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Fractional order differential equations with iterations of linear modification of the argument

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Abstract

In this paper, we apply Picard operator theory to investigate a class of fractional differential equations with iterations of linear modification of the argument. For that, two useful work spaces $C_L(J,J)$ and $C_{L,\theta}(J,J)$ with three powerful norms $\|\cdot\|_B$, $\|\cdot\|_\beta$ and $\|\cdot\|_C$ are used, respectively. Some existence and uniqueness results are presented. Here, we introduce a new norm $\|\cdot\|_\beta$ and give another direct way to deal with the iterative term in the nonlinear term, which can be regarded as the main novelty in this paper.

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Keywords: fractional order differential equations; iterations; existence; Picard operator

1 Introduction

Integer order differential equations with iterations give a good approach to search for approximative solutions and have been discussed by many researchers [1–9] owning to their wide applications in engineer control and computational mathematics.

Recently, fractional order differential equations have appeared naturally in the fields such as viscoelasticity, electrical circuits, nonlinear oscillation of earthquake, *etc.* There are some remarkable monographs that provide the main theoretical tools for the qualitative analysis of fractional order differential equations and, at the same time, show the interconnection as well as the contrast between integer order differential models and fractional order differential models, [10–17].

In [9], the author discussed a first-order differential equation with iterations of linear modification of the argument

$$\begin{cases} x'(t) = f(t, x(t), x(\lambda t), x(\lambda x(\lambda t))), & 0 < \lambda < 1, t \in J_b := [0, b], b > 0, \\ x(0) = 0. \end{cases} \tag{1}$$

The existence, existence and uniqueness, and data dependence for the solutions of equation (1) were analyzed by using Picard operators and weakly Picard operators methods. The importance of iterations of linear modification of the argument will help us to find a simple way and suitable parameter to find the solution.

In [18], the authors extended to study fractional order case $q \in (0,1)$:

$$\begin{cases} {}^cD^q_{a,t}x(t) = f(t, x(t), x(x^\nu(t))) + \lambda, \quad t \in [a,b], \nu \in R \setminus \{0\}, q \in (0,1), \lambda \in R, \\ x(t) = \varphi(t), \quad t \in [a_1, a], \\ x(t) = \psi(t), \quad t \in [b, b_1], \end{cases} \tag{2}$$

where ${}^cD^q_{a,t}$ is the Caputo fractional derivative of order q with the lower limit a and $a_1 \le a < b \le b_1$, $a_1 \le a_1^\nu$ and $b_1^\nu \le b_1$, $f \in C([a,b] \times [a_1, b_1]^2, R)$ and $\varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$.

Here, we extend to another fractional order case $q \in (1,2)$:

$$\begin{cases} {}^cD^q_{0,t}x(t) = f(t, x(t), x(\lambda t), x(\lambda x(\lambda t))), \quad 0 < \lambda < 1, t \in J := [0,1], \\ x(0) = 0, \qquad x'(0) = 0, \end{cases} \tag{3}$$

where $1 < q < 2$ and f is a Carathéodory function satisfying some assumptions that will be specified later. Clearly, equation (3) is a generalization of equations (1) and (2).

For the existence results of solutions for problem (3), we emphasize that the main difficulty from the fractional order derivative ${}^cD^q_{0,t}x(\cdot)$ and iterative term $x(\lambda x(\lambda \cdot))$ in f. Compared with the results and methods in [9, 18], one can find that: (i) we introduce a new norm $\|\cdot\|_\beta$ and give another direct way to deal with the iterative term $x(\lambda x(\lambda \cdot))$ in f, which can be regarded as the main novelty in this paper; (ii) we derive new existence and uniqueness results for a general class of fractional order differential equations.

2 Preliminaries

We recall the definitions of fractional integrals and derivatives. For more details, one can refer to Kilbas *et al.* [12].

Definition 2.1 The fractional order integral of the function $h \in L^1(J, \mathbb{R})$ of order $q \in \mathbb{R}^+$ is defined by

$$I^q_{0,t}h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds,$$

where Γ is the gamma function.

Definition 2.2 For a function h given on the interval J, the qth Riemann-Liouville fractional order derivative of h is defined by

$${}^L\big(D^q_{0,t}h\big)(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} h(s)\,ds,$$

here $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3 The Caputo derivative of order q for a function $f : J \to \mathbb{R}$ can be written as

$${}^cD^q_{0,t}h(t) = {}^LD^q_{0,t}\left(h(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} h^{(k)}(0)\right), \quad t > 0, n-1 < q < n.$$

Like in the proof of Lemma 3.1 in [19], one can see that a function $x \in C(J,J)$ given by

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\, ds$$

is a unique solution of the following problem:

$$\begin{cases} {}^cD_{0,t}^q x(t) = h(t), & t \in J, 1 < q < 2, \\ x(0) = 0, & x'(0) = 0, \end{cases}$$

where $h \in C(J,J)$.

Next, we collect some notions and results from the weakly Picard operator theory (for more details, see Rus [20, 21]).

Let (X,d) be a metric space, $A : X \to X$ be an operator, and $F_A = \{x \in X : A(x) = x\}$ be the fixed point set of A.

Definition 2.4 (Rus [22, 23]) Let (X,d) be a metric space. An operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that $F_A = \{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.

Definition 2.5 (Jung *et al.* [24]) Suppose that E is a vector space over $\mathbb{K}$. A function $\|\cdot\|_\beta$ $(0 < \beta \le 1) : E \to [0,\infty)$ is called a β-norm if and only if it satisfies (i) $\|x\|_\beta = 0$ if and only if $x = 0$; (ii) $\|\lambda x\|_\beta = |\lambda|^\beta \|x\|_\beta$ for all $\lambda \in \mathbb{K}$ and all $x \in E$; (iii) $\|x+y\|_\beta \le \|x\|_\beta + \|y\|_\beta$.

Let $C(J,J)$ be the space of all continuous functions from J into J. For some $L > 0$ and $0 < \theta \le 1$, we consider the following spaces:

$$\begin{aligned} C_L(J,J) &:= \bigl\{x \in C(J,J) : \bigl|x(t_1) - x(t_2)\bigr| \le L|t_1 - t_2| \text{ for all } t_1, t_2 \in J\bigr\}, \\ C_{L,\theta}(J,J) &:= \bigl\{x \in C_L(J,J) : x(t) \le \theta t \text{ for all } t \in J\bigr\}. \end{aligned}$$

Meanwhile, we introduce three powerful norms $\|\cdot\|_B$, $\|\cdot\|_\beta$ and $\|\cdot\|_C$ in the space of $C(J,\mathbb{R})$ which are defined by

$$\begin{aligned} &\|x\|_B := \max_{t\in J} \bigl|x(t)\bigr| e^{-\tau t} \quad (\tau > 0), \qquad \|x\|_\beta := \max_{t\in J} \bigl|x(t)\bigr|^\beta \quad (0 < \beta < 1), \\ &\|x\|_C := \max_{t\in J} \bigl|x(t)\bigr|. \end{aligned}$$

Let d_B, d_β and d_C be their corresponding metrics, respectively.

Obviously, if $d \in \{d_C, d_B, d_\beta\}$, then the spaces $(C_L(J,J), d)$, $(C_L^q(J,J), d)$ and $(C_{L,\theta}(J,J), d)$ are complete metric spaces.

3 First results in $(C_{L,\theta}(J,J), \|\cdot\|_B)$

We will use $\|\phi\|_{L^p(J)}$ to denote the $L^p(J,\mathbb{R}_+)$ norm of ϕ whenever $\phi \in L^p(J,\mathbb{R}_+)$ for some p with $1 < p < \infty$. Let $q_i \in (0,1)$, $i = 1,2$, and $\eta(\cdot) \in L^{\frac{1}{q_2}}(J,\mathbb{R}_+)$. For brevity, let $N := \|\eta\|_{L^{\frac{1}{q_2}}(J)}$, $\gamma := \frac{q-1}{1-q_2} \in (-1,0)$.

We introduce the following assumptions:

(C1) $f : J^4 \to \mathbb{R}$ is a Carathéodory function.
(C2) $f \in C(J^4, R)$ and there exist $L_u, L_v, L_w > 0$ such that

$$\bigl|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)\bigr| \le L_u|u_1 - u_2| + L_v|v_1 - v_2| + L_w|w_1 - w_2|$$

for all $t, u_i, v_i, w_i \in J$, $i = 1, 2$.
(C3) There are m_f, M_f, $0 < \theta \le 1$ such that

$$0 \le m_f < M_f \le \theta\Gamma(q+1)$$

and

$$m_f \le f(t,u,v,w) \le M_f \quad \text{for all } t \in J.$$

(C4) There exists a constant $L > 0$ such that $L \ge \frac{\max\{|m_f|,|M_f|\}}{\Gamma(q)}$.
(C5) For some $\tau > 0$ and $0 < q_1, \lambda < 1$,

$$L_A := \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(L_u + \frac{L_v + L_w L\lambda}{\lambda^{q_1}} + \frac{L_w}{(\lambda^2\theta)^{q_1}}\right)\left(\frac{q_1}{\tau}\right)^{q_1} < 1.$$

Theorem 3.1 *Assume that* (C1)-(C5) *are satisfied. Then problem* (3) *has a unique solution in* $(C_{L,\theta}(J,J), \|\cdot\|_B)$.

Proof Consider the operator

$$A : \bigl(C_{L,\theta}(J,J), \|\cdot\|_B\bigr) \to \bigl(C(J,\mathbb{R}), \|\cdot\|_B\bigr)$$

defined by

$$A(x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f\bigl(s, x(s), x(\lambda s), x\bigl(\lambda x(\lambda s)\bigr)\bigr)\,ds, \quad t \in J. \tag{4}$$

It is clear that $(C_{L,\theta}(J,J), \|\cdot\|_B)$ is a nonempty bounded closed convex subset of the Banach space $(C(J,\mathbb{R}), \|\cdot\|_B)$.

We firstly prove that $C_{L,\theta}(J,J)$ is an invariant subset for A. In fact, we obtain $0 \le Ax(t) \le 1$ and $Ax(t) \le \theta t$ for all $t \in J$ clearly due to (C2) and (C3).

Moreover, consider $0 < s_1 < s_2 \le 1$. Then

$$\begin{aligned}
&\bigl|A(x)(s_2) - A(x)(s_1)\bigr| \\
&\quad\le \frac{1}{\Gamma(q)}\int_0^{s_1} \bigl|(s_2-s)^{q-1} - (s_1-s)^{q-1}\bigr| \bigl|f\bigl(s, x(s), x(\lambda s), x\bigl(\lambda x(\lambda s)\bigr)\bigr)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(q)}\int_{s_1}^{s_2} (s_2-s)^{q-1}\bigl|f\bigl(s, x(s), x(\lambda s), x\bigl(\lambda x(\lambda s)\bigr)\bigr)\bigr|\,ds \\
&\quad\le \frac{\max\{|m_f|, |M_f|\}}{\Gamma(q)}\left(\int_0^{s_1}\bigl[(s_2-s)^{q-1} - (s_1-s)^{q-1}\bigr]\,ds + \int_{s_1}^{s_2}(s_2-s)^{q-1}\,ds\right) \\
&\quad\le \frac{\max\{|m_f|, |M_f|\}}{\Gamma(q)}|s_2 - s_1|.
\end{aligned}$$

Moreover, for $0 = s_1 < s_2 \le 1$, we have

$$\begin{aligned}\big|A(x)(s_2) - A(x)(s_1)\big| &\le \left|\frac{1}{\Gamma(q)}\int_{s_1}^{s_2}(s_2 - s)^{q-1} f\big(s, x(s), x(\lambda s), x\big(\lambda x(\lambda s)\big)\big)\,ds\right| \\ &\le \frac{\max\{|m_f|, |M_f|\}}{\Gamma(q+1)}|s_2 - s_1|.\end{aligned}$$

Thus, we have $C_{L,\theta}(J,J)$ is an invariant subset for the operator A.

From condition (C5) it follows that A is a Picard mapping. Indeed, for all $t \in J$, we get

$$\begin{aligned}&\big|A(x_1)(t) - A(x_2)(t)\big| \\ &\quad\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[L_u\big|x_1(s) - x_2(s)\big| + L_v\big|x_1(\lambda s) - x_2(\lambda s)\big| \\ &\qquad + L_w\big|x_1\big(\lambda x_1(\lambda s)\big) - x_2\big(\lambda x_2(\lambda s)\big)\big|\big]\,ds \\ &\quad\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[L_u\big|x_1(s) - x_2(s)\big| + L_v\big|x_1(\lambda s) - x_2(\lambda s)\big| \\ &\qquad + L_w\big|x_1\big(\lambda x_1(\lambda s)\big) - x_1\big(\lambda x_2(\lambda s)\big)\big| \\ &\qquad + L_w\big|x_1\big(\lambda x_2(\lambda s)\big) - x_2\big(\lambda x_2(\lambda s)\big)\big|\big]\,ds \\ &\quad\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[L_u\big|x_1(s) - x_2(s)\big| + L_v\big|x_1(\lambda s) - x_2(\lambda s)\big| \\ &\qquad + L_w L\big|\lambda x_1(\lambda s) - \lambda x_2(\lambda s)\big| \\ &\qquad + L_w\big|x_1\big(\lambda x_2(\lambda s)\big) - x_2\big(\lambda x_2(\lambda s)\big)\big|\big]\,ds \\ &\quad\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[L_u\big|x_1(s) - x_2(s)\big|e^{-\tau s}e^{\tau s} \\ &\qquad + L_v\big|x_1(\lambda s) - x_2(\lambda s)\big|e^{-\tau\lambda s}e^{\tau\lambda s} + L_w L\lambda\big|x_1(\lambda s) - x_2(\lambda s)\big|e^{-\tau\lambda s}e^{\tau\lambda s} \\ &\qquad + L_w\big|x_1\big(\lambda x_2(\lambda s)\big) - x_2\big(\lambda x_2(\lambda s)\big)\big|e^{-\tau\lambda x_2(\lambda s)}e^{\tau\lambda x_2(\lambda s)}\big]\,ds \\ &\quad\le \frac{1}{\Gamma(q)}\bigg[L_u\int_0^t (t-s)^{q-1}e^{\tau s}\,ds + L_v\int_0^t (t-s)^{q-1}e^{\tau\lambda s}\,ds \\ &\qquad + L_w L\lambda\int_0^t (t-s)^{q-1}e^{\tau\lambda s}\,ds + L_w\int_0^t (t-s)^{q-1}e^{\tau\lambda x_2(\lambda s)}\,ds\bigg]\|x_1 - x_2\|_B \\ &\quad\le \frac{1}{\Gamma(q)}\bigg[L_u\int_0^t (t-s)^{q-1}e^{\tau s}\,ds + L_v\int_0^t (t-s)^{q-1}e^{\tau\lambda s}\,ds \\ &\qquad + L_w L\lambda\int_0^t (t-s)^{q-1}e^{\tau\lambda s}\,ds \\ &\qquad + L_w\int_0^t (t-s)^{q-1}e^{\tau\lambda^2\theta s}\,ds\bigg]\|x_1 - x_2\|_B \\ &\quad\le \frac{1}{\Gamma(q)}\bigg[L_u\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau}\right)^{q_1}e^{t\tau} + L_v\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau\lambda}\right)^{q_1}e^{t\tau\lambda} \\ &\qquad + L_w L\lambda\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau\lambda}\right)^{q_1}e^{t\tau\lambda} \\ &\qquad + L_w\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau\lambda^2\theta}\right)^{q_1}e^{t\tau\lambda^2\theta}\bigg]\|x_1 - x_2\|_B\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau}\right)^{q_1} e^{t\tau}\biggl[L_u + \frac{L_v e^{t\tau(\lambda-1)}}{\lambda^{q_1}} \\
&\quad + \frac{L_w L\lambda e^{t\tau(\lambda-1)}}{\lambda^{q_1}} + \frac{L_w e^{t\tau(\lambda^2\theta-1)}}{(\lambda^2\theta)^{q_1}}\biggr]\|x_1 - x_2\|_B \\
&\leq \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(L_u + \frac{L_v + L_w L\lambda}{\lambda^{q_1}} + \frac{L_w}{(\lambda^2\theta)^{q_1}}\right)\left(\frac{q_1}{\tau}\right)^{q_1} e^{t\tau}\|x_1 - x_2\|_B,
\end{aligned}$$

which yields that

$$\begin{aligned}
&\bigl|A(x_1)(t) - A(x_2)(t)\bigr| e^{-\tau t} \\
&\quad \leq \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(L_u + \frac{L_v + L_w L\lambda}{\lambda^{q_1}} + \frac{L_w}{(\lambda^2\theta)^{q_1}}\right)\left(\frac{q_1}{\tau}\right)^{q_1}\|x_1 - x_2\|_B.
\end{aligned}$$

Thus

$$\begin{aligned}
&\bigl\|A(x_1) - A(x_2)\bigr\|_B \\
&\quad \leq \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(L_u + \frac{L_v + L_w L\lambda}{\lambda^{q_1}} + \frac{L_w}{(\lambda^2\theta)^{q_1}}\right)\left(\frac{q_1}{\tau}\right)^{q_1}\|x_1 - x_2\|_B,
\end{aligned}$$

where we use the inequality

$$\begin{aligned}
\int_0^t (t-s)^{q-1} e^{\tau s}\,ds &\leq \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}\,ds\right)^{1-q_1}\left(\int_0^t e^{\frac{\tau s}{q_1}}\,ds\right)^{q_1} \\
&\leq \left(\frac{1-q_1}{q-q_1} t^{\frac{q-q_1}{1-q_1}}\right)^{1-q_1}\left(\frac{q_1}{\tau}\bigl(e^{\frac{t\tau}{q_1}} - 1\bigr)\right)^{q_1} \\
&\leq \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau}\right)^{q_1}\bigl(e^{\frac{t\tau}{q_1}}\bigr)^{q_1} \\
&= \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(\frac{q_1}{\tau}\right)^{q_1} e^{t\tau}.
\end{aligned}$$

So we get

$$\bigl\|A(x_1) - A(x_2)\bigr\|_B \leq L_A\|x_1 - x_2\|_B,$$

where

$$L_A = \frac{1}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(L_u + \frac{L_v + L_w L\lambda}{\lambda^{q_1}} + \frac{L_w}{(\lambda^2\theta)^{q_1}}\right)\left(\frac{q_1}{\tau}\right)^{q_1}.$$

Thus, A is of Lipschitz type with a constant $L_A \in (0,1)$ due to (C5). By applying the contraction principle, we obtain that A is a Picard operator. This completes the proof. □

4 Second results in $(C_L(J,J), \|\cdot\|_C)$

We give the following necessary assumptions.

(C1′) There are m_f, M_f such that

$$0 \leq m_f < M_f \leq \Gamma(q+1)$$

and

$$m_f \le f(t,u,w) \le M_f \quad \text{for all } t \in J.$$

(C2′) There is $0 < \lambda < 1$ such that $L'_A := \frac{L_u+L_v+L_w+L_wL\lambda}{\Gamma(q+1)} < 1$.

Theorem 4.1 *Assume that* (C1), (C2), (C4), (C1′) *and* (C2′) *are satisfied. Then problem* (3) *has a unique solution in* $(C_L(J,J), \|\cdot\|_C)$.

Proof Consider the operator

$$A : \bigl(C_L(J,J), \|\cdot\|_C\bigr) \to \bigl(C(J,\mathbb{R}), \|\cdot\|_C\bigr)$$

defined by (4).

It is clear that $(C_L(J,J), \|\cdot\|_C)$ is a nonempty bounded closed convex subset of the Banach space $(C(J,\mathbb{R}), \|\cdot\|_C)$.

Step 1. According to (C1), (C2), (C4), (C1′), $C_L(J,J)$ is clearly an invariant subset for A.

Step 2. We prove that A is of Lipschitz type with a constant

$$L'_A = \frac{L_u + L_v + L_w + L_wL\lambda}{\Gamma(q+1)}.$$

Indeed, for all $t \in J$, taking into account (C1), we get

$$\begin{aligned}
&\bigl|A(x_1)(t) - A(x_2)(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\bigl[L_u\bigl|x_1(s)-x_2(s)\bigr| + L_v\bigl|x_1(\lambda s)-x_2(\lambda s)\bigr| \\
&\qquad + L_w\bigl|x_1\bigl(\lambda x_1(\lambda s)\bigr) - x_2\bigl(\lambda x_2(\lambda s)\bigr)\bigr|\bigr]\,ds \\
&\quad\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\bigl[L_u\bigl|x_1(s)-x_2(s)\bigr| + L_v\bigl|x_1(\lambda s)-x_2(\lambda s)\bigr| \\
&\qquad + L_w\bigl|x_1\bigl(\lambda x_1(\lambda s)\bigr) - x_1\bigl(\lambda x_2(\lambda s)\bigr)\bigr| + L_w\bigl|x_1\bigl(\lambda x_2(\lambda s)\bigr) - x_2\bigl(\lambda x_2(\lambda s)\bigr)\bigr|\bigr]\,ds \\
&\quad\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\bigl[L_u\|x_1-x_2\|_C + L_v\|x_1-x_2\|_C \\
&\qquad + L_wL\bigl|\lambda x_1(\lambda s) - \lambda x_2(\lambda s)\bigr| + L_w\|x_1-x_2\|_C\bigr]\,ds \\
&\quad\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}\bigl[L_u\|x_1-x_2\|_C + L_v\|x_1-x_2\|_C \\
&\qquad + L_wL\lambda\|x_1-x_2\|_C + L_w\|x_1-x_2\|_C\bigr]\,ds \\
&\quad\le \frac{L_u + L_v + L_w + L_wL\lambda}{\Gamma(q+1)}\|x_1-x_2\|_C \\
&\quad= L'_A\|x_1-x_2\|_C.
\end{aligned}$$

So we get

$$\bigl\|A(x_1) - A(x_2)\bigr\|_C \le L'_A\|x_1-x_2\|_C.$$

Thus, A is of Lipschitz type with a constant $L'_A \in (0,1)$ due to (C2′). By applying the contraction principle, we obtain that A is a Picard operator. This completes the proof. □

5 Third results in $(C_L(J,J), \|\cdot\|_\beta)$

We impose the following condition:

(C1″) There is $0 < \lambda < 1$ such that $L''_A := \frac{L_u^\beta + L_v^\beta + (L_w L\lambda)^\beta + L_w^\beta}{\Gamma^\beta(q+1)} < 1$.

Theorem 5.1 *Assume that* (C1), (C2), (C4), (C1′), (C1″) *are satisfied. Then problem* (3) *has a unique solution in* $(C_L(J,J), \|\cdot\|_\beta)$.

Proof Consider $A : (C_L(J,J), \|\cdot\|_\beta) \to (C(J,R), \|\cdot\|_\beta)$ given by (4).

It is clear that $(C_L(J,J), \|\cdot\|_\beta)$ is a nonempty bounded closed convex subset of the Banach space $(C(J,R), \|\cdot\|_\beta)$.

Similar to the proof of Step 1 in Theorem 3.1, one can easily verify that $C_L(J,J)$ is an invariant subset for A due to (C1), (C2), (C4), (C1′).

Next, we have to prove that A is a Lipschitz-type operator.

For all $x, z \in C_L(J,J)$, by using our conditions, we have

$$\begin{aligned}
&\bigl|A(x)(t) - A(z)(t)\bigr|^\beta \\
&\quad\le \left(\frac{1}{\Gamma(q)}\right)^\beta \biggl|\int_0^t (t-s)^{q-1}\bigl[L_u\bigl|x_1(s) - x_2(s)\bigr| + L_v\bigl|x_1(\lambda s) - x_2(\lambda s)\bigr| \\
&\qquad + L_w\bigl|x_1\bigl(\lambda x_1(\lambda s)\bigr) - x_2\bigl(\lambda x_2(\lambda s)\bigr)\bigr|\bigr]\,ds\biggr|^\beta \\
&\quad\le \left(\frac{1}{\Gamma(q)}\right)^\beta \biggl|\int_0^t (t-s)^{q-1}\bigl[L_u\bigl|x_1(s) - x_2(s)\bigr| + L_v\bigl|x_1(\lambda s) - x_2(\lambda s)\bigr| \\
&\qquad + L_w\bigl|x_1\bigl(\lambda x_1(\lambda s)\bigr) - x_1\bigl(\lambda x_2(\lambda s)\bigr)\bigr| + L_w\bigl|x_1\bigl(\lambda x_2(\lambda s)\bigr) - x_2\bigl(\lambda x_2(\lambda s)\bigr)\bigr|\bigr]\,ds\biggr|^\beta \\
&\quad\le \left(\frac{1}{\Gamma(q)}\right)^\beta \biggl|\int_0^t (t-s)^{q-1}\bigl[L_u\bigl|x_1(s) - x_2(s)\bigr| + L_v\bigl|x_1(\lambda s) - x_2(\lambda s)\bigr| \\
&\qquad + L_w L\lambda\bigl|x_1(\lambda s) - x_2(\lambda s)\bigr| + L_w\bigl|x_1\bigl(\lambda x_2(\lambda s)\bigr) - x_2\bigl(\lambda x_2(\lambda s)\bigr)\bigr|\bigr]\,ds\biggr|^\beta \\
&\quad\le \left(\frac{1}{\Gamma(q)}\right)^\beta \biggl|\int_0^t (t-s)^{q-1}\bigl[L_u\|x_1 - x_2\|_\beta^{\frac{1}{\beta}} + L_v\|x_1 - x_2\|_\beta^{\frac{1}{\beta}} \\
&\qquad + L_w L\lambda\|x_1 - x_2\|_\beta^{\frac{1}{\beta}} + L_w\|x_1 - x_2\|_\beta^{\frac{1}{\beta}}\bigr]\,ds\biggr|^\beta \\
&\quad\le \frac{1}{\Gamma^\beta(q+1)}\bigl[L_u^\beta\|x_1 - x_2\|_\beta + L_v^\beta\|x_1 - x_2\|_\beta \\
&\qquad + (L_w L\lambda)^\beta\|x_1 - x_2\|_\beta + L_w^\beta\|x_1 - x_2\|_\beta\bigr] \\
&\quad\le \frac{L_u^\beta + L_v^\beta + (L_w L\lambda)^\beta + L_w^\beta}{\Gamma^\beta(q+1)}\|x_1 - x_2\|_\beta,
\end{aligned}$$

where we use the inequality $(a+b)^n \le a^n + b^n$ for any $n < 1$ and nonnegative a, b.

So we get

$$\left\|A(x_1)-A(x_2)\right\|_\beta \le L''_A\|x_1-x_2\|_\beta.$$

Thus, A is of Lipschitz type with a constant $L''_A \in (0,1)$ due to (C1″). By applying the contraction principle, we obtain that A is a Picard operator. This completes the proof. □

6 Example

Let us consider the following problem:

$$\begin{cases} {}^cD^{1.5}_{0,t}x(t) = 0.5x(0.5x(0.5t)), & t \in [0,1], \\ x(0) = 0, \qquad x'(0) = 0. \end{cases} \tag{5}$$

We have the following two propositions.

Proposition 6.1 *Problem* (5) *has a unique solution in* $(C_{0.56419}([0,1],[0,1]), \|\cdot\|_C)$.

Proof By Theorem 4.1, we choose $\lambda = \frac{1}{2}$, $q = \frac{3}{2}$, $L = 0.56419$, $m_f = 0$ and $M_f = \frac{1}{2}$. Clearly, one can verify that conditions (C1), (C2), (C4), (C1′) and (C2′) from Theorem 4.1 hold. □

Proposition 6.2 *Problem* (5) *has a unique solution in* $(C_{0.56419}([0,1],[0,1]), \|\cdot\|_{\frac{1}{2}})$.

Proof By Theorem 5.1, we choose $\beta = \frac{1}{2}$, $\lambda = \frac{1}{2}$, $q = \frac{3}{2}$, $L = 0.56419$, $m_f = 0$ and $M_f = \frac{1}{2}$. Clearly, one can verify that conditions (C1), (C2), (C4), (C1′) and (C1″) from Theorem 5.1 hold. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. JRW raised these interesting problems in this research. JRW and JHD proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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