3} + u^{\Delta}(t)), & t \in [0,1]_{\mathbb{T}}, \\ u(0) = \frac{1}{3}u(\frac{1}{4}) + \frac{2}{3}u(\frac{1}{2}), & u^{\Delta}(1) = 0. \end{cases} \tag{6.1}$$

It is clear that $f(t,u,v) = \frac{1}{6}(50 + 2t^2 + u\sin t + 20t^2v^{1/3} + v)$, $0 \in \mathbb{T}$, $1 \in \mathbb{T}^k$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2} \in \mathbb{T}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$, and $\alpha_1 + \alpha_2 = 1$, thus BVP (6.1) is resonant.

In what follows, we try to show that all the conditions in Theorem 4.1 are satisfied.

Let $\delta(t) = \frac{t^2+25}{3}$, $p(t) = \frac{|t|}{6}$, $q(t) = \frac{1}{6}$, $r(t) = \frac{10t^2}{3}$, $\varepsilon = \frac{1}{3}$. We can see that

$$\big|f(t,u,v)\big| \le \delta(t) + p(t)|u| + q(t)|v| + r(t)|v|^{\varepsilon}$$

holds, which implies that (H_2) is satisfied.

After a series of calculations, we obtain

$$\begin{aligned} C_1 &= \int_0^1 s\nabla s - \sum_{i=1}^{m}\alpha_i\int_{\xi_i}^1 (s-\xi_i)\nabla s \\ &= \left(\int_0^{1/4} + \int_{1/2}^{3/4}\right)s\,ds + \left(\int_{1/4}^{1/2} + \int_{3/4}^{1}\right)s\nabla s \\ &\quad - \frac{1}{3}\left(\left(\int_{1/4}^{1/2} + \int_{3/4}^{1}\right)\left(s-\frac{1}{4}\right)\nabla s + \int_{1/2}^{3/4}\left(s-\frac{1}{4}\right)ds\right) \\ &\quad - \frac{2}{3}\left(\int_{1/2}^{3/4}\left(s-\frac{1}{2}\right)ds + \int_{3/4}^{1}\left(s-\frac{1}{2}\right)\nabla s\right) = \frac{11}{32} \ne 0. \end{aligned}$$

For $u \in \operatorname{dom} L_1 \setminus \operatorname{Ker} L_1$, $u(t) = at$, $u^{\Delta}(t) = a$, then we have

$$\begin{aligned} 6C_1Q_1Nu &= \int_0^1 sf(s,as,a)\nabla s - \sum_{i=1}^{m}\alpha_i\int_{\xi_i}^1 (s-\xi_i)f(s,as,a)\nabla s \\ &= \frac{80{,}789}{4{,}608} + \frac{44{,}313{,}960{,}092{,}071{,}337}{36{,}028{,}797{,}018{,}963{,}968}a + \frac{23{,}915}{2{,}304}a^{1/3} \\ &\approx 17.5323 + 1.23a + 10.3798a^{1/3}. \end{aligned}$$

Let $A = 3$, then when $|u(t)| = |a| > 3$, $Q_1Nu \neq 0$, which implies that (H_1) holds while

$$\begin{aligned} &b\left(\int_0^1 sf(s,b,0)\nabla s - \sum_{i=1}^{m}\alpha_i\int_{\xi_i}^1 (s-\xi_i)f(s,b,0)\nabla s\right) \\ &\quad = \frac{1}{6}\left(\frac{47{,}545}{2{,}304}b + \frac{38{,}324{,}624{,}432{,}429{,}689}{72{,}057{,}594{,}037{,}927{,}936}b^2\right) \\ &\quad \approx \frac{1}{6}\left(20.6359b + 0.5319b^2\right). \end{aligned}$$

Let $B = 39$, then when $|b| > B$, (4.2a) or (4.2b) holds, which implies that (H_3) is satisfied.

Finally, it is obvious that $\|p\|_1 + \|q\|_1 < \frac{1}{2}$. Thus, all the conditions in Theorem 4.1 are satisfied, then BVP (6.1) has at least one solution.

Example 6.2 Let $\mathbb{T} = [0, \frac{2}{3}] \cup \{1\}$. We consider the following BVP on $\mathbb{T}$.

$$\begin{cases} u^{\Delta\nabla}(t) = 100 + t + \frac{t^2u}{4} + t^2u^{1/5} + \frac{tu^{\Delta}(t)}{8}, & t \in [0,1]_{\mathbb{T}}, \\ u(0) = u(\frac{1}{2}), \qquad u(1) = 0. \end{cases} \tag{6.2}$$

It is clear that $f(t,u,v) = 100 + t + \frac{t^2u}{4} + t^2u^{1/5} + \frac{tu^{\Delta}(t)}{8}$, $0 \in \mathbb{T}$, $1 \in \mathbb{T}^k$, $\xi_1 = \frac{1}{2} \in \mathbb{T}$, $\alpha_1 = 1$, thus BVP (6.2) is resonant.

In what follows, we try to show that all the conditions in Theorem 5.1 are satisfied.

Let $\delta(t) = 100 + t$, $p(t) = \frac{t^2}{4}$, $q(t) = \frac{t}{8}$, $r(t) = t^2$, $\varepsilon = \frac{1}{5}$. We can see that

$$\left|f(t,u,v)\right| \leq \delta(t) + p(t)|u| + q(t)|v| + r(t)|u|^{\varepsilon}$$

holds, which implies that (H_2) is satisfied.

After a series of calculations, we obtain

$$\begin{aligned} C_2 &= \int_0^1 s(s-1)\nabla s - \int_{1/2}^1\left(s-\frac{1}{2}\right)(s-1)\nabla s \\ &= \int_0^{2/3} s(s-1)\,ds + \int_{2/3}^1 s(s-1)\nabla s - \int_{1/2}^{2/3}\left(s-\frac{1}{2}\right)(s-1)\,ds - \int_{2/3}^1\left(s-\frac{1}{2}\right)(s-1)\nabla s \\ &= -0.1181 \neq 0. \end{aligned}$$

For $u \in \operatorname{dom} L_2 \setminus \operatorname{Ker} L_2$, $u(t) = a$, $u^{\Delta}(t) = 0$, then we have

$$\begin{aligned} C_2Q_2Nu &= \int_0^1 sf(s,a,0)\nabla s - \int_{1/2}^1\left(s-\frac{1}{2}\right)f(s,a,0)\nabla s \\ &= \frac{2{,}875}{20{,}736}a + \frac{2{,}875}{5{,}184}a^{1/5} + \frac{5{,}437}{144} \\ &\approx 0.1386a + 0.5546a^{1/5} + 37.7569. \end{aligned}$$

Let $C = 300$, then when $|u^{\Delta}(t)| = |a| > 300$, $Q_2Nu \neq 0$, which implies that (H_4) holds, while

$$\begin{aligned}
&d\biggl(\int_0^1 sf\bigl(s, d(s-1), d\bigr)\nabla s - \int_{1/2}^1 \biggl(s - \frac{1}{2}\biggr) f\bigl(s, d(s-1), d\bigr)\nabla s\biggr) \\
&\quad = \frac{4{,}937{,}627{,}091{,}458{,}903}{72{,}057{,}594{,}037{,}927{,}936} d^2 - \frac{4{,}954{,}847{,}807{,}426{,}245}{1{,}152{,}921{,}504{,}606{,}846{,}976} d^{6/5} \\
&\qquad + \frac{7{,}490{,}236{,}786{,}645{,}797}{140{,}737{,}488{,}355{,}328} d \\
&\quad \approx 0.0685 d^2 - 0.0043 d^{6/5} + 53.2213 d.
\end{aligned}$$

Let $D = 780$, then when $|d| > D$, (5.1a) or (5.1b) holds, which implies that (H_5) is satisfied.

Finally, it is obvious that $\|p\|_1 + \|q\|_1 < \frac{1}{2}$. Thus, all the conditions in Theorem 5.1 are satisfied, then BVP (6.2) has at least one solution.

Competing interests
The authors declare that there are no competing interests regarding the publication of this article.

Authors' contributions
BZ and HL conceived of the study and participated in its coordination. JZ drafted the manuscript. All authors read and approved the final manuscript.

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