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Hermite and poly-Bernoulli mixed-type polynomials

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Abstract
In this paper, we consider Hermite and poly-Bernoulli mixed-type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities associated with Stirling numbers, Bernoulli and Frobenius-Euler polynomials of higher order.

1 Introduction

For $r \in \mathbb{Z}_{\geq 0}$, as is well known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\sum_{n=0}^{\infty} \frac{\mathbb{B}_n^{(r)}(x)}{n!} t^n = \left(\frac{t}{e^t - 1}\right)^r e^{xt} \quad \text{(see [1–16])}. \tag{1.1}$$

For $k \in \mathbb{Z}$, the polylogarithm is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}. \tag{1.2}$$

Note that $\mathrm{Li}_1(x) = -\log(1 - x)$.

The poly-Bernoulli polynomials are defined by the generating function to be

$$\frac{\mathrm{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} \quad \text{(see [5, 8])}. \tag{1.3}$$

When $x = 0$, $B_n^{(k)} = B_n^{(k)}(0)$ are called the poly-Bernoulli numbers (of index k).

For $\nu\ (\neq 0) \in \mathbb{R}$, the Hermite polynomials of order ν are given by the generating function to be

$$e^{-\frac{\nu t^2}{2}} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\nu)}(x) \frac{t^n}{n!} \quad \text{(see [6, 12, 13])}. \tag{1.4}$$

When $x = 0$, $H_n^{(\nu)} = H_n^{(\nu)}(0)$ are called the Hermite numbers of order ν.

In this paper, we consider the Hermite and poly-Bernoulli mixed-type polynomials $HB_n^{(\nu,k)}(x)$ which are defined by the generating function to be

$$e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} HB_n^{(\nu,k)}(x) \frac{t^n}{n!}, \tag{1.5}$$

where $k \in \mathbb{Z}$ and $\nu\ (\neq 0) \in \mathbb{R}$.

When $x = 0$, $HB_n^{(\nu,k)} = HB_n^{(\nu,k)}(0)$ are called the Hermite and poly-Bernoulli mixed-type numbers.

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ as follows:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.6}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$.

$\langle L|p(x)\rangle$ denotes the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad (n \geq 0). \tag{1.7}$$

Then, by (1.6) and (1.7), we get

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{1.8}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $O(f)$ of the power series $f(t) \neq 0$ is the smallest integer for which a_k does not vanish. If $O(f) = 0$, then $f(t)$ is called an invertible series. If $O(f) = 1$, then $f(t)$ is called a delta series. For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle. \tag{1.9}$$

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k \quad \text{(see [8, 9, 11, 13, 14])}. \tag{1.10}$$

By (1.10), we get

$$p^{(k)}(0) = \langle t^k|p(x)\rangle = \langle 1|p^{(k)}(x)\rangle, \tag{1.11}$$

where $p^{(k)}(0) = \frac{d^k p(x)}{dx^k}|_{x=0}$.

From (1.11), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [8, 9, 13])}. \tag{1.12}$$

By (1.12), we easily get

$$e^{yt} p(x) = p(x+y), \qquad \langle e^{yt} | p(x) \rangle = p(y). \tag{1.13}$$

For $O(f(t)) = 1$, $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k | x^n \rangle = n!\delta_{n,k}$ $(n, k \geq 0)$.

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

Let $p(x) \in \mathbb{P}$, $f(t) \in \mathcal{F}$. Then we see that

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = \left\langle \frac{df(t)}{dt} \middle| p(x) \right\rangle. \tag{1.14}$$

For $s_n(x) \sim (g(t), f(t))$, we have the following equations:

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) | s_k(x) \rangle}{k!} g(t) f(t)^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k | p(x) \rangle}{k!} s_k(x), \tag{1.15}$$

where $h(t) \in \mathcal{F}$, $p(x) \in \mathbb{P}$,

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(y) \frac{t^n}{n!}, \tag{1.16}$$

where $\bar{f}(t)$ is the compositional inverse for $f(t)$ with $f(\bar{f}(t)) = t$,

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y) p_{n-k}(x), \quad \text{where } p_n(x) = g(t) s_n(x), \tag{1.17}$$

$$f(t) s_n(x) = n s_{n-1}(x), \qquad s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x), \tag{1.18}$$

and the conjugate representation is given by

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j. \tag{1.19}$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, we have

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \tag{1.20}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle \quad \text{(see [8, 9, 13])}. \tag{1.21}$$

In this paper, we consider Hermite and poly-Bernoulli mixed-type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities associated with Bernoulli and Frobenius-Euler polynomials of higher order.

2 Hermite and poly-Bernoulli mixed-type polynomials

From (1.5) and (1.16), we note that

$$HB_n^{(\nu,k)}(x) \sim \left(e^{\frac{\nu t^2}{2}} \frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t\right), \tag{2.1}$$

and, by (1.3), (1.4) and (1.16), we get

$$B_n^{(k)}(x) \sim \left(\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t\right), \tag{2.2}$$

$$H_n^{(\nu)}(x) \sim \left(e^{\frac{\nu t^2}{2}}, t\right), \quad \text{where } n \geq 0. \tag{2.3}$$

From (1.18), (2.1), (2.2) and (2.3), we have

$$tB_n^{(k)}(x) = nB_{n-1}^{(k)}(x), \qquad tH_n^{(\nu)}(x) = nH_{n-1}^{(\nu)}(x), \qquad tHB_n^{(\nu,k)}(x) = nHB_{n-1}^{(\nu,k)}(x). \tag{2.4}$$

By (1.5), (1.8) and (2.1), we get

$$\begin{aligned} HB_n^{(\nu,k)}(x) &= e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n = e^{-\frac{\nu t^2}{2}} B_n^{(k)}(x) \\ &= \sum_{m=0}^{[\frac{n}{2}]} \frac{1}{m!}\left(-\frac{\nu}{2}\right)^m (n)_{2m} B_{n-2m}^{(k)}(x) \\ &= \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} \frac{(2m)!}{m!}\left(-\frac{\nu}{2}\right)^m B_{n-2m}^{(k)}(x). \end{aligned} \tag{2.5}$$

Therefore, by (2.5), we obtain the following proposition.

Proposition 1 *For $n \geq 0$, we have*

$$HB_n^{(\nu,k)}(x) = \sum_{m=0}^{[\frac{n}{2}]} \binom{n}{2m} \frac{(2m)!}{m!}\left(-\frac{\nu}{2}\right)^m B_{n-2m}^{(k)}(x).$$

From (1.5), we can also derive

$$\begin{aligned} HB_n^{(\nu,k)}(x) &= \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-\frac{\nu t^2}{2}} x^n = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} H_n^{(\nu)}(x) = \sum_{m=0}^{\infty} \frac{(1-e^{-t})^m}{(m+1)^k} H_n^{(\nu)}(x) \\ &= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} \binom{m}{j} (-1)^j e^{-jt} H_n^{(\nu)}(x) \\ &= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} \binom{m}{j} (-1)^j H_n^{(\nu)}(x-j). \end{aligned} \tag{2.6}$$

Therefore, by (2.6), we obtain the following theorem.

Theorem 2 *For* $n \geq 0$, *we have*

$$HB_n^{(\nu,k)}(x) = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} \binom{m}{j} (-1)^j H_n^{(\nu)}(x-j).$$

By (1.5), we get

$$\begin{aligned} HB_n^{(\nu,k)}(x) &= e^{-\frac{\nu t^2}{2}} B_n^{(k)}(x) = \sum_{l=0}^{\infty} \frac{1}{l!} \left(-\frac{\nu}{2}\right)^l t^{2l} B_n^{(k)}(x) \\ &= \sum_{l=0}^{[\frac{n}{2}]} \frac{1}{l!} \left(-\frac{\nu}{2}\right)^l \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} t^{2l} (x-j)^n \\ &= \sum_{l=0}^{[\frac{n}{2}]} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \binom{n}{2l} \frac{(2l)!}{l!} \left(-\frac{\nu}{2}\right)^l \frac{(-1)^j \binom{m}{j}}{(m+1)^k} \right\} (x-j)^{n-2l}. \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 3 *For* $n \geq 0$, *we have*

$$HB_n^{(\nu,k)}(x) = \sum_{l=0}^{[\frac{n}{2}]} \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \binom{n}{2l} \frac{(2l)!}{l!} \left(-\frac{\nu}{2}\right)^l \frac{(-1)^j \binom{m}{j}}{(m+1)^k} \right\} (x-j)^{n-2l}.$$

By (2.6), we get

$$\begin{aligned} HB_n^{(\nu,k)}(x) &= \sum_{m=0}^{n} \frac{(1-e^{-t})^m}{(m+1)^k} H_n^{(\nu)}(x) \\ &= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{a=0}^{n-m} \frac{m!}{(a+m)!} (-1)^a S_2(a+m,m) (n)_{a+m} H_{n-a-m}^{(\nu)}(x) \\ &= \sum_{m=0}^{n} \sum_{a=0}^{n-m} \frac{(-1)^{n-a-m} m!}{(m+1)^k} \binom{n}{n-a} S_2(n-a,m) H_a^{(\nu)}(x) \\ &= (-1)^n \sum_{a=0}^{n} \left\{ \sum_{m=0}^{n-a} \frac{(-1)^{m+a} m!}{(m+1)^k} \binom{n}{a} S_2(n-a,m) \right\} H_a^{(\nu)}(x), \end{aligned} \tag{2.8}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

Therefore, by (2.8), we obtain the following theorem.

Theorem 4 *For* $n \geq 0$, *we have*

$$HB_n^{(\nu,k)}(x) = (-1)^n \sum_{a=0}^{n} \left\{ \sum_{m=0}^{n-a} \frac{(-1)^{a+m} m!}{(m+1)^k} \binom{n}{a} S_2(n-a,m) \right\} H_a^{(\nu)}(x).$$

From (1.19) and (2.1), we have

$$\begin{aligned} HB_n^{(\nu,k)}(x) &= \sum_{j=0}^{n}\binom{n}{j}\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle| x^{n-j}\right\rangle x^j \\ &= \sum_{j=0}^{n}\binom{n}{j}\left\langle e^{-\frac{\nu t^2}{2}}\middle| B_{n-j}^{(k)}(x)\right\rangle x^j \\ &= \sum_{j=0}^{n}\binom{n}{j}\sum_{l=0}^{[\frac{n-j}{2}]}\frac{(-\frac{\nu}{2})^l}{l!}(n-j)_{2l}\left\langle 1\middle| B_{n-j-2l}^{(k)}(x)\right\rangle x^j \\ &= \sum_{j=0}^{n}\binom{n}{j}\sum_{l=0}^{[\frac{n-j}{2}]}\frac{1}{l!}\left(-\frac{\nu}{2}\right)^l(n-j)_{2l}B_{n-j-2l}^{(k)}x^j \\ &= \sum_{j=0}^{n}\left\{\sum_{l=0}^{[\frac{n-j}{2}]}\binom{n}{j}\binom{n-j}{2l}\frac{(2l)!}{l!}\left(-\frac{\nu}{2}\right)^l B_{n-j-2l}^{(k)}\right\}x^j. \end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$HB_n^{(\nu,k)}(x) = \sum_{j=0}^{n}\left\{\sum_{l=0}^{[\frac{n-j}{2}]}\binom{n}{j}\binom{n-j}{2l}\frac{(2l)!}{l!}\left(-\frac{\nu}{2}\right)^l B_{n-j-2l}^{(k)}\right\}x^j.$$

Remark By (1.17) and (2.1), we easily get

$$HB_n^{(\nu,k)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}HB_j^{(\nu,k)}(x)y^{n-j}. \tag{2.10}$$

We note that

$$HB_n^{(\nu,k)}(x) \sim \left(g(t) = e^{\frac{\nu t^2}{2}}\frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, f(t) = t\right). \tag{2.11}$$

From (1.18) and (2.11), we have

$$HB_{n+1}^{(\nu,k)}(x) = \left(x - \frac{g'(t)}{g(t)}\right)HB_n^{(\nu,k)}(x). \tag{2.12}$$

Now, we observe that

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \left(\log\left(g(t)\right)\right)' \\ &= \left(\log e^{\frac{\nu t^2}{2}} + \log\left(1-e^{-t}\right) - \log\left(\mathrm{Li}_k\left(1-e^{-t}\right)\right)\right)' \\ &= \nu t + \frac{e^{-t}}{1-e^{-t}}\left(1 - \frac{\mathrm{Li}_{k-1}(1-e^{-t})}{\mathrm{Li}_k(1-e^{-t})}\right). \end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we get

$$
\begin{aligned}
&HB_{n+1}^{(\nu,k)}(x) \\
&= xHB_n^{(\nu,k)}(x) - \frac{g'(t)}{g(t)} HB_n^{(\nu,k)}(x) \\
&= xHB_n^{(\nu,k)}(x) - \nu n HB_{n-1}^{(\nu,k)}(x) - e^{-\frac{\nu t^2}{2}} \frac{t}{e^t-1} \frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})} x^n. \qquad (2.14)
\end{aligned}
$$

It is easy to show that

$$
\begin{aligned}
\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}} &= \sum_{m=2}^{\infty} \left(\frac{1}{m^k} - \frac{1}{m^{k-1}} \right) \left(1-e^{-t}\right)^{m-1} \\
&= \left(\frac{1}{2^k} - \frac{1}{2^{k-1}} \right) t + \cdots . \qquad (2.15)
\end{aligned}
$$

Thus, by (2.15), we get

$$
\frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})} x^n = \frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{1-e^{-t}} \frac{x^{n+1}}{n+1}. \qquad (2.16)
$$

From (2.16), we can derive

$$
\begin{aligned}
&e^{-\frac{\nu t^2}{2}} \frac{t}{e^t-1} \frac{\mathrm{Li}_k(1-e^{-t}) - \mathrm{Li}_{k-1}(1-e^{-t})}{t(1-e^{-t})} x^n \\
&= \frac{1}{n+1} \left(\sum_{l=0}^{\infty} \frac{B_l}{l!} t^l \right) \left(HB_{n+1}^{(\nu,k)}(x) - HB_{n+1}^{(\nu,k-1)}(x) \right) \\
&= \frac{1}{n+1} \sum_{l=0}^{n+1} \frac{B_l}{l!} t^l \left(HB_{n+1}^{(\nu,k)}(x) - HB_{n+1}^{(\nu,k-1)}(x) \right) \\
&= \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left(HB_{n+1-l}^{(\nu,k)}(x) - HB_{n+1-l}^{(\nu,k-1)}(x) \right). \qquad (2.17)
\end{aligned}
$$

Therefore, by (2.14) and (2.17), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$
\begin{aligned}
&HB_{n+1}^{(\nu,k)}(x) \\
&= xHB_n^{(\nu,k)}(x) - \nu n HB_{n-1}^{(\nu,k)}(x) \\
&\quad - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left\{ HB_{n+1-l}^{(\nu,k)}(x) - HB_{n+1-l}^{(\nu,k-1)}(x) \right\}. \qquad (2.18)
\end{aligned}
$$

Let us take t on the both sides of (2.18). Then we have

$$
\begin{aligned}
&(n+1) HB_n^{(\nu,k)}(x) \\
&= (xt+1) HB_n^{(\nu,k)}(x) - \nu n(n-1) HB_{n-2}^{(\nu,k)}(x)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} (n+1-l) B_l \bigl\{ HB_{n-l}^{(\nu,k)}(x) - HB_{n-l}^{(\nu,k-1)}(x) \bigr\} \\
&= nxHB_{n-1}^{(\nu,k)}(x) + HB_n^{(\nu,k)}(x) - \nu n(n-1) HB_{n-2}^{(\nu,k)}(x) \\
&\quad - \sum_{l=0}^{n} \binom{n}{l} B_l \bigl(HB_{n-l}^{(\nu,k)}(x) - HB_{n-l}^{(\nu,k-1)}(x) \bigr),
\end{aligned} \tag{2.19}
$$

where $n \geq 3$.

Thus, by (2.19), we obtain the following theorem.

Theorem 7 *For $n \geq 3$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n} \binom{n}{l} B_l HB_{n-l}^{(\nu,k-1)}(x) \\
&\quad = (n+1) HB_n^{(\nu,k)}(x) - n\left(x + \frac{1}{2}\right) HB_{n-1}^{(\nu,k)}(x) \\
&\qquad + n(n-1)\left(\nu + \frac{1}{12}\right) HB_{n-2}^{(\nu,k)}(x) \\
&\qquad + \sum_{l=0}^{n-3} \binom{n}{l} B_{n-l} HB_l^{(\nu,k)}(x).
\end{aligned}
$$

By (1.5) and (1.8), we get

$$
\begin{aligned}
&HB_n^{(\nu,k)}(y) \\
&\quad = \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^n \right\rangle \\
&\quad = \left\langle \partial_t \left(e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \right) \middle| x^{n-1} \right\rangle \\
&\quad = \left\langle \bigl(\partial_t e^{-\frac{\nu t^2}{2}}\bigr) \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\
&\qquad + \left\langle e^{-\frac{\nu t^2}{2}} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle \\
&\qquad + \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \bigl(\partial_t e^{yt}\bigr) \middle| x^{n-1} \right\rangle \\
&\quad = -\nu(n-1) \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-2} \right\rangle \\
&\qquad + y \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{yt} \middle| x^{n-1} \right\rangle \\
&\qquad + \left\langle e^{-\frac{\nu t^2}{2}} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle \\
&\quad = -\nu(n-1) HB_{n-2}^{(\nu,k)}(y) + y HB_{n-1}^{(\nu,k)}(y) \\
&\qquad + \left\langle e^{-\frac{\nu t^2}{2}} \left(\partial_t \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{yt} \middle| x^{n-1} \right\rangle.
\end{aligned} \tag{2.20}
$$

Now, we observe that

$$\partial_t\left(\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right)=\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{(1-e^{-t})^2}e^{-t}. \tag{2.21}$$

From (2.21), we have

$$\begin{aligned}
&\left\langle e^{-\frac{\nu t^2}{2}}\left(\partial_t\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\right)e^{yt}\middle| x^{n-1}\right\rangle\\
&=\left\langle e^{-\frac{\nu t^2}{2}}\left(\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{(1-e^{-t})^2}\right)e^{-t}e^{yt}\middle|\frac{1}{n}tx^n\right\rangle\\
&=\frac{1}{n}\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|\frac{t}{e^t-1}x^n\right\rangle\\
&=\frac{1}{n}\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|B_n(x)\right\rangle\\
&=\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_l\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_{k-1}(1-e^{-t})-\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^{n-l}\right\rangle\\
&=\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_l\left\{HB_{n-l}^{(\nu,k-1)}(y)-HB_{n-l}^{(\nu,k)}(y)\right\},
\end{aligned} \tag{2.22}$$

where B_n are the ordinary Bernoulli numbers which are defined by the generating function to be

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}t^n.$$

Therefore, by (2.20) and (2.22), we obtain the following theorem.

Theorem 8 *For $n\geq 2$, we have*

$$\begin{aligned}
HB_n^{(\nu,k)}(x)&=-\nu(n-1)HB_{n-2}^{(\nu,k)}(x)+xHB_{n-1}^{(\nu,k)}(x)\\
&\quad+\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_l\left(HB_{n-l}^{(\nu,k-1)}(x)-HB_{n-l}^{(\nu,k)}(x)\right).
\end{aligned}$$

Now, we compute

$$\left\langle e^{-\frac{\nu t^2}{2}}\mathrm{Li}_k\left(1-e^{-t}\right)|x^{n+1}\right\rangle$$

in two different ways.

On the one hand,

$$\begin{aligned}
&\left\langle e^{-\frac{\nu t^2}{2}}\mathrm{Li}_k\left(1-e^{-t}\right)|x^{n+1}\right\rangle\\
&=\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\left(1-e^{-t}\right)\middle|x^{n+1}\right\rangle\\
&=\left\langle e^{-\frac{\nu t^2}{2}}\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}\middle|\left(1-e^{-t}\right)x^{n+1}\right\rangle
\end{aligned}$$

$$= \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^{n+1} - (x-1)^{n+1} \right\rangle$$
$$= \sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} \left\langle e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \middle| x^m \right\rangle$$
$$= \sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} HB_m^{(\nu,k)}. \tag{2.23}$$

On the other hand,

$$\left\langle e^{-\frac{\nu t^2}{2}} \mathrm{Li}_k\left(1-e^{-t}\right) | x^{n+1} \right\rangle$$
$$= \left\langle \mathrm{Li}_k\left(1-e^{-t}\right) | e^{-\frac{\nu t^2}{2}} x^{n+1} \right\rangle$$
$$= \left\langle \int_0^t \left(\mathrm{Li}_k\left(1-e^{-s}\right)\right)' ds \middle| e^{-\frac{\nu t^2}{2}} x^{n+1} \right\rangle$$
$$= \left\langle \int_0^t e^{-s} \frac{\mathrm{Li}_{k-1}(1-e^{-s})}{1-e^{-s}} ds \middle| e^{-\frac{\nu t^2}{2}} x^{n+1} \right\rangle$$
$$= \left\langle \sum_{l=0}^{\infty} \left(\sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} B_m^{(k-1)} \frac{t^{l+1}}{(l+1)!} \right) \middle| H_{n+1}^{(\nu)}(x) \right\rangle$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} B_m^{(k-1)} \frac{1}{(l+1)!} \left\langle t^{l+1} | H_{n+1}^{(\nu)}(x) \right\rangle$$
$$= \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_m^{(k-1)} H_{n-l}^{(\nu)}. \tag{2.24}$$

Therefore, by (2.23) and (2.24), we obtain the following theorem.

Theorem 9 *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} HB_m^{(\nu,k)}$$
$$= \sum_{m=0}^{n} \sum_{l=m}^{n} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B_m^{(k-1)} H_{n-l}^{(\nu)}.$$

Let us consider the following two Sheffer sequences:

$$HB_n^{(\nu,k)}(x) \sim \left(e^{\frac{\nu t^2}{2}} \frac{1-e^{-t}}{\mathrm{Li}_k(1-e^{-t})}, t \right) \tag{2.25}$$

and

$$\mathbb{B}_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t} \right)^r, t \right) \quad (r \in \mathbb{Z}_{\geq 0}). \tag{2.26}$$

Let us assume that

$$HB_n^{(\nu,k)}(x) = \sum_{m=0}^{n} C_{n,m} \mathbb{B}_m^{(r)}(x). \tag{2.27}$$

Then, by (1.20) and (1.21), we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^t-1}{t}\right)^r t^m \,\middle|\, e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \left(\frac{e^t-1}{t}\right)^r \,\middle|\, t^m HB_n^{(\nu,k)}(x) \right\rangle = \frac{1}{m!}(n)_m \left\langle \left(\frac{e^t-1}{t}\right)^r \,\middle|\, HB_{n-m}^{(\nu,k)}(x) \right\rangle \\
&= \binom{n}{m} \sum_{l=0}^{\infty} \frac{r!}{(l+r)!} S_2(l+r,r) \langle t^l | HB_{n-m}^{(\nu,k)}(x) \rangle \\
&= \binom{n}{m} \sum_{l=0}^{n-m} (n-m)_l \frac{r!}{(l+r)!} S_2(l+r,r) HB_{n-m-l}^{(\nu,k)} \\
&= \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+r}{r}} S_2(l+r,r) HB_{n-m-l}^{(\nu,k)}.
\end{aligned} \tag{2.28}$$

Therefore, by (2.27) and (2.28), we obtain the following theorem.

Theorem 10 *For $n, r \in \mathbb{Z}_{\geq 0}$, we have*

$$HB_n^{(\nu,k)}(x) = \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+r}{r}} S_2(l+r,r) HB_{n-m-l}^{(\nu,k)} \right\} \mathbb{B}_m^{(r)}(x).$$

For $\lambda\ (\neq 1) \in \mathbb{C}$, $r \in \mathbb{Z}_{\geq 0}$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!} \quad \text{(see [1, 4, 7, 9, 10])}. \tag{2.29}$$

From (1.16) and (2.29), we note that

$$H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r, t \right). \tag{2.30}$$

Let us assume that

$$HB_n^{(\nu,k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(r)}(x|\lambda). \tag{2.31}$$

By (1.21), we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^t-\lambda}{1-\lambda}\right)^r t^m \,\middle|\, e^{-\frac{\nu t^2}{2}} \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} x^n \right\rangle \\
&= \frac{(n)_m}{m!(1-\lambda)^r} \left\langle \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{r-l} e^{lt} \,\middle|\, HB_{n-m}^{(\nu,k)}(x) \right\rangle \\
&= \binom{n}{m} \frac{1}{(1-\lambda)^r} \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{r-l} \langle 1 | e^{lt} HB_{n-m}^{(\nu,k)}(x) \rangle \\
&= \frac{\binom{n}{m}}{(1-\lambda)^r} \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{r-l} HB_{n-m}^{(\nu,k)}(l).
\end{aligned} \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 11 *For $n, r \in \mathbb{Z}_{\geq 0}$, we have*

$$HB_n^{(\nu,k)}(x) = \frac{1}{(1-\lambda)^r} \sum_{m=0}^{n} \binom{n}{m} \left\{ \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{r-l} HB_{n-m}^{(\nu,k)}(l) \right\} H_m^{(r)}(x|\lambda).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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