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Growth and fixed points of solutions to second-order LDE with certain analytic coefficients in the unit disc

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Abstract
In this article, the authors investigate the growth and fixed points of solutions of certain second-order linear differential equations with analytic coefficients in the unit disc and obtain some results which improve and generalize previous results.
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Keywords: unit disc; hyper-order; fixed points

1 Introduction and results

In this paper, we shall assume that the readers are familiar with the fundamental results and standard notations of the Nevanlinna value distribution theory in the complex plane $\mathbb{C}$ and in the unit disc $\Delta = \{z : |z| < 1\}$ (see [1–6]). Before we state our main results, we need to recall some definitions and notations.

Definition 1.1 ([1, 2, 5]) For a meromorphic function $f(z)$ in Δ, the order of $f(z)$ is defined by

$$\sigma(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the characteristic function of $f(z)$. And for an analytic function $f(z)$ in Δ, we define $\sigma_M(f)$ by

$$\sigma_M(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ \log^+ M(r,f)}{\log \frac{1}{1-r}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ is the maximum modulus function of $f(z)$.

Remark 1.2 ([5]) If $f(z)$ is an analytic function in Δ, then

$$\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1.$$

Definition 1.3 ([7, 8]) Let $f(z)$ be a meromorphic function in Δ, the hyper-order of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+ \log^+ T(r,f)}{\log \frac{1}{1-r}} = \overline{\lim_{r\to 1^-}} \frac{\log_2^+ T(r,f)}{\log \frac{1}{1-r}}.$$

If $f(z)$ is an analytic function in Δ, then the hyper-order about maximum modulus of $f(z)$ is also defined by

$$\sigma_{M,2}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_3^+ M(r,f)}{\log \frac{1}{1-r}}.$$

Definition 1.4 Let $f(z)$ be a meromorphic function in Δ, the hyper-lower-order of $f(z)$ is defined by

$$\mu_2(f) = \underline{\lim_{r\to 1^-}} \frac{\log_2^+ T(r,f)}{\log \frac{1}{1-r}}.$$

If $f(z)$ is an analytic function in Δ, we define $\mu_{M,2}(f)$ by

$$\mu_{M,2}(f) = \underline{\lim_{r\to 1^-}} \frac{\log_3^+ M(r,f)}{\log \frac{1}{1-r}}.$$

Remark 1.5 ([7]) If $f(z)$ is an analytic function in Δ, then

$$\text{(i)} \quad \sigma_2(f) = \sigma_{M,2}(f), \qquad \text{(ii)} \quad \mu_2(f) = \mu_{M,2}(f).$$

Definition 1.6 The hyper convergence exponent and the hyper-lower convergence exponent of fixed points of a meromorphic function f in Δ are defined by

$$\lambda_2(f-z) = \overline{\lim_{r\to 1^-}} \frac{\log_2^+ N(r, \frac{1}{f-z})}{\log \frac{1}{1-r}}, \qquad \underline{\lambda}_2(f-z) = \underline{\lim_{r\to 1^-}} \frac{\log_2^+ N(r, \frac{1}{f-z})}{\log \frac{1}{1-r}}.$$

And we also define $\overline{\lambda}_2(f-z)$ and $\underline{\overline{\lambda}}_2(f-z)$, respectively, by

$$\overline{\lambda}_2(f-z) = \overline{\lim_{r\to 1^-}} \frac{\log_2^+ \overline{N}(r, \frac{1}{f-z})}{\log \frac{1}{1-r}} \quad \text{and} \quad \underline{\overline{\lambda}}_2(f-z) = \underline{\lim_{r\to 1^-}} \frac{\log_2^+ \overline{N}(r, \frac{1}{f-z})}{\log \frac{1}{1-r}}.$$

Many authors investigate the linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ are entire functions (*e.g.*, see [9–12]). In [10], Chen proved that if $ab \neq 0$ and $a \neq b$, then every solution $f(z) \not\equiv 0$ of (1.1) is of infinite order; furthermore, if $ab \neq 0$, $a \neq b$, $A(z) \equiv 1$, $B(z)$ is a polynomial, then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_2(f) = 1$. In 2012, Hamouda investigated the equation

$$f'' + A(z)e^{\frac{a}{(z_0-z)^\mu}} f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0, \tag{1.2}$$

where $A(z)$ and $B(z)$ are analytic functions in Δ, and he obtained the following results.

Theorem 1.7 ([13]) *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ is a real constant, a, b and z_0 are complex numbers such that $ab \neq 0$, $\arg a \neq \arg b$, $|z_0| = 1$. If $A(z)$ and $B(z)$ are analytic on z_0, then every solution $f(z) \not\equiv 0$ of* (1.2) *is of infinite order.*

Theorem 1.8 ([13]) *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ is a real constant, a, b and z_0 are complex numbers such that $ab \neq 0$, $a = cb$ $(0 < c < 1)$, $|z_0| = 1$. If $A(z)$ and $B(z)$ are analytic on z_0, then every solution $f(z) \not\equiv 0$ of* (1.2) *is of infinite order.*

Remark 1.9 Throughout this paper, we choose the principal branch of logarithm of the function $e^{\frac{\lambda}{(z_0-z)^\mu}}$ if μ is not an integer $(\lambda \in \mathbb{C} \setminus 0)$.

In this paper, we focus on studying the hyper-order and fixed points of the solutions of (1.2) and obtain the following results.

Theorem 1.10 *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc, and let $\mu > 1$ be a real constant, a, b and z_0 be complex numbers such that $ab \neq 0$, $\arg a \neq \arg b$, $|z_0| = 1$. If $A(z)$ and $B(z)$ satisfy one of the following conditions*:

(1) $\max\{\sigma_M(A), \sigma_M(B)\} \leq \mu$, *$A(z)$ and $B(z)$ are analytic on z_0*;
(2) $\sigma_M(A) < \mu$, $\sigma_M(B) \leq \mu$ *and $B(z)$ is analytic on z_0*;

then every solution $f(z) \not\equiv 0$ of (1.2) *satisfies*

(i) $\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu$;
(ii) $\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu$.

Theorem 1.11 *Under the assumptions of Theorem* 1.10, *with the exception that $ab \neq 0$, $a = cb$ $(0 < c < 1)$, $|z_0| = 1$, every solution $f(z) \not\equiv 0$ of* (1.2) *satisfies*

(i) $\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu$;
(ii) $\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu$.

Corollary 1.12 *Let $A(z), B(z) \not\equiv 0$, $C(z)$ and $D(z)$ be analytic functions in the unit disc, and let a, b and z_0 be complex numbers such that $ab \neq 0$, $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$, $|z_0| = 1$. If one of the following conditions holds,*

(1) $\max\{\sigma_M(A), \sigma_M(B), \sigma_M(C), \sigma_M(D)\} \leq \mu$ *and $A(z)$, $B(z)$, $C(z)$, $D(z)$ are analytic on z_0*;
(2) $\max\{\sigma_M(A), \sigma_M(C), \sigma_M(D)\} < \mu$, $\sigma_M(B) \leq \mu$ *and $B(z)$ is analytic on z_0*;

then every solution $f(z) \not\equiv 0$ of

$$f'' + \bigl(A(z)e^{\frac{a}{(z_0-z)^\mu}} + C(z)\bigr)f' + \bigl(B(z)e^{\frac{b}{(z_0-z)^\mu}} + D(z)\bigr)f = 0 \tag{1.3}$$

satisfies

(i) $\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu$;
(ii) $\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu$.

Theorem 1.13 *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ and ν are real constants, a, b, z_0 and z_1 are complex numbers such that $b \neq 0$, $z_0 \neq z_1$ and $|z_0| = |z_1| = 1$. If $A(z)$ and $B(z)$ satisfy one of the following conditions*:

(1) *$A(z)$ and $B(z)$ are analytic on z_0*;
(2) $\sigma_M(A) < \mu$ *and $B(z)$ is analytic on z_0*;

then every solution $f(z) \not\equiv 0$ *of*

$$f'' + A(z)e^{\frac{a}{(z_1-z)^\nu}} f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0 \tag{1.4}$$

satisfies $\mu_{M,2}(f) \geq \mu_2(f) \geq \mu$.

Theorem 1.14 *Let* $A(z)$ *and* $B(z) \not\equiv 0$ *be analytic functions in the unit disc. Suppose that* μ, ν ($\mu > 1$, $\mu \geq \nu$) *are real constants,* a, b, z_0 *and* z_1 *are complex numbers such that* $b \neq 0$, $z_0 \neq z_1$ *and* $|z_0| = |z_1| = 1$. *If* $A(z)$ *and* $B(z)$ *satisfy one of the following conditions*:

(1) $\max\{\sigma_M(A), \sigma_M(B)\} \leq \mu$, $A(z)$ *and* $B(z)$ *are analytic on* z_0;

(2) $\sigma_M(A) < \mu$, $\sigma_M(B) \leq \mu$ *and* $B(z)$ *is analytic on* z_0;

then every solution $f(z) \not\equiv 0$ *of* (1.4) *satisfies*

(i) $\mu_2(f) = \mu_{M,2}(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu$;

(ii) $\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu$.

2 Lemmas

Lemma 2.1 ([14]) *Let* k *and* j *be integers satisfying* $k > j \geq 0$, *and let* $\varepsilon > 0$. *If* $f(z)$ *is meromorphic in* Δ *such that* $f^{(j)}$ *does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \cdot \max\left\{\log\frac{1}{1-|z|}, T\big(s(|z|), f\big)\right\}\right)^{k-j} \quad \big(|z| \notin E_1\big),$$

where $E_1 \subset [0,1)$ *is a set with* $\int_{E_1} \frac{1}{1-r}\,dr < \infty$ *and* $s(|z|) = 1 - d(1-|z|)$.

Lemma 2.2 ([13]) *Let* $A(z)$ *be an analytic function on a point* $z_0 \in \mathbb{C}$, *set* $g(z) = A(z)e^{\frac{a}{(z-z_0)^\mu}}$ ($\mu > 0$ *is a real constant*), $a = \alpha + i\beta \neq 0$, $z_0 - z = Re^{i\varphi}$, $\delta_a(\varphi) = \alpha\cos(\mu\varphi) + \beta\sin(\mu\varphi)$ *and* $H = \{\varphi \in [0,2\pi) : \delta_a(\varphi) = 0\}$ (*obviously,* H *is of linear measure zero*). *Then, for any given* $\varepsilon > 0$ *and for any* $\varphi \in [0,2\pi) \setminus H$, *there exists* $R_0 > 0$ *such that for* $0 < R < R_0$, *we have*

(i) *if* $\delta_a(\varphi) > 0$, *then*

$$\exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\} \leq \big|g(z)\big| \leq \exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\};$$

(ii) *if* $\delta_a(\varphi) < 0$, *then*

$$\exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\} \leq \big|g(z)\big| \leq \exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\}.$$

Remark 2.3 ([13]) Set $\delta_a(\varphi) = \gamma\cos(\mu\varphi + \varphi_0)$, where $\gamma = \sqrt{\alpha^2+\beta^2}$, $\mu > 1$, $\varphi \in [0,2\pi)$. It is easy to know that $\delta_a(\varphi)$ changes its sign on each interval $I \subset [0,2\pi)$ satisfying $\mu \cdot mI > \pi$, where mI denotes the linear measure of the interval I.

Lemma 2.4 ([15]) *Let* $g : (0,1) \longrightarrow R$ *and* $h : (0,1) \longrightarrow R$ *be monotone increasing functions such that* $g(r) \leq h(r)$ *holds outside of an exceptional set* $E_2 \subset [0,1)$, *for which* $\int_{E_2} \frac{1}{1-r}\,dr < \infty$. *Then there exists a constant* $d \in (0,1)$ *such that if* $s(r) = 1 - d(1-r)$, *then* $g(r) \leq h(s(r))$ *for all* $r \in [0,1)$.

Lemma 2.5 ([7]) *If $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ are analytic functions of finite order in the unit disc, then every solution $f \not\equiv 0$ of*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0 \tag{2.1}$$

satisfies

$$\sigma_2(f) = \sigma_{M,2}(f) \le \max\{\sigma_M(A_j) : j = 0, 1, \ldots, k-1\}.$$

Remark 2.6 Lemma 2.5 is a special case of Theorem 2.1 in [7].

Lemma 2.7 ([4]) *Let f be a meromorphic function in the unit disc, and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r,f) = O(\log^+ T(r,f)) + O(\log(\frac{1}{1-r}))$, possibly outside a set $E_3 \subset [0,1)$ with $\int_{E_3} \frac{1}{1-r}\,dr < \infty$.

Lemma 2.8 ([8]) *Suppose that $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ are meromorphic functions in Δ, and let $f(z)$ be a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = F(z), \tag{2.2}$$

such that $\max\{\sigma_i(F), \sigma_i(A_j) : j = 0, 1, \ldots, k-1\} < \sigma_i(f)$, *where $i = 1, 2$, then*

$$\overline{\lambda}_i(f) = \lambda_i(f) = \sigma_i(f).$$

Lemma 2.9 *Suppose that $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ are meromorphic functions in Δ, and let $f(z)$ be a meromorphic solution of equation* (2.2) *such that* $\max\{\sigma_i(F), \sigma_i(A_j) : j = 0, 1, \ldots, k-1\} < \mu_i(f)$, *where $i = 1, 2$, then*

$$\underline{\overline{\lambda}}_i(f) = \underline{\lambda}_i(f) = \mu_i(f).$$

Proof Suppose that $f(z) \not\equiv 0$ is a solution of (2.2), by (2.2), we get

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right), \tag{2.3}$$

it is easy to see that if f has a zero at z_0 of order α ($\alpha > k$), and $A_0, \ldots, A_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \le k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right)$$

and

$$N\left(r, \frac{1}{f}\right) \le k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{2.4}$$

By Lemma 2.7 and (2.3), we have

$$m\left(r,\frac{1}{f}\right)\leq m\left(r,\frac{1}{F}\right)+\sum_{j=0}^{k-1}m(r,A_j)+O\left\{\log^+ T(r,f)+\log\left(\frac{1}{1-r}\right)\right\}\quad (r\notin E_3),\tag{2.5}$$

where $\int_{E_3}\frac{1}{1-r}\,dr<\infty$. By (2.4)-(2.5), we get

$$\begin{aligned}T\left(r,\frac{1}{f}\right)&\leq k\overline{N}\left(r,\frac{1}{f}\right)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)\\&\quad+O\left\{\log^+ T(r,f)+\log\left(\frac{1}{1-r}\right)\right\}\quad (r\notin E_3).\end{aligned}\tag{2.6}$$

Since $\max\{\sigma_i(F),\sigma_i(A_j):j=0,1,\ldots,k-1\}<\mu_i(f)$, then we have

$$\max\left\{\frac{T(r,F)}{T(r,f)},\frac{T(r,A_j)}{T(r,f)}\right\}\longrightarrow 0\quad \left(r\to 1^-,j=0,\ldots,k-1\right).\tag{2.7}$$

By (2.6)-(2.7) and by Lemma 2.4, for all $|z|=r\in[0,1)$, we have

$$\left(1-o(1)\right)T(r,f)\leq k\overline{N}\left(s(r),\frac{1}{f}\right)+O\left\{\log^+ T\left(s(r),f\right)+\log\left(\frac{1}{1-s(r)}\right)\right\},\tag{2.8}$$

where $s(r)=1-d(1-r)$, $d\in(0,1)$. By (2.8), we have

$$\overline{\lambda}_i(f)=\underline{\lambda}_i(f)=\mu_i(f)\quad (i=1,2).\qquad\square$$

3 Proofs of theorems

Proof of Theorem 1.10 (i) Suppose that $f(z)\not\equiv 0$ is a solution of (1.2), we obtain

$$\left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right|\leq\left|\frac{f''(z)}{f(z)}\right|+\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right|\left|\frac{f'(z)}{f(z)}\right|.\tag{3.1}$$

From Lemma 2.1, for any given $\varepsilon>0$, there exists a set $E_1\subset[0,1)$ with $\int_{E_1}\frac{1}{1-r}\,dr<\infty$ such that for all $z\in\Delta$ satisfying $|z|=r\notin E_1$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq\left(\frac{1}{1-r}\right)^M\left[T\left(s(r),f\right)\right]^k\quad (k=1,2),\tag{3.2}$$

where $s(r)=1-d(1-r)$, $d\in(0,1)$, $M>0$ is a constant, not necessarily the same at each occurrence. Set $I=\{\theta:z-z_0=Re^{i\theta},z\in\Delta\}\subset[0,2\pi)$, we have $mI\to\pi$ as $R\to 0$. Since $\mu>1$, there exists R_0 (R_0 is sufficiently small and not necessarily the same at each occurrence) such that for $0<R<R_0$, we have $\mu\cdot mI>\pi$. By Remark 2.3 and $\arg a\neq\arg b$, for all $0<R<R_0$, there exists some $\varphi\in I$ such that $\delta_b(\varphi)>0$ and $\delta_a(\varphi)<0$. From Lemma 2.2, for any given ε $(0<\varepsilon<1)$ and for all $z\in\{z:z-z_0=Re^{i\varphi},z\in\Delta\}$, there exists R_0 such that for $0<R<R_0$, we have

$$\left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right|\geq\exp\left\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\right\}.\tag{3.3}$$

By condition (1) that $A(z)$ is analytic on z_0 and by Lemma 2.2, for all $z \in E_4 = \{z : z - z_0 = Re^{i\varphi}, 0 < R < R_0, z \in \Delta\}$, we have

$$\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right| \le \exp\left\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\} < M. \tag{3.4}$$

By the metric relations in the triangle $\triangle oz_0z$, we have

$$|z|^2 = 1 + R^2 - 2R\cos\varphi^* \quad \left(\varphi^* = \angle oz_0z \in \left[0, \frac{\pi}{2}\right)\right). \tag{3.5}$$

By (3.5), for all $z \in \{z : z - z_0 = Re^{i\theta}, 0 < R < R_0, \theta \in I\}$, there exists certain ε_0 $(0 < \varepsilon_0 < 1)$ such that

$$\frac{1}{1-|z|} \ge \frac{1}{R} = \frac{1}{1-|z|}\frac{2\cos\varphi^* - R}{1+|z|} > \frac{\varepsilon_0}{1-|z|}. \tag{3.6}$$

Equation (3.6) implies $R \to 0 \Longleftrightarrow r \to 1^-$. By condition (2), $\sigma_M(A) < \mu$ and $\delta_a(\varphi) < 0$, for all $z \in \Delta$, $|z| = r \to 1^-$ and $0 < R < R_0$, we have

$$\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right| = \left|A(z)\right|\exp\left\{\frac{\delta_a(\varphi)}{R^\mu}\right\} \le M\exp\left\{\frac{1}{(1-r)^{\mu_0}}\right\}, \tag{3.7}$$

where μ_0 satisfies $\sigma_M(A) < \mu_0 < \mu$. By (3.1)-(3.4), (3.6) and (3.7), for all $z \in E_4$ and $|z| = r \notin E_1 \to 1^-$, we have

$$\exp\left\{(1-\varepsilon)\delta_b(\varphi)\frac{{\varepsilon_0}^\mu}{(1-r)^\mu}\right\} \le M\left(\frac{1}{1-r}\right)^M \exp\left\{\frac{1}{(1-r)^{\mu_0}}\right\} \cdot \left[T\left(s(r), f\right)\right]^2, \tag{3.8}$$

where $s(r) = 1 - d(1-r)$, $d \in (0,1)$. By (3.8) and Lemma 2.4, we have

$$\mu \le \mu_2(f) = \mu_{M,2}(f). \tag{3.9}$$

On the other hand, by Lemma 2.5, we have

$$\sigma_2(f) = \sigma_{M,2}(f) \le \mu. \tag{3.10}$$

By (3.9) and (3.10), we have

$$\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu.$$

(ii) Set $g(z) = f(z) - z$, $z \in \Delta$, where $f(z) \not\equiv 0$ is a solution of (1.2). It is obvious that $\overline{\lambda}_2(g) = \overline{\lambda}_2(f - z)$, $\lambda_2(g) = \lambda_2(f - z)$, $\sigma_2(g) = \sigma_2(f - z) = \sigma_2(f) = \mu$. Then equation (1.2) becomes

$$g'' + A(z)e^{\frac{a}{(z_0-z)^\mu}}g' + B(z)e^{\frac{b}{(z_0-z)^\mu}}g = -\left(A(z)e^{\frac{a}{(z_0-z)^\mu}} + zB(z)e^{\frac{b}{(z_0-z)^\mu}}\right). \tag{3.11}$$

By (3.3), (3.4) and (3.7), it is easy to see $A(z)e^{\frac{a}{(z_0-z)^\mu}} + zB(z)e^{\frac{b}{(z_0-z)^\mu}} \not\equiv 0$ by modulus estimation. By Lemma 2.8 and (3.11), we have

$$\overline{\lambda}_2(g) = \lambda_2(g) = \sigma_2(g) = \sigma_2(f) = \mu.$$

Also, by Lemma 2.9 and (3.11), we deduce $\overline{\underline{\lambda}}_2(g) = \underline{\lambda}_2(g) = \mu_2(g) = \mu$. Therefore, we obtain

$$\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu. \qquad \square$$

Proof of Theorem 1.11 (i) Similar to the proof of Theorem 1.10, we can obtain (3.1)-(3.3). Since $a = cb$ $(0 < c < 1)$ and $\delta_b(\varphi) > 0$, we have $\delta_a(\varphi) = c\delta_b(\varphi) > 0$. From conditions (1)-(2) and by (3.7), Lemma 2.2, for any given $\varepsilon > 0$ and for all $z \in E_4 = \{z : z - z_0 = Re^{i\varphi}, 0 < R < R_0, z \in \Delta\}$, we have

$$\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right| \le \exp\left\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\right\}. \tag{3.12}$$

By (3.1)-(3.3) and (3.12), for any given ε $(0 < \varepsilon < \frac{1-c}{1+c})$ and for all $z \in E_4 = \{z : z - z_0 = Re^{i\varphi}, 0 < R < R_0, z \in \Delta\}$, we have

$$\exp\left\{\left[\left(1-c-(1+c)\varepsilon\right)\right]\delta_b(\varphi)\frac{1}{R^u}\right\} \le M\left(\frac{1}{1-r}\right)^M \cdot \left[T\left(s(r),f\right)\right]^2 \quad (r \notin E_1),$$

where $s(r) = 1 - d(1-r)$, $d \in (0,1)$ and $\int_{E_1} \frac{1}{1-r}\,dr < \infty$. By (3.6) and Lemma 2.4, we obtain

$$\exp\left\{\left[\left(1-c-(1+c)\varepsilon\right)\right]\delta_b(\varphi)\frac{{\varepsilon_0}^\mu}{(1-r)^\mu}\right\} \le \left(\frac{1}{1-s(r)}\right)^M \cdot \left[T\left(s_1(r),f\right)\right]^2 \quad \left(r \to 1^-\right), \tag{3.13}$$

where $s_1(r) = 1 - d^2(1-r)$, $d \in (0,1)$. By (3.13) and Lemma 2.4, we have

$$\mu \le \mu_{M,2}(f) = \mu_2(f).$$

On the other hand, by Lemma 2.5, we have $\sigma_2(f) = \sigma_{M,2}(f) \le \mu$. Therefore, we obtain

$$\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu.$$

(ii) By the similar proof in case (ii) of Theorem 1.10, we have that

$$\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu$$

holds for every solution $f(z) \not\equiv 0$ of (1.2). $\square$

Proof of Theorem 1.13 Suppose that $f \not\equiv 0$ is a solution of (1.4), from (1.4), we obtain

$$\left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right| \le \left|\frac{f''(z)}{f(z)}\right| + \left|A(z)e^{\frac{a}{(z_1-z)^\nu}}\right|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.14}$$

Since $A(z)$ is analytic on z_0 or $\sigma_M(A) < \mu_0 < \mu$, for z near enough z_0 and $z \in \Delta$, we have

$$\left|A(z)\right| < M, \quad \text{or} \quad \left|A(z)\right| < \exp\left\{\frac{1}{(1-r)^{\mu_0}}\right\}. \tag{3.15}$$

Since $z_1 \neq z_0$, for all z near enough z_0 and $z \in \Delta$, we have

$$|z_1 - z| \ge |z_1 - z_0| - |z_0 - z| \ge \frac{|z_0 - z_1|}{2}, \qquad \left|e^{\frac{a}{(z_1-z)^\nu}}\right| \le e^{\frac{2^\nu|a|}{|z_1-z_0|^\nu}} \le M. \tag{3.16}$$

Using (3.1)-(3.3), (3.6) and (3.15)-(3.16), for all $z \in E_4$ and $|z| = r \notin E_1 \to 1^-$, we obtain

$$\exp\left\{(1-\varepsilon)\delta_b(\varphi)\frac{\varepsilon_0^{\mu}}{(1-r)^{\mu}}\right\} \le M\left(\frac{1}{1-r}\right)^M \exp\left\{\frac{1}{(1-r)^{\mu_0}}\right\} \cdot \left[T\big(s(r),f\big)\right]^2. \tag{3.17}$$

By (3.17) and Lemma 2.4, we have

$$\mu \le \mu_2(f) = \mu_{M,2}(f). \qquad \square$$

Proof of Theorem 1.14 (i) From Theorem 1.13 we have that every solution $f(z) \not\equiv 0$ of (1.4) satisfies

$$\mu \le \mu_2(f) = \mu_{M,2}(f).$$

On the other hand, by Lemma 2.5, we have that every solution $f(z) \not\equiv 0$ of (1.4) satisfies

$$\sigma_2(f) = \sigma_{M,2}(f) \le \mu.$$

Therefore every solution $f(z) \not\equiv 0$ of (1.4) satisfies

$$\mu_{M,2}(f) = \mu_2(f) = \sigma_2(f) = \sigma_{M,2}(f) = \mu.$$

(ii) Set $g(z) = f(z) - z$, $z \in \Delta$, equation (1.4) becomes

$$g'' + A(z)e^{\frac{a}{(z_1-z)^{\nu}}}g' + B(z)e^{\frac{b}{(z_0-z)^{\mu}}}g = -\Big(A(z)e^{\frac{a}{(z_1-z)^{\nu}}} + zB(z)e^{\frac{b}{(z_0-z)^{\mu}}}\Big).$$

It is easy to see $A(z)e^{\frac{a}{(z_1-z)^{\mu}}} + zB(z)e^{\frac{b}{(z_0-z)^{\mu}}} \not\equiv 0$ by (3.3), (3.15) and (3.16). By the similar proof in case (ii) of Theorem 1.10, we have that every solution $f(z) \not\equiv 0$ of (1.4) satisfies

$$\overline{\underline{\lambda}}_2(f-z) = \underline{\lambda}_2(f-z) = \overline{\lambda}_2(f-z) = \lambda_2(f-z) = \mu. \qquad \square$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JT, TYP, HYX, HZ and GYD completed the main part of this article, JT corrected the main theorems. All authors read and approved the final manuscript.

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