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Consensus for wireless sensor networks with Markovian switching topology and stochastic communication noises

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Abstract

In this paper, we consider the problem of average consensus for wireless sensor networks (WSNs). By applying the Lyapunov function and the M-matrix approach, several sufficient conditions to ensure the average consensus in pth moment for WSNs with Markovian switching topology are derived. Also we study the stochastic average consensus in the large for WSNs with Markovian switching topology, whose transition jump rates matrix is not precisely known. We finally show the numerical simulations to illustrate the effectiveness of the results derived in this paper.

Keywords: wireless sensor networks; consensus in pth moment; Markovian switching topology; stochastic communication noises; stochastic average consensus in the large

1 Introduction

With the increasing development of microelectronics, wireless communication and computing, wireless sensor networks (WSNs) have been more widely used in such areas as robotics, surveillance, environment monitoring, information collection, and so on.

From a network-theoretic perspective, a WSN can be looked upon as a multi-agent system (MAS) with each node representing a sensor and each edge performing information exchange between sensors. In some cases, the agreement is a common value which may be the average of the initial states of the system, and which is often called average consensus and has wide application background in the areas such as formation control [1], distributed filtering [2] and distributed computation [3]. It means to achieve the accordance of the states of MAS. In [4], the authors considered the average consensus control for the directed and undirected networks with fixed and switching topologies. In [5], the authors extended the results of [4] to the discrete-time models and weakened the condition of instantaneous strong connectivity. In [6], the authors considered the distributed averaging consensus of the networks with fixed and undirected topologies.

In the real communication surroundings, the topology of the WSN will change among some modes along with the time. Due to the limited energy, the sensor nodes often switch the states between sleep and awake to save energy. So we make use of Markov switching topology to describe this situation. Zhang [7] designed a distributed consensus protocol to analyze the multi-agent systems in uncertain communication environments including the communication noises and Markov topology switches. The asymptotic unbiased mean

square average consensus and asymptotic unbiased almost sure average consensus were obtained. Also they considered the effect of agent dependent gain functions on the consensus of multi-agent systems. Zhou *et al.* [8] studied the problem of the mode and delay-dependent adaptive exponential synchronization in pth moment for stochastic delayed neural networks with Markovian switching. By applying a novel nonnegative function and the M-matrix approach, several sufficient conditions to ensure the mode and delay dependent adaptive exponential synchronization in pth moment for stochastic delayed neural networks were derived. Zhou *et al.* [9] considered the problem of target synchronization for the WSNs with multiple time-varying delays, Markov switching topology and the stochastic disturbance. The communication topology graphs they studied in this paper were undirected.

In real networked control systems, the packet dropouts and channel delays usually arise in the communication among the sensor nodes. The situations can be modeled by the Markov chain with the assumption of completely known or incompletely known transition probabilities. Therefore, from control perspectives, it is important and necessary to discuss the systems with partly unknown transition probabilities. Zhang and Boukas [10] studied the stability and stabilization problems of a class of continuous-time and discrete-time Markovian jump linear systems with partly unknown transition probabilities. Zhang and Lam [11] considered the Markov jump linear system with incomplete transition descriptions and got the necessary and sufficient criteria in both continuous-time domain and discrete-time cases. Zhu *et al.* [12] discussed the asymptotic stability of nonlinear stochastic differential equations with Markovian switching. The transition jump rates matrix Γ is not precisely known for the equations.

Moreover, to study the average consensus for the WSNs with Markovian switching topology and stochastic noises, we take advantage of the pth moment exponential stability theory [13] which is widely applied to the system stability analysis [14–16]. Specially, the case $p = 2$ is the mean square exponential stability which is frequently used in different research works such as [17–19]. In [19], the authors showed that a necessary and sufficient condition for asymptotically unbiased mean square average consensus is that the consensus gains satisfy the step rule similar to that of classical stochastic approximation.

Next, we introduce our main contributions. To the best of our knowledge, the method combining the Lyapunov function and the M-matrix is rarely used in the researching area of the average consensus in pth moment for WSNs with Markovian switching topology and stochastic noises. Also, the stochastic average consensus in the large for WSNs with Markovian switching topology, whose transition jump rates matrix is not precisely known, is rarely discussed. The present paper considers the consensus in pth moment for WSNs with Markovian switching topology and stochastic noises in communication. Making use of the pth moment exponential stability theory, we propose a sufficient condition of the pth moment exponential consensus for the WSNs. Different from [9], the communication topology graphs discussed are directed. We also consider the stochastic average consensus in the large for the WSN, whose transition jump rates matrix Γ is not precisely known.

The remainder of the paper is organized as follows. In Section 2, some concepts in graph theory are described, and the problem to be investigated is formulated. In Section 3, the main results are presented. In Section 4, two numerical examples show the reliability of the main results. In Section 5, some conclusions are given.

The following notations will be used throughout this paper : $|x|$ represents the Euclidean norm of a vector x in $\mathbb{R}^n$, $|A| := \sqrt{\text{trace}(A^T A)}$, *i.e.*, the trace norm of a matrix A. 'T' represents the transpose of a matrix or a vector, I is an $n \times n$ dimensional identity matrix, 1_n is an n-dimensional column vector with all ones, $\mathbb{E}$ stands for the mathematical expectation operator. If A is a matrix, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of A, respectively.

2 Problem formulation and preliminaries

2.1 Concepts in graph theory

Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$ be a weighted digraph, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of nodes, node i represents the ith sensor node, $\mathcal{E}$ is the set of edges, and an edge in $\mathcal{G}$ is denoted by an ordered pair (j, i). $(j, i) \in \mathcal{E}$ if and only if the jth sensor node can send information to the ith sensor node directly. The neighborhood of the ith sensor node is denoted by $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$.

$\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called the weighted adjacency matrix of $\mathcal{G}$. For any $i, j \in V$, $a_{ij} \geq 0$, and $a_{ij} > 0 \Leftrightarrow j \in N_i$. $\deg_{in}(i) = \sum_{j=1}^{n} a_{ij}$ is called the in-degree of i; $\deg_{out}(i) = \sum_{j=1}^{n} a_{ij}$ is called the out-degree of i; $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is called the Laplacian matrix of $\mathcal{G}$, where $\mathcal{D} = \text{diag}(\deg_{in}(1), \ldots, \deg_{in}(n))$.

2.2 Average consensus for WSNs

In this paper, we study the average consensus control for a WSN with dynamics

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i(t) \in \mathbb{R}$ is the state of the ith sensor, and $u_i(t) \in \mathbb{R}$ is the control input. The initial state $x_i(0)$ is deterministic.

The ith sensor can receive information from its neighbors

$$y_{ji}(t) = x_j(t) + \sigma_{ji}(x_i(t), t)\omega_{ji}(t), \tag{2}$$

where $y_{ji}(t)$ denotes the measurement of the jth sensor's state $x_j(t)$ by the ith sensors, $\{\omega_{ji}(t) | i, j = 1, 2, \ldots, n\}$ are the communication noises, $\sigma_{ji}(x_i(t), t)$ is the noise intensity function.

Denote $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $(\mathcal{G}, X)$ is called a dynamic network.

For the dynamic network $(\mathcal{G}, X)$, we use the distributed protocol

$$u_i(t) = \sum_{j \in N_i} a_{ij}(t)\big(y_{ji}(t) - x_i(t)\big). \tag{3}$$

Substituting protocol (3) into system (1) leads to

$$dX(t) = -L(t)X(t)\, dt + G\big(X(t), t\big)\, dW(t), \tag{4}$$

where $L(t)$ is the Laplacian matrix of $\mathcal{G}$, $G(X(t), t)$ is the noise intensity matrix, $W(t) = (W_1(t), \ldots, W_n(t))^T$ is an n-dimensional Brownian motion.

In WSNs, each sensor node communicates with other sensor nodes through the unreliable networks. If the communication channel between sensors i and j is $(j, i) \in \mathcal{E}_f$, $\mathcal{E}_f$ is the

set of the communication channel which probably lost the signal, then the time-varying topologies under link failure or creation can be described by Markov switching topology.

Let $\{r(t)_{t \geq 0}\}$ be a right-continuous Markovian chain on the probability space taking values in a finite state set $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases} \tag{5}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$$

and $r(0) = r_0$.

We denote the directed communication graph by $\mathcal{G} = \{\mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(N)\}$, where $\mathcal{G}(k) = \{\mathcal{V}, \mathcal{E}_{\mathcal{G}(k)}, \mathcal{A}_{\mathcal{G}(k)}\}$ is the weighted digraph. Denote the topology graph as $\mathcal{G}_t$ at moment t ($t \geq 0$), so $r(t) = k$ if and only if $\mathcal{G}_t = \mathcal{G}(k)$.

Under Markovian switching topology, we have

$$dX(t) = -L(r(t))X(t)\,dt + G(X(t), r(t))\,dW(t). \tag{6}$$

Next, we consider the average consensus control protocol for system (6) as follows:

$$d\delta(t) = -L(r(t))\delta(t)\,dt + (I - J)G(\delta(t), r(t))\,dW(t), \tag{7}$$

where $\delta(t) = X(t) - JX(t)$, $J = \frac{1}{n}1_n 1_n^T$.

If we define

$$f(\delta, t, i) = -L(r(t))\delta(t), \qquad g(\delta, t, i) = (I - J)G(\delta(t), r(t)), \tag{8}$$

then (7) becomes

$$d\delta(t) = f(\delta, t, i)\,dt + g(\delta, t, i)\,dW(t). \tag{9}$$

Definition 1 Dynamic system (6) is said to be average consensus in pth moment if error system (7) satisfies that $\limsup_{t \to \infty} E|\delta(t)|^p = 0$. If $p = 2$, we called it reach mean-square average consensus.

Definition 2 Dynamic system (6) is said to reach stochastic average consensus in the large if error system (7) is stochastic stable and, moreover,

$$\mathbb{P}\left\{\lim_{t \to \infty} \delta(t; t_0, \delta_0, i) = 0\right\} = 1.$$

To get the main results, we need the following assumptions.

(A1) $\mathcal{G}$ is a balanced digraph;

(A2) The union of $\{\mathcal{G}(i) | i \in S\}$ contains a spanning tree.

(A3) There exists a positive constant H such that the noise intensity matrix $g(\cdot, \cdot, \cdot)$ satisfies

$$\text{trace}\big(g^T(\delta, t, i)g(\delta, t, i)\big) \leq H|\delta(t)|^2.$$

To this end, we introduce some concepts and lemmas which will be used in the proofs of our main results.

Consider an n-dimensional stochastic differential equation with Markovian switching

$$dx(t) = \tilde{f}\big(x(t), r_t, t\big)\, dt + \tilde{g}\big(x(t), r_t, t\big)\, dW(t) \tag{10}$$

on $t \in [0, \infty)$ with the initial data $x(t_0) = x_0 \in L^p_{\mathcal{L}_0}(\Omega; R^n)$ and $r(t_0) = r_0$.

For $V \in C^{2,1}(R^n \times S \times R_+; R_+)$, define an operator $\mathcal{L}$ from $R^n \times S \times R_+$ to R by

$$
\begin{aligned}
\mathcal{L}V\big(x(t), i, t\big) &= V_t\big(x(t), i, t\big) + V_x\big(x(t), i, t\big)\tilde{f}\big(x(t), i, t\big) \\
&\quad + \frac{1}{2}\text{trace}\big(\tilde{g}^T\big(x(t), i, t\big)V_{xx}\big(x(t)i, t\big)\tilde{g}\big(x(t), i, t\big)\big) \\
&\quad + \sum_{j=1}^N \gamma_{ij} V\big(x(t), j, t\big),
\end{aligned}
$$

where

$$
\begin{aligned}
V_t\big(x(t), i, t\big) &= \frac{\partial V(x(t), i, t)}{\partial t}, \\
V_x\big(x(t), i, t\big) &= \left(\frac{\partial V(x(t), i, t)}{\partial x_1}, \frac{\partial V(x(t), i, t)}{\partial x_2}, \dots, \frac{\partial V(x(t), i, t)}{\partial x_n}\right), \\
V_{xx}\big(x(t), i, t\big) &= \left(\frac{\partial^2 V(x(t), i, t)}{\partial x_j\, \partial x_k}\right)_{n \times n}.
\end{aligned}
$$

Lemma 1 [13] *Let $V \in C^{2,1}(R^n \times R_+ \times S, R_+)$ and τ_1, τ_2 be bounded stopping times such that $0 \leq \tau_1 \leq \tau_2$ a.s. If $V(x(t), t, r(t))$ and $\mathcal{L}V(x(t), t, r(t))$ etc., are bounded on $t \in [\tau_1, \tau_2]$ with probability 1, then*

$$
\begin{aligned}
\mathbb{E}V\big(\tau_2, r(\tau_2), x(\tau_2)\big) &= \mathbb{E}V\big(\tau_1, r(\tau_1), x(\tau_1)\big) \\
&\quad + \mathbb{E}\int_{\tau_1}^{\tau_2} \mathcal{L}V\big(s, r(s), x(s)\big)\, ds. \tag{11}
\end{aligned}
$$

Lemma 2 [20] *Let $N_r(t)$ be the number of switches on the interval $(t_0, t]$, then we have*

$$P\big(N_r(t) = k\big) \leq e^{-\tilde{q}t}\frac{(\bar{q}t)^k}{k!}, \quad \forall k \in \mathbb{N},$$

where $\tilde{q} = \max\{\gamma_{ij} : i, j \in S\}$, $\bar{q} = \max\{|\gamma_{ii}| : i \in S\}$.

Definition 3 [21] A square matrix $M = (m_{ij})_{n \times n}$ is called a nonsingular M-matrix if M can be expressed in the form of $M = sI_n - G$ with some $G \geq 0$ (*i.e.*, each element of G is nonnegative) and $s > \rho(G)$, where $\rho(G)$ is the spectral radius of G.

Lemma 3 [13] *If $M = (m_{ij})_{n \times n} \in R^{n \times n}$ with $m_{ij} < 0$ $(i \neq j)$, then the following statements are equivalent*:

(1) *M is a nonsingular M-matrix;*

(2) *Every real eigenvalue of M is positive;*

(3) *M is positive stable. That is, M^{-1} exists and $M^{-1} > 0$ (i.e., $M^{-1} \geq 0$ and at least one element of M^{-1} is positive).*

Lemma 4 [13] (Young inequality) *Let $a, b \in R$ and $\beta \in [0, 1]$. Then*

$$|a|^{\beta} |b|^{(1-\beta)} \leq \beta |a| + (1 - \beta)|b|.$$

Lemma 5 [13] (Gronwall inequality) *Let $T > 0$ and $u(\cdot)$ be a Borel measurable bounder nonnegative function on $[0, T]$. If*

$$u(t) \leq \alpha(t) + \int_0^t \beta u(s)\, ds, \quad \forall 0 \leq t \leq T$$

for the non-decreasing function $\alpha(t)$ and the constant β, then

$$u(t) \leq \alpha(t) \exp(\beta t), \quad \forall 0 \leq t \leq T.$$

3 Main results

In this section, we give criteria of average consensus for system (6).

3.1 Average consensus in pth moment for the WSNs

Theorem 1 *Assume that there is a function $V(t, i, \delta) \in C^{2,1}(R_+ \times S \times R^n; R_+)$ and positive constants p, c_1, c_2, λ and η such that*

$$c_1 |\delta|^p \leq V(t, i, \delta) \leq c_2 |\delta|^p, \tag{12}$$

$$\mathcal{L}V(t, i, \delta) \leq -\lambda |\delta|^p + \eta \tag{13}$$

for all $t \geq 0$. $i \in S$ and $\delta \in R^n$. Then (7) is exponential stable in pth moment.

Proof For the function $V(t, i, \delta)$, applying Lemma 1 and using the above conditions, we obtain that

$$c_1 \mathbb{E}|\delta|^p \leq \mathbb{E}V\big(0, r_0, \xi(0)\big) + \mathbb{E}\int_0^t \mathcal{L}V\big(s, r(s), \delta(s)\big)\, ds$$

$$\leq \mathbb{E}V\big(0, r_0, \xi(0)\big) + \mathbb{E}\int_0^t \big(-\lambda |\delta|^p + \eta\big)\, ds. \tag{14}$$

So,

$$\mathbb{E}|\delta|^p \leq \frac{\mathbb{E}V(0, r_0, \xi(0))}{c_1} + \int_0^t \frac{-\lambda \mathbb{E}|\delta|^p + \eta}{c_1}\, ds$$

$$\leq c(t) + \int_0^t \nu \mathbb{E}|\delta|^p\, ds, \tag{15}$$

where

$$c(t) = \frac{\mathbb{E}V(0, r_0, \xi(0)) + \eta t}{c_1} \to \infty,$$

$$\nu = -\frac{\lambda}{c_1}.$$

It can be seen that $c(t) > 0$ and $\nu < 0$. By using Gronwall's inequality, we have

$$\mathbb{E}|\delta|^p \leq c(t) \exp(\nu t).$$

Therefore

$$\limsup_{t \to \infty} \frac{1}{t} \log\left(\mathbb{E}\big|\delta(t, \xi)\big|^p\right) \leq \nu < 0.$$

Thus (7) is exponential stable in pth moment. This completes the proof. □

Now we give a criterion of average consensus in pth moment for system (6).

Theorem 2 *Assume that for each $i \in S$, there is a pair of real numbers $\alpha > 0$ and β_i such that*

$$\delta^T f(\delta, t, i) + \frac{p-1}{2}\big|g(\delta, t, i)\big|^2 \leq \alpha + \beta_i|\delta|^2$$

for all $(\delta, t) \in R^n \times R_+$, where $p \geq 2$.

Assume also that

$$M := -\operatorname{diag}(p\beta_1, \ldots, p\beta_N) - \Gamma$$

is a nonsingular M-matrix, in this case,

$$(q_1, \ldots, q_n)^T := M^{-1}\vec{1} \gg 0,$$

where $\vec{1} = (1, \ldots, 1)^T$. Then system (6) reaches average consensus in pth moment.

Proof We can see that

$$\delta^T f(\delta, t, i) + \frac{p-1}{2}\big|g(\delta, t, i)\big|^2$$

$$= -\delta^T L_i \delta + \frac{p-1}{2}\big|(I - J)G_i\big|^2$$

$$\leq \left(\frac{p-1}{2}\big|(I - J)\big|^2 H - \lambda_{\min}(L_i)\right)|\delta|^2$$

$$\leq \alpha + \beta_i|\delta|^2,$$

where $\alpha \in R^+$ and $\beta_i = \frac{p-1}{2}|(I - J)|^2 H - \lambda_{\min}(L_i)$.

Define the function $V : R_+ \times S \times R^n \to R_+$ by

$$V(\delta, t, i) = q_i |\delta|^p. \tag{16}$$

Clearly, V obeys Theorem 1 with $c_1 = \min_{i \in S} q_i$ and $c_2 = \max_{i \in S} q_i$, and $c_1 |\delta|^p \leq V(\delta, t, i) \leq c_2 |\delta|^p$.

We compute the operator $\mathcal{L}V$ as follows:

$$\begin{aligned}
\mathcal{L}V(\delta, t, i) &= pq_i |\delta|^{p-2} \delta^T f(\delta, t, i) + \frac{1}{2} pq_i |\delta|^{p-2} |g(\delta, t, i)|^2 \\
&\quad + \frac{1}{2} p(p-2) q_i |\delta|^{p-4} |\delta^T g(\delta, t, i)|^2 + \sum_{j=1}^{N} \gamma_{ij} q_j |\delta|^p \\
&= pq_i |\delta|^{p-2} \left(\delta^T f + \frac{p-1}{2} |g|^2 \right) + \sum_{j=1}^{N} \gamma_{ij} q_j |\delta|^p \\
&\leq pq_i |\delta|^{p-2} \left(\beta_i |\delta|^2 + \alpha \right) + \sum_{j=1}^{N} \gamma_{ij} q_j |\delta|^p \\
&\leq \left(p\beta_i q_i + \sum_{j=1}^{N} \gamma_{ij} q_j \right) |\delta|^p + \alpha pq_i |\delta|^{p-2} \\
&= -|\delta|^p + \alpha pq_i |\delta|^{p-2}. \tag{17}
\end{aligned}$$

By Young's inequality,

$$\begin{aligned}
\alpha pq_i |\delta|^{p-2} &= \alpha pq_i 2^{(p-2)/p} \left(\frac{1}{2} |\delta|^p \right)^{(p-2)/p} \\
&= \left[(\alpha pq_i)^{p/2} 2^{(p-2)/2} \right]^{2/p} \left(\frac{1}{2} |\delta|^p \right)^{(p-2)/p} \\
&\leq \frac{2}{p} (\alpha pq_i)^{p/2} 2^{(p-2)/2} + \frac{(p-2)}{2p} |\delta|^p. \tag{18}
\end{aligned}$$

Substituting this into (17) gives

$$\mathcal{L}V(t, i, \delta) \leq -\frac{p+1}{p} |\delta|^p + \frac{2}{p} (\alpha pq_i)^{p/2} 2^{(p-2)/2}. \tag{19}$$

Let

$$\lambda = \frac{p+1}{p},$$

$$\eta = \max_{i \in S} \left[\frac{2}{p} (\alpha pq_i)^{p/2} 2^{(p-2)/2} \right] = \frac{2}{p} (\alpha pc_2)^{p/2} 2^{(p-2)/2}.$$

We can see that λ, η are constants, so by Theorem 1,

$$\limsup_{t \to \infty} \frac{1}{t} \log \left(\mathbb{E} |\delta(t, i, \xi)|^p \right) \leq \nu < 0.$$

It means that

$$\limsup_{t \to \infty} \mathbb{E}\big|\delta(t,i,\xi)\big|^p \le \exp(vt) \overset{t \to \infty}{\longrightarrow} 0.$$

So system (6) reaches the average consensus in pth moment. This completes the proof. $\square$

Remark 1 In Theorem 2, we can easily check the condition obtained by using the new method if the dynamic system is given and the positive constant m is well chosen. To the best of our knowledge, the method combining the Lyapunov function and the M-matrix in this paper is rarely used in the researching area of the average consensus in pth moment for WSNs with Markovian switching topology and stochastic noises.

Corollary 1 *If the functions f, g satisfy*

$$\delta^T f \le k\big(\alpha + \beta_i |\delta|^2\big),$$

$$\frac{p-1}{2}|g|^2 \le (1-k)\big(\alpha + \beta_i |\delta|^2\big),$$

where $0 < k < 1$, $k \in \mathbb{R}$, then we get

$$\delta^T f(\delta,t,i) + \frac{p-1}{2}\big|g(\delta,t,i)\big|^2 \le \alpha + \beta_i |\delta|^2.$$

Following Theorem 2, system (6) *reaches consensus in pth moment.*

3.2 The stochastic average consensus in the large for the WSNs under Γ is not precisely known

In the real communication networks, the part of the elements in the desired transition probabilities matrix are difficult to obtain. Therefore, it is important and necessary, from control perspectives, to further study the WSNs with partly unknown transition probabilities.

Consequently, in this part, we consider stochastic average consensus in the large for the system, even though the transition jump rates matrix $\Gamma = (\gamma_{ij})_{N \times N}$ is not precisely known.

Theorem 3 *Assume that there exist functions $V \in C^2(R^n \times S; R_+)$, positive constants α_1, α_2 and for each $i \in S$, there exist a real number $\lambda > 0$ and symmetric positive-definite matrices P_i such that*

(i)

$$\alpha_1 \big|\delta(t)\big|^2 \le V(\delta,i) \le \alpha_2 \big|\delta(t)\big|^2, \quad \forall i \in S; \tag{20}$$

(ii)

$$2\delta^T P_i f(\delta,i) + \mathrm{trace}\big(g^T(\delta,i)P_i g(\delta,i)\big) \le -\lambda|\delta|^2; \tag{21}$$

(iii)

$$V(\delta,i) \le \frac{\alpha_2}{\alpha_1} V(\delta,j), \quad \forall i,j \in S; \tag{22}$$

(iv)

$$1 < \frac{\alpha_2}{\alpha_1} < \frac{\lambda + \tilde{q}}{\tilde{q}}, \tag{23}$$

then the system reaches stochastic average consensus in the large.

Proof Define $V(\delta, i) = \delta^T P_i \delta$ for all $i \in S$. Let $\alpha_1 = \min\{\lambda_{\min}(P_i) : i \in S\}$, $\alpha_2 = \max\{\lambda_{\max}(P_i) : i \in S\}$. Obviously,

$$\alpha_1 |\delta(t)|^2 \le V(\delta, i) \le \alpha_2 |\delta(t)|^2.$$

Also we can see that

$$\begin{aligned}
\tilde{\mathcal{L}} V(\delta, i) &= 2\delta^T P_i f(\delta, i) + \text{trace}\big(g^T(\delta, i) P_i g(\delta, i)\big) \\
&= -2\delta^T P_i L_i \delta + \text{trace}\big(g^T(\delta, i) P_i g(\delta, i)\big) \\
&\le -2\lambda_{\max}(P_i)\lambda_{\min}(L_i)|\delta|^2 + H\lambda_{\max}(P_i)|\delta|^2 \\
&= \big(-2\lambda_{\min}(L_i) + H\big)\lambda_{\max}(P_i)|\delta|^2.
\end{aligned}$$

Let $\lambda_i = (-2\lambda_{\min}(L_i) + H)\lambda_{\max}(P_i)$ and $\lambda = \min\{\lambda_i : i \in S\}$, we have $\tilde{\mathcal{L}} V(\delta, i) < -\lambda_i V(\delta, i) < -\lambda V(\delta, i)$.

On the other hand,

$$V(\delta, i) \le \frac{\alpha_2}{\alpha_1} V(\delta, j), \quad \forall i, j \in S.$$

Obviously, $1 < u = \frac{\alpha_2}{\alpha_1} = \max\{\frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)} : i, j \in S\} < \frac{\lambda + \tilde{q}}{\tilde{q}}$.

Consequently, according to Theorem 3.1 in [12], it is obtained that error system (7) is stochastic stable and

$$\mathbb{P}\left\{\lim_{t \to \infty} \delta(t; t_0, \delta_0, i) = 0\right\} = 1.$$

It means that system (6) reaches stochastic average consensus in the large. This completes the proof. □

Remark 2 In contrast with the case that the transition probabilities of the underlying Markov chain are assumed to be completely known, the circumstance that transition jump rates matrix Γ is not precisely known for the equation (7) is more general in the real control systems.

Remark 3 There are several disadvantages associated with the result derived in Theorem 3. (1) Hypothesis (22), it may reduce the set of systems to which the results in this section can be used. (2) The constructive method for choosing the best Lyapunov functions is currently unknown.

Figure 1 The topologies of the WSN in states 1, 2, 3.
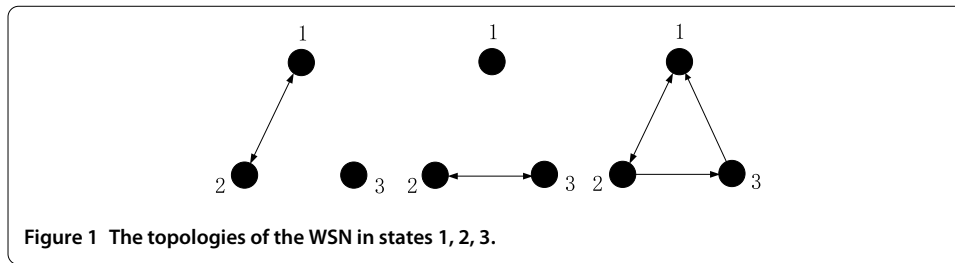
4 Numerical examples

In this section, we give two examples to examine the average consensus of system (6).

Consider a WSN composed of three sensors in which each dynamic state of the sensor is $\dot{x}_i = u_i$, where $i = 1, 2, 3$ (see the topologies in Figure 1), the initial state is $X(0) = (3, -2, 4)^T$, the switching topology of the three sensors is determined by the Markov chain $r(t)$ whose state space is $S = \{1, 2, 3\}$. The related topology graph is $\mathcal{G}(i) = \{\mathcal{V}(i), \mathcal{E}(i), \mathcal{A}(i)\}$, the adjacent matrices are:

$$\mathcal{A}(1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

$$\mathcal{A}(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{A}(3) = \begin{pmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}. \tag{25}$$

The related Laplacian matrices are:

$$\mathcal{L}(1) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{26}$$

$$\mathcal{L}(2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \mathcal{L}(3) = \begin{pmatrix} 0.5 & -0.5 & 0 \\ -1 & 1 & 0 \\ 0 & -0.5 & 0.5 \end{pmatrix}. \tag{27}$$

Example 1 Suppose that the Markovian transition matrix is

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}. \tag{28}$$

We let $p = 5$, $\alpha = 2.1|\delta|^2$, $\beta_i = 1$, $k = \frac{1}{2}$, $|g| = 0.15|\delta|$, it can be checked that assumptions (A1), (A2), and the conditions of Corollary 1 are satisfied and the matrix M is a nonsingular M-matrix. So the WSN reaches average consensus by Corollary 1.

We can draw the dynamic curve of the states of the sensors by Matlab as Figure 2. It shows us that the three sensor nodes reach the average consensus in 5th moment.

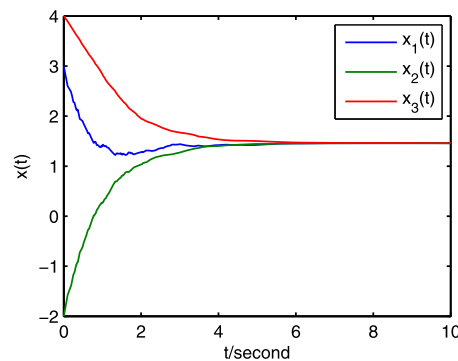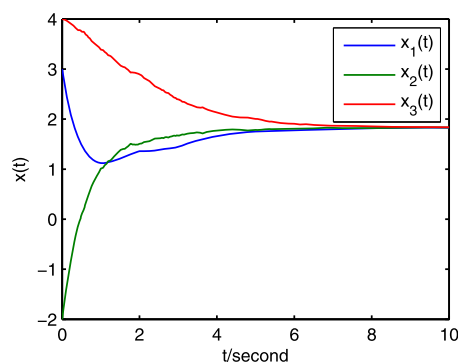**Figure 2 The trajectories of the state vectors $x_1(t)$, $x_2(t)$, $x_3(t)$.**

Figure 3 The trajectories of the state vectors $x_1(t)$, $x_2(t)$, $x_3(t)$.

Example 2 Suppose that the Markovian transition matrix with partly unknown jump rates is

$$\Gamma = \begin{pmatrix} -a & b & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \tag{29}$$

where $a, b > 0$ denotes the unknown transition jump rates subject to $a \leq 2$, $b \leq 1$, respectively. We can see that $\tilde{q} = 2$, $\bar{q} = 1$.

Letting $|g| = 0.25|\delta|$ and according to Theorem 3, we can draw the dynamic curve of the states of the sensors by Matlab as Figure 3. It shows us that the three sensor nodes reach stochastic average consensus in the large.

5 Conclusions

In this paper, we have dealt with the problem of average consensus in pth moment of the WSNs. By using the Lyapunov function and the M-matrix approach, we have obtained several sufficient conditions to ensure the average consensus in pth moment for WSNs with Markovian switching topology. Also, a sufficient condition for the stochastic average consensus in the large for the WSNs with partly known transition jump rates matrix Γ is derived. There are many other topics such as the time-delay cases and the adaptive control of the WSN's consensus in pth moment, *etc.* worth investigating.

Authors' contributions
WZ and CJ carried out the main part of this manuscript. WZ participated in the discussion and corrected the main theorem. All authors read and approved the final manuscript.

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