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Solvability for a coupled system of fractional differential equations with integral boundary conditions at resonance

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Abstract

By constructing suitable operators, we investigate the existence of solutions for a coupled system of fractional differential equations with integral boundary conditions at resonance. Our analysis relies on the coincidence degree theory due to Mawhin. An example is given to illustrate our main result.

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1 Introduction

Fractional differential equations arise in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, electron-analytical chemistry, biology, control theory *etc.* (see [1, 2]). Recently, more and more authors have paid their close attention to them (see [3–24]). The existence of solutions for differential equations at resonance has been studied by many authors (see [19–23, 25–29] and references cited therein). In papers [19–22], the authors investigated the fractional differential equations with multi-point boundary conditions at resonance. In paper [23], the authors discussed a coupled system of fractional differential equations with two-point boundary condition at resonance. In paper [24], the authors showed the existence of solutions for higher-order fractional differential inclusions with multi-strip fractional integral boundary conditions. In paper [26], the authors studied solvability of integer-order differential equations with integral boundary conditions at resonance, which was the generalization of two, three, multi-point and nonlocal boundary value problems.

Motivated by the excellent results mentioned above, in this paper, we discuss the existence of solutions for a coupled system of fractional differential equations with integral boundary conditions at resonance

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f_1(t, x(t), y(t), D_{0^+}^{\alpha-1}x(t)), \quad 0 < t < 1, \\ D_{0^+}^{\beta}y(t) = f_2(t, x(t), y(t), D_{0^+}^{\beta-1}y(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad D_{0^+}^{\alpha-1}x(0) = \int_0^1 h_1(t)D_{0^+}^{\alpha-1}x(t)\,dt, \\ D_{0^+}^{\alpha-1}x(1) = \int_0^1 h_2(t)D_{0^+}^{\alpha-1}x(t)\,dt, \\ y(0) = 0, \qquad D_{0^+}^{\beta-1}y(0) = \int_0^1 g_1(t)D_{0^+}^{\beta-1}y(t)\,dt, \\ D_{0^+}^{\beta-1}y(1) = \int_0^1 g_2(t)D_{0^+}^{\beta-1}y(t)\,dt, \end{cases} \tag{1.1}$$

where $2 < \alpha, \beta \leq 3$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $D_{0^+}^{\gamma}u(\xi) := D_{0^+}^{\gamma}u(t)|_{t=\xi}$. To the best of our knowledge, this is the first paper to study the boundary value problems of a coupled system of fractional differential equations with integral boundary conditions at resonance with $\dim \operatorname{Ker} L = 4$.

In this paper, we will always suppose that the following conditions hold.

(H_1) $2 < \alpha, \beta \leq 3$, $h_i, g_i \in L[0,1]$, $\int_0^1 h_i(t)\,dt = 1$, $\int_0^1 g_i(t)\,dt = 1$, $i = 1, 2$.

(H_2)

$$\Delta_1 = \begin{vmatrix} \int_0^1 th_1(t)\,dt & 1 - \int_0^1 th_2(t)\,dt \\ \frac{1}{2}\int_0^1 t^2 h_1(t)\,dt & \frac{1}{2}(1 - \int_0^1 t^2 h_2(t)\,dt) \end{vmatrix} := \begin{vmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{vmatrix} \neq 0,$$

$$\Delta_2 = \begin{vmatrix} \int_0^1 tg_1(t)\,dt & 1 - \int_0^1 tg_2(t)\,dt \\ \frac{1}{2}\int_0^1 t^2 g_1(t)\,dt & \frac{1}{2}(1 - \int_0^1 t^2 g_2(t)\,dt) \end{vmatrix} := \begin{vmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{vmatrix} \neq 0.$$

(H_3) $f_i : [0,1] \times R^3 \to R$ satisfies the Carathéodory conditions and there exist functions $a_{0i}(t), b_{0i}(t), c_{0i}(t), d_{0i}(t), r_i(t) \in L[0,1]$ and constants $\eta_1, \eta_2 \in (0,1)$ with $c_0 < 1$, $c_0' < 1$, $\frac{1}{\Gamma(\alpha)(1-c_0)}(2 + \frac{1}{\eta_1^{\alpha-2}})a_0 < 1$, $\frac{1}{\Gamma(\beta)(1-c_0')}(2 + \frac{1}{\eta_2^{\beta-2}})b_0' < 1$, $A_1A_2a_0'b_0 < 1$ such that

$$\bigl|f_1(t,x,y,z)\bigr| \leq a_{01}(t)|x| + b_{01}(t)|y| + c_{01}(t)|z| + d_{01}(t)|x|^{\theta_1} + r_1(t),$$
$$\bigl|f_2(t,x,y,z)\bigr| \leq a_{02}(t)|x| + b_{02}(t)|y| + c_{02}(t)|z| + d_{02}(t)|y|^{\theta_2} + r_2(t),$$

where $a_0 = \int_0^1 a_{01}(t)\,dt$, $b_0 = \int_0^1 b_{01}(t)\,dt$, $c_0 = \int_0^1 c_{01}(t)\,dt$, $d_0 = \int_0^1 d_{01}(t)\,dt$, $r_0 = \int_0^1 r_1(t)\,dt$, $a_0' = \int_0^1 a_{02}(t)\,dt$, $b_0' = \int_0^1 b_{02}(t)\,dt$, $c_0' = \int_0^1 c_{02}(t)\,dt$, $d_0' = \int_0^1 d_{02}(t)\,dt$, $r_0' = \int_0^1 r_2(t)\,dt$, $0 \leq \theta_1$, $\theta_2 < 1$, $A_1 = \frac{2\eta_1^{\alpha-2}+1}{\Gamma(\alpha)(1-c_0)\eta_1^{\alpha-2} - a_0(2\eta_1^{\alpha-2}+1)}$, $A_2 = \frac{2\eta_2^{\beta-2}+1}{\Gamma(\beta)(1-c_0')\eta_2^{\beta-2} - b_0'(2\eta_2^{\beta-2}+1)}$.

2 Preliminaries

For convenience, we introduce some notations and a theorem. For more details, see [30].

Let X and Y be real Banach spaces and $L : \operatorname{dom}(L) \subset X \to Y$ be a Fredholm operator with index zero, let $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L, \qquad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P.

Assume that Ω is an open bounded subset of X, $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. The map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 [30] *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(1) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$;

(3) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, *where* $Q : Y \to Y$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then the equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

The following definitions and lemmas can be found in [1, 2].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \tag{2.1}$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds, \tag{2.2}$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$.

Lemma 2.1 *Assume* $f \in L[0,1]$, $q \ge p \ge 0$, $q > 1$, *then*

$$D_{0^+}^{p} I_{0^+}^{q} f(t) = I_{0^+}^{q-p} f(t).$$

Lemma 2.2 *Assume* $\alpha > 0$, $\lambda > -1$, *then*

$$D_{0^+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(n+\lambda-\alpha+1)} \frac{d^n}{dt^n} \left(t^{n+\lambda-\alpha}\right),$$

where n *is the smallest integer greater than or equal to* α.

Lemma 2.3 $D_{0^+}^{\alpha} u(t) = 0$ *if and only if*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where n *is the smallest integer greater than or equal to* α, $c_i \in R$, $i = 1,2,\ldots,n$.

Take $X = C^{\alpha-1}[0,1] \times C^{\beta-1}[0,1]$ with the norm

$$\left\|(x,y)\right\| = \max\left\{\|x\|_\infty, \|y\|_\infty, \left\|D_{0^+}^{\alpha-1}x\right\|_\infty, \left\|D_{0^+}^{\beta-1}y\right\|_\infty\right\},$$

where $C^{\alpha-1}[0,1] = \{x \mid x, D_{0^+}^{\alpha-1}x \in C[0,1]\}$, $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Set $Y = L[0.1] \times L[0.1]$ with the norm

$$\left\|(f,g)\right\| = \max\left\{\int_0^1 \left|f(x)\right| dx, \int_0^1 \left|g(x)\right| dx\right\}.$$

Define operators $L : \operatorname{dom} L \subset X \to Y$, $N : X \to Y$ as follows:

$$L(x,y) = \left(D_{0^+}^{\alpha}x, D_{0^+}^{\beta}y\right), \quad (x,y) \in \operatorname{dom} L, \qquad N(x,y) = \left(N_1(x,y), N_2(x,y)\right), \quad (x,y) \in X,$$

where

$$\operatorname{dom} L = \Big\{(x,y) \mid (x,y) \in X, \big(D_{0^+}^{\alpha}x, D_{0^+}^{\beta}y\big) \in Y, x(0) = y(0) = 0,$$
$$D_{0^+}^{\alpha-1}x(0) = \int_0^1 h_1(t)D_{0^+}^{\alpha-1}x(t)\,dt, D_{0^+}^{\alpha-1}x(1) = \int_0^1 h_2(t)D_{0^+}^{\alpha-1}x(t)\,dt,$$
$$D_{0^+}^{\beta-1}y(0) = \int_0^1 g_1(t)D_{0^+}^{\beta-1}y(t)\,dt, D_{0^+}^{\beta-1}y(1) = \int_0^1 g_2(t)D_{0^+}^{\beta-1}y(t)\,dt\Big\},$$

$N_1(x,y) = f_1(t, x(t), y(t), D_{0^+}^{\alpha-1}x(t))$, $N_2(x,y) = f_2(t, x(t), y(t), D_{0^+}^{\beta-1}y(t))$. Then problem (1.1) is $L(x,y) = N(x,y)$.

By Lemma 2.3 in [20], we get that X is a Banach space.

Definition 2.3 $(x,y) \in \operatorname{dom} L$ is a solution of problem (1.1) if it satisfies (1.1), *i.e.*, $L(x,y) = N(x,y)$.

3 Main result

Define operators $T_i : L[0,1] \to R$, $i = 1,2,3,4$, and $Q_j : L[0,1] \to L[0,1]$, $j = 1,2$ as follows:

$$T_1u = \int_0^1 u(t)\int_t^1 h_1(s)\,ds\,dt, \qquad T_2u = \int_0^1 u(t)\int_0^t h_2(s)\,ds\,dt,$$
$$T_3u = \int_0^1 u(t)\int_t^1 g_1(s)\,ds\,dt, \qquad T_4u = \int_0^1 u(t)\int_0^t g_2(s)\,ds\,dt,$$
$$Q_1u = \frac{1}{\Delta_1}(\Delta_{22}T_1u - \Delta_{21}T_2u) + \frac{1}{\Delta_1}(\Delta_{11}T_2u - \Delta_{12}T_1u)t,$$
$$Q_2u = \frac{1}{\Delta_2}(\delta_{22}T_3u - \delta_{21}T_4u) + \frac{1}{\Delta_2}(\delta_{11}T_4u - \delta_{12}T_3u)t.$$

It is clear that $\Delta_{11} = T_1 1$, $\Delta_{12} = T_2 1$, $\Delta_{21} = T_1 t$, $\Delta_{22} = T_2 t$.

Lemma 3.1 *If* (H_1) *and* (H_2) *hold, then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero, the linear continuous projectors* $P : X \to X$ *and* $Q : Y \to Y$ *can be defined as*

$$P(x,y) = \left(\frac{D_{0^+}^{\alpha-1}x(0)}{\Gamma(\alpha)}t^{\alpha-1} + \frac{D_{0^+}^{\alpha-2}x(0)}{\Gamma(\alpha-1)}t^{\alpha-2}, \frac{D_{0^+}^{\beta-1}y(0)}{\Gamma(\beta)}t^{\beta-1} + \frac{D_{0^+}^{\beta-2}y(0)}{\Gamma(\beta-1)}t^{\beta-2}\right),$$
$$Q(u,v) = (Q_1u, Q_2v),$$

respectively, and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$K_P(u,v) = \big(I_{0^+}^{\alpha}u, I_{0^+}^{\beta}v\big).$$

Proof We can easily get that

$$\operatorname{Ker} L = \big\{\big(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\big) \mid c_1, c_2, d_1, d_2 \in R\big\}.$$

Obviously, $\operatorname{Im} P = \operatorname{Ker} L$, $P^2(u,v) = P(u,v)$.

By a simple calculation, we obtain that

$$\operatorname{Im} L = \{(u,v) \in Y \mid T_1 u = T_2 u = T_3 v = T_4 v = 0\}$$

and $Q^2(u,v) = Q(u,v)$. By (H_2), we have $\operatorname{Im} L = \operatorname{Ker} Q$. It is clear that

$$X = \operatorname{Ker} P \oplus \operatorname{Ker} L, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

This means that L is a Fredholm operator of index zero.

For $(u,v) \in \operatorname{Im} L$, we can easily get that $K_P(u,v) = (I_{0^+}^{\alpha} u, I_{0^+}^{\beta} v) \in \operatorname{dom} L \cap \operatorname{Ker} P$. Obviously, $LK_P(u,v) = (u,v)$, $(u,v) \in \operatorname{Im} L$. For $(x,y) \in \operatorname{dom} L \cap \operatorname{Ker} P$, by Lemma 2.3 and $K_P L(x,y) \in \operatorname{dom} L$, we get that

$$K_P L(x,y) = \bigl(x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, y(t) + d_1 t^{\beta-1} + d_2 t^{\beta-2}\bigr).$$

It follows from $(x,y) \in \operatorname{Ker} P$ that $D_{0^+}^{\alpha-1} x(0) = D_{0^+}^{\alpha-2} x(0) = D_{0^+}^{\beta-1} y(0) = D_{0^+}^{\beta-2} y(0) = 0$. This, together with $K_P L(x,y) \in \operatorname{Ker} P$, means that $c_1 = c_2 = d_1 = d_2 = 0$. So, $K_P L(x,y) = (x,y)$. Therefore, $K_P = (L|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. The proof is completed. □

Lemma 3.2 *Suppose that* (H_1), (H_2) *and* (H_3) *hold. If* $\Omega \subset X$ *is an open bounded subset and* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, *then* N *is* L-*compact on* $\overline{\Omega}$.

Proof Since Ω is bounded, there exists a constant $r > 0$ such that $\|(x,y)\| < r$, $(x,y) \in \overline{\Omega}$. It follows from condition (H_3) that there exist functions $\Phi_i \in L[0,1]$ such that $|f_i(t,x,y,z)| \leq \Phi_i(t)$ for all $|x|, |y|, |z| \in [0,r]$, a.e. $t \in [0,1]$, $i = 1,2$. Thus,

$$\begin{aligned}
\bigl|T_1 N_1(x,y)\bigr| &= \biggl|\int_0^1 N_1(x,y) \int_t^1 h_1(s)\,ds\,dt\biggr| \\
&\leq \int_0^1 \Phi_1(t)\,dt \int_0^1 \bigl|h_1(s)\bigr|\,ds < +\infty, \quad (x,y) \in \overline{\Omega}, \\
\bigl|T_2 N_1(x,y)\bigr| &= \biggl|\int_0^1 N_1(x,y) \int_0^t h_2(s)\,ds\,dt\biggr| \\
&\leq \int_0^1 \Phi_1(t)\,dt \int_0^1 \bigl|h_2(s)\bigr|\,ds < +\infty, \quad (x,y) \in \overline{\Omega}, \\
\bigl|T_3 N_2(x,y)\bigr| &= \biggl|\int_0^1 N_2(x,y) \int_t^1 g_1(s)\,ds\,dt\biggr| \\
&\leq \int_0^1 \Phi_2(t)\,dt \int_0^1 \bigl|g_1(s)\bigr|\,ds < +\infty, \quad (x,y) \in \overline{\Omega}, \\
\bigl|T_4 N_2(x,y)\bigr| &= \biggl|\int_0^1 N_2(x,y) \int_0^t g_2(s)\,ds\,dt\biggr| \\
&\leq \int_0^1 \Phi_2(t)\,dt \int_0^1 \bigl|g_2(s)\bigr|\,ds < +\infty, \quad (x,y) \in \overline{\Omega}.
\end{aligned}$$

These mean that there exist constants $a_i > 0$, $b_i > 0$, $i = 1,2$, such that

$$\bigl|Q_1 N_1(x,y)\bigr| \leq a_1 + b_1 t, \qquad \bigl|Q_2 N_2(x,y)\bigr| \leq a_2 + b_2 t, \quad (x,y) \in \overline{\Omega}, t \in [0,1],$$

i.e., $QN(\overline{\Omega}) \subset Y$ is bounded. Now we will prove that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Obviously, $K_P(I-Q)N(\overline{\Omega})$ is bounded. For $0 \le t_1 < t_2 \le 1$, $(x,y) \in \overline{\Omega}$, we have

$$\begin{aligned}
&K_P(I-Q)N(x,y)(t_2) - K_P(I-Q)N(x,y)(t_1) \\
&\quad = \bigl(I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_2), I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_2)\bigr) \\
&\qquad - \bigl(I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_1), I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_1)\bigr) \\
&\quad = \bigl(I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_2) - I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_1), \\
&\qquad I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_2) - I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_1)\bigr),
\end{aligned}$$

where $I_0 : L[0,1] \to L[0,1]$ is an identical mapping.

It follows from

$$\begin{aligned}
&\bigl|I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_2) - I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_1)\bigr| \\
&\quad = \frac{1}{\Gamma(\alpha)}\biggl|\int_0^{t_2}(t_2-s)^{\alpha-1}(I_0-Q_1)N_1\bigl(x(s),y(s)\bigr)\,ds \\
&\qquad - \int_0^{t_1}(t_1-s)^{\alpha-1}(I_0-Q_1)N_1\bigl(x(s),y(s)\bigr)\,ds\biggr| \\
&\quad \le \frac{1}{\Gamma(\alpha)}\biggl[\int_0^{t_1}\bigl((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr)\bigl(\Phi_1(s)+a_1+b_1s\bigr)\,ds \\
&\qquad + \int_{t_1}^{t_2}\bigl(\Phi_1(s)+a_1+b_1s\bigr)\,ds\biggr], \\
&\bigl|D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_2) - D_{0^+}^{\alpha-1}I_{0^+}^{\alpha}(I_0-Q_1)N_1(x,y)(t_1)\bigr| \\
&\quad = \biggl|\int_0^{t_2}(I_0-Q_1)N_1\bigl(x(s),y(s)\bigr)\,ds - \int_0^{t_1}(I_0-Q_1)N_1\bigl(x(s),y(s)\bigr)\,ds\biggr| \\
&\quad \le \int_{t_1}^{t_2}\bigl(\Phi_1(s)+a_1+b_1s\bigr)\,ds, \\
&\bigl|I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_2) - I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_1)\bigr| \\
&\quad = \frac{1}{\Gamma(\beta)}\biggl|\int_0^{t_2}(t_2-s)^{\beta-1}(I_0-Q_2)N_2\bigl(x(s),y(s)\bigr)\,ds \\
&\qquad - \int_0^{t_1}(t_1-s)^{\beta-1}(I_0-Q_2)N_2\bigl(x(s),y(s)\bigr)\,ds\biggr| \\
&\quad \le \frac{1}{\Gamma(\beta)}\biggl[\int_0^{t_1}\bigl((t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\bigr)\bigl(\Phi_2(s)+a_2+b_2s\bigr)\,ds \\
&\qquad + \int_{t_1}^{t_2}\bigl(\Phi_2(s)+a_2+b_2s\bigr)\,ds\biggr], \\
&\bigl|D_{0^+}^{\beta-1}I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_2) - D_{0^+}^{\beta-1}I_{0^+}^{\beta}(I_0-Q_2)N_2(x,y)(t_1)\bigr| \\
&\quad = \biggl|\int_0^{t_2}(I_0-Q_2)N_2\bigl(x(s),y(s)\bigr)\,ds \\
&\qquad - \int_0^{t_1}(I_0-Q_2)N_2\bigl(x(s),y(s)\bigr)\,ds\biggr| \\
&\quad \le \int_{t_1}^{t_2}\bigl(\Phi_2(s)+a_2+b_2s\bigr)\,ds,
\end{aligned}$$

the uniform continuity of $(t-s)^{\alpha-1}$ and $(t-s)^{\beta-1}$ on $[0,1]\times[0,1]$, the absolute continuity of integral of $\Phi_i + a_i + b_i t$ on $[0,1]$, $i=1,2$, and the Ascoli-Arzela theorem that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. The proof is completed. □

In order to obtain our main results, we present the following conditions.

(H_4) There exist constants $M_i > 0$, $L_i > 0$, $i = 1,2$, such that if either

$$\min_{t\in[\eta_1,1]} \left|x(t)\right| > M_1 \quad \text{or} \quad \min_{t\in[\eta_1,1]} \left|D_{0^+}^{\alpha-1}x(t)\right| > L_1,$$

then either

$$\int_0^1 f_1\left(t, x(t), y(t), D_{0^+}^{\alpha-1}x(t)\right) \int_t^1 h_1(s)\,ds\,dt \neq 0$$

or

$$\int_0^1 f_1\left(t, x(t), y(t), D_{0^+}^{\alpha-1}x(t)\right) \int_0^t h_2(s)\,ds\,dt \neq 0,$$

and if either

$$\min_{t\in[\eta_2,1]} \left|y(t)\right| > M_2 \quad \text{or} \quad \min_{t\in[\eta_2,1]} \left|D_{0^+}^{\beta-1}y(t)\right| > L_2,$$

then either

$$\int_0^1 f_2\left(t, x(t), y(t), D_{0^+}^{\beta-1}y(t)\right) \int_t^1 g_1(s)\,ds\,dt \neq 0$$

or

$$\int_0^1 f_2\left(t, x(t), y(t), D_{0^+}^{\beta-1}y(t)\right) \int_0^t g_2(s)\,ds\,dt \neq 0,$$

where η_i, $i = 1,2$, are the same as in (H_3).

(H_5) For $(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}) \in \operatorname{Ker} L$, there exist constants k_1, k_2, l_1, l_2 such that either (1) or (2) holds, where

$$\begin{aligned}
(1)\quad & c_1T_1N_1\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) > 0, \quad \text{if } |c_1| > k_1,\\
& c_2T_2N_1\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) > 0, \quad \text{if } |c_1| \le k_1, |c_2| > k_2,\\
& d_1T_3N_2\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) > 0, \quad \text{if } |d_1| > l_1,\\
& d_2T_4N_2\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) > 0, \quad \text{if } |d_1| \le l_1, |d_2| > l_2.\\
(2)\quad & c_1T_1N_1\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) < 0, \quad \text{if } |c_1| > k_1,\\
& c_2T_2N_1\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) < 0, \quad \text{if } |c_1| \le k_1, |c_2| > k_2,\\
& d_1T_3N_2\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) < 0, \quad \text{if } |d_1| > l_1,\\
& d_2T_4N_2\left(c_1t^{\alpha-1} + c_2t^{\alpha-2}, d_1t^{\beta-1} + d_2t^{\beta-2}\right) < 0, \quad \text{if } |d_1| \le l_1, |d_2| > l_2.
\end{aligned}$$

Lemma 3.3 *Suppose that* (H_1)-(H_4) *hold, then the set*

$$\Omega_1 = \{(x,y) \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid L(x,y) = \lambda N(x,y), \lambda \in (0,1)\}$$

is bounded in X.

Proof Take $(x,y) \in \Omega_1$. By $L(x,y) = \lambda N(x,y)$, we get

$$\begin{cases} x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, x(s), y(s), D_{0^+}^{\alpha-1} x(s))\, ds + a_1 t^{\alpha-1} + a_2 t^{\alpha-2}, \\ y(t) = \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f_2(s, x(s), y(s), D_{0^+}^{\beta-1} y(s))\, ds + b_1 t^{\beta-1} + b_2 t^{\beta-2}. \end{cases} \tag{3.1}$$

By Lemmas 2.1, 2.2 and (3.1), we have

$$\begin{cases} D_{0^+}^{\alpha-1} x(t) = \lambda \int_0^t f_1(s, x(s), y(s), D_{0^+}^{\alpha-1} x(s))\, ds + a_1 \Gamma(\alpha), \\ D_{0^+}^{\beta-1} y(t) = \lambda \int_0^t f_2(s, x(s), y(s), D_{0^+}^{\beta-1} y(s))\, ds + b_1 \Gamma(\beta). \end{cases} \tag{3.2}$$

It follows from $N(x,y) \in \operatorname{Im} L$ that

$$\int_0^1 f_1(t, x(t), y(t), D_{0^+}^{\alpha-1} x(t)) \int_t^1 h_1(s)\, ds\, dt = 0,$$
$$\int_0^1 f_1(t, x(t), y(t), D_{0^+}^{\alpha-1} x(t)) \int_0^t h_2(s)\, ds\, dt = 0,$$
$$\int_0^1 f_2(t, x(t), y(t), D_{0^+}^{\beta-1} y(t)) \int_t^1 g_1(s)\, ds\, dt = 0,$$
$$\int_0^1 f_2(t, x(t), y(t), D_{0^+}^{\beta-1} y(t)) \int_0^t g_2(s)\, ds\, dt = 0.$$

These, together with (H_4), mean that there exist constants $t_0, t_1 \in [\eta_1, 1]$ and $t_0', t_1' \in [\eta_2, 1]$ such that

$$|x(t_0)| \le M_1, \qquad |D_{0^+}^{\alpha-1} x(t_1)| \le L_1, \qquad |y(t_0')| \le M_2, \qquad |D_{0^+}^{\beta-1} y(t_1')| \le L_2. \tag{3.3}$$

By (3.2), we get

$$D_{0^+}^{\alpha-1} x(t) = \lambda \int_{t_1}^t f_1(s, x(s), y(s), D_{0^+}^{\alpha-1} x(s))\, ds + D_{0^+}^{\alpha-1} x(t_1),$$
$$D_{0^+}^{\beta-1} y(t) = \lambda \int_{t_1'}^t f_2(s, x(s), y(s), D_{0^+}^{\beta-1} y(s))\, ds + D_{0^+}^{\beta-1} y(t_1').$$

By (3.3) and (H_3), we obtain that

$$\begin{cases} \|D_{0^+}^{\alpha-1} x\|_\infty \le \frac{1}{1-c_0} (a_0 \|x\|_\infty + b_0 \|y\|_\infty + d_0 \|x\|_\infty^{\theta_1} + r_0 + L_1), \\ \|D_{0^+}^{\beta-1} y\|_\infty \le \frac{1}{1-c_0'} (a_0' \|x\|_\infty + b_0' \|y\|_\infty + d_0' \|y\|_\infty^{\theta_2} + r_0' + L_2). \end{cases} \tag{3.4}$$

Instead of t by t_0, t_0' in (3.1) and t_1, t_1' in (3.2), respectively, we get

$$\begin{cases} x(t) = \frac{\lambda}{\Gamma(\alpha)}[\int_0^t (t-s)^{\alpha-1} f_1(s,x(s),y(s),D_{0^+}^{\alpha-1}x(s))\,ds \\ \qquad + t^{\alpha-2}(t_0-t)\int_0^{t_1} f_1(s,x(s),y(s),D_{0^+}^{\alpha-1}x(s))\,ds \\ \qquad - \frac{t^{\alpha-2}}{t_0^{\alpha-2}}\int_0^{t_0}(t_0-s)^{\alpha-1} f_1(s,x(s),y(s),D_{0^+}^{\alpha-1}x(s))\,ds] \\ \qquad + \frac{t^{\alpha-2}}{\Gamma(\alpha)} D_{0^+}^{\alpha-1}x(t_1)(t-t_0) + \frac{t^{\alpha-2}}{t_0^{\alpha-2}}x(t_0), \\ y(t) = \frac{\lambda}{\Gamma(\beta)}[\int_0^t (t-s)^{\beta-1} f_2(s,x(s),y(s),D_{0^+}^{\beta-1}y(s))\,ds \\ \qquad + t^{\beta-2}(t_0'-t)\int_0^{t_1'} f_2(s,x(s),y(s),D_{0^+}^{\beta-1}y(s))\,ds \\ \qquad - \frac{t^{\beta-2}}{t_0'^{\beta-2}}\int_0^{t_0'}(t_0'-s)^{\beta-1} f_2(s,x(s),y(s),D_{0^+}^{\beta-1}y(s))\,ds] \\ \qquad + \frac{t^{\beta-2}}{\Gamma(\beta)} D_{0^+}^{\beta-1}y(t_1')(t-t_0') + \frac{t^{\beta-2}}{t_0'^{\beta-2}}y(t_0'). \end{cases} \tag{3.5}$$

It follows from (3.4), (3.5) and (H_3) that

$$\begin{aligned} |x(t)| &\le \frac{1}{\Gamma(\alpha)}\left(2+\frac{1}{\eta_1^{\alpha-2}}\right)\int_0^1 \left|f_1\left(s,x(s),y(s),D_{0^+}^{\alpha-1}x(s)\right)\right|ds + \left(\frac{L_1}{\Gamma(\alpha)}+\frac{M_1}{\eta_1^{\alpha-2}}\right) \\ &\le \frac{1}{\Gamma(\alpha)}\left(2+\frac{1}{\eta_1^{\alpha-2}}\right)\left(a_0\|x\|_\infty + b_0\|y\|_\infty + c_0\left\|D_{0^+}^{\alpha-1}x\right\|_\infty + d_0\|x\|_\infty^{\theta_1} + r_0\right) \\ &\quad + \left(\frac{L_1}{\Gamma(\alpha)}+\frac{M_1}{\eta_1^{\alpha-2}}\right) \\ &\le \frac{1}{\Gamma(\alpha)(1-c_0)}\left(2+\frac{1}{\eta_1^{\alpha-2}}\right)\left(a_0\|x\|_\infty + b_0\|y\|_\infty + d_0\|x\|_\infty^{\theta_1} + r_0 + c_0L_1\right) \\ &\quad + \left(\frac{L_1}{\Gamma(\alpha)}+\frac{M_1}{\eta_1^{\alpha-2}}\right) \end{aligned}$$

and

$$\begin{aligned} |y(t)| &\le \frac{1}{\Gamma(\beta)}\left(2+\frac{1}{\eta_2^{\beta-2}}\right)\int_0^1 \left|f_2\left(s,x(s),y(s),D_{0^+}^{\beta-1}y(s)\right)\right|ds + \left(\frac{L_2}{\Gamma(\beta)}+\frac{M_2}{\eta_2^{\beta-2}}\right) \\ &\le \frac{1}{\Gamma(\beta)}\left(2+\frac{1}{\eta_2^{\beta-2}}\right)\left(a_0'\|x\|_\infty + b_0'\|y\|_\infty + c_0'\left\|D_{0^+}^{\beta-1}y\right\|_\infty + d_0'\|y\|_\infty^{\theta_2} + r_0'\right) \\ &\quad + \left(\frac{L_2}{\Gamma(\beta)}+\frac{M_2}{\eta_2^{\beta-2}}\right) \\ &\le \frac{1}{\Gamma(\beta)(1-c_0')}\left(2+\frac{1}{\eta_2^{\beta-2}}\right)\left(a_0'\|x\|_\infty + b_0'\|y\|_\infty + d_0'\|y\|_\infty^{\theta_2} + r_0' + c_0'L_2\right) \\ &\quad + \left(\frac{L_2}{\Gamma(\beta)}+\frac{M_2}{\eta_2^{\beta-2}}\right). \end{aligned}$$

Thus,

$$\|x\|_\infty \le A_1\left[b_0\|y\|_\infty + d_0\|x\|_\infty^{\theta_1}\right] + M_0, \tag{3.6}$$

$$\|y\|_\infty \le A_2\left[a_0'\|x\|_\infty + d_0'\|y\|_\infty^{\theta_2}\right] + M_0', \tag{3.7}$$

where $M_0 = A_1[(r_0 + c_0 L_1) + (\frac{L_1}{\Gamma(\alpha)} + \frac{M_1}{\eta_1^{\alpha-2}})/\frac{1}{\Gamma(\alpha)(1-c_0)}(2 + \frac{1}{\eta_1^{\alpha-2}})]$, $M_0' = A_2[(r_0' + c_0' L_2) + (\frac{L_2}{\Gamma(\beta)} + \frac{M_2}{\eta_2^{\beta-2}})/\frac{1}{\Gamma(\beta)(1-c_0')}(2 + \frac{1}{\eta_2^{\beta-2}})]$.

By (H_3), (3.4), (3.6) and (3.7), we can get that Ω_1 is bounded in X. The proof is completed. □

Lemma 3.4 *Suppose that* (H_1), (H_2), (H_3) *and* (H_5) *hold, then the set*

$$\Omega_2 = \{(x,y) \mid (x,y) \in \operatorname{Ker} L, N(x,y) \in \operatorname{Im} L\}$$

is bounded in X.

Proof For $(x,y) \in \Omega_2$, we have $(x,y) = (c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2})$, $c_1, c_2, d_1, d_2 \in R$ and $T_1 N_1(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) = T_2 N_1(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) = T_3 N_2(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) = T_4 N_2(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) = 0$. By (H_5), we get that $|c_1| \le k_1$, $|c_2| \le k_2$, $|d_1| \le l_1$, $|d_2| \le l_2$. These imply that Ω_2 is bounded in X. □

Lemma 3.5 *Suppose that* (H_1), (H_2), (H_3) *and* (H_5) *hold. The set*

$$\Omega_3 = \{(x,y) \in \operatorname{Ker} L \mid \lambda J(x,y) + (1-\lambda)\theta QN(x,y) = (0,0), \lambda \in [0,1]\}$$

is bounded in X, *where* $J : \operatorname{Ker} L \to \operatorname{Im} Q$ *is a linear isomorphism given by*

$$\begin{aligned} &J(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) \\ &\quad = \biggl(\frac{1}{\Delta_1}(\Delta_{22} c_1 - \Delta_{21} c_2) + \frac{1}{\Delta_1}(\Delta_{11} c_2 - \Delta_{12} c_1)t, \\ &\qquad \frac{1}{\Delta_2}(\delta_{22} d_1 - \delta_{21} d_2) + \frac{1}{\Delta_2}(\delta_{11} d_2 - \delta_{12} d_1)t\biggr), \\ &\theta = \begin{cases} 1, & \textit{if } (H_5)(1) \textit{ holds}, \\ -1, & \textit{if } (H_5)(2) \textit{ holds}. \end{cases} \end{aligned}$$

Proof For $(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) \in \Omega_3$, there exists $\lambda \in [0,1]$ such that

$$\lambda J(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) = -(1-\lambda)\theta QN(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}).$$

This means that

$$\begin{aligned} \lambda c_1 &= -(1-\lambda)\theta T_1 N_1(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}), \\ \lambda c_2 &= -(1-\lambda)\theta T_2 N_1(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}), \\ \lambda d_1 &= -(1-\lambda)\theta T_3 N_2(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}), \\ \lambda d_2 &= -(1-\lambda)\theta T_4 N_2(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}). \end{aligned}$$

If $\lambda = 0$, by (H_5), we get $|c_1| \le k_1$, $|c_2| \le k_2$, $|d_1| \le l_1$, $|d_2| \le l_2$. If $\lambda = 1$, then $c_1 = c_2 = d_1 = d_2 = 0$. For $\lambda \in (0,1)$, if $|c_1| > k_1$, we can get

$$\lambda c_1^2 = -(1-\lambda)\theta c_1 T_1 N_1(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}) < 0,$$

a contradiction. If $|c_1| \leq k_1$ and $|c_2| > k_2$, we can get

$$\lambda c_2^2 = -(1-\lambda)\theta c_2 T_2 N_1\bigl(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, d_1 t^{\beta-1} + d_2 t^{\beta-2}\bigr) < 0.$$

This is a contradiction, too. Thus, $|c_i| \leq k_i$, $i = 1, 2$. By the same methods, we can obtain that $|d_i| \leq l_i$, $i = 1, 2$. This means that Ω_3 is bounded in X. □

Theorem 3.1 *Suppose that* (H_1)-(H_5) *hold. Then problem* (1.1) *has at least one solution in* X.

Proof Let $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i} \cup \{(0,0)\}$ be a bounded open subset of X. It follows from Lemma 3.2 that N is L-compact on $\overline{\Omega}$. By Lemmas 3.3 and 3.4, we get

(1) $L(x,y) \neq \lambda N(x,y)$ for every $(x,y,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$,

(2) $N(x,y) \notin \operatorname{Im} L$ for every $(x,y) \in \operatorname{Ker} L \cap \partial\Omega$.

We need only to prove

(3) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, (0,0)) \neq 0$.

Take

$$H(x,y,\lambda) = \lambda J(x,y) + \theta(1-\lambda) QN(x,y).$$

According to Lemma 3.5, we know $H(x,y,\lambda) \neq (0,0)$ for $(x,y) \in \partial\Omega \cap \operatorname{Ker} L$. By the homotopy of degree, we get that

$$\begin{aligned}\deg\bigl(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, (0,0)\bigr) &= \deg\bigl(\theta H(\cdot,0), \Omega \cap \operatorname{Ker} L, (0,0)\bigr) \\ &= \deg\bigl(\theta H(\cdot,1), \Omega \cap \operatorname{Ker} L, (0,0)\bigr) \\ &= \deg\bigl(\theta J, \Omega \cap \operatorname{Ker} L, (0,0)\bigr) \neq 0.\end{aligned}$$

By Theorem 2.1, we can get that $L(x,y) = N(x,y)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, *i.e.*, (1.1) has at least one solution in X. The proof is completed. □

4 Example

Let us consider the following system of fractional differential equations at resonance:

$$\begin{cases} D_{0^+}^{\frac{5}{2}} x(t) = f_1(t, x(t), y(t), D_{0^+}^{\frac{3}{2}} x(t)), \quad 0 < t < 1, \\ D_{0^+}^{\frac{5}{2}} y(t) = f_2(t, x(t), y(t), D_{0^+}^{\frac{3}{2}} y(t)), \quad 0 < t < 1, \\ x(0) = 0, \quad D_{0^+}^{\frac{3}{2}} x(0) = \int_0^1 h_1(t) D_{0^+}^{\frac{3}{2}} x(t)\, dt, \\ D_{0^+}^{\frac{3}{2}} x(1) = \int_0^1 h_2(t) D_{0^+}^{\frac{3}{2}} x(t)\, dt, \\ y(0) = 0, \quad D_{0^+}^{\frac{3}{2}} y(0) = \int_0^1 g_1(t) D_{0^+}^{\frac{3}{2}} y(t)\, dt, \\ D_{0^+}^{\frac{3}{2}} y(1) = \int_0^1 g_2(t) D_{0^+}^{\frac{3}{2}} y(t)\, dt, \end{cases} \tag{4.1}$$

where

$$f_1(t,x,y,z) = \begin{cases} \frac{1}{4} t \sin x + \frac{1}{8} t^3 \sin y, & t \in [0, \frac{1}{4}), \\ \frac{1}{4} t \sin x + \frac{1}{8} t^3 \sin y + tz, & t \in [\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{4} tx + \frac{1}{8} t^3 \sin y + t \sin z, & t \in [\frac{1}{2}, 1], \end{cases}$$

$$f_2(t,x,y,z)=\begin{cases}\frac{1}{8}t^3\sin x+\frac{1}{10}\sin y, & t\in[0,\frac{1}{9}),\\ \frac{1}{8}t^3\sin x+\frac{1}{10}\sin y+tz, & t\in[\frac{1}{9},\frac{1}{4}),\\ \frac{1}{8}t^3\sin x+\frac{1}{10}y+t\sin z, & t\in[\frac{1}{4},1],\end{cases}$$

$$h_1(t)=\begin{cases}2, & t\in[0,\frac{1}{2}),\\ 0, & t\in[\frac{1}{2},1],\end{cases}\qquad h_2(t)=\begin{cases}0, & t\in[0,\frac{1}{2}),\\ 2, & t\in[\frac{1}{2},1],\end{cases}$$

$$g_1(t)=\begin{cases}4, & t\in[0,\frac{1}{4}),\\ 0, & t\in[\frac{1}{4},1],\end{cases}\qquad g_2(t)=\begin{cases}0, & t\in[0,\frac{1}{4}),\\ \frac{4}{3}, & t\in[\frac{1}{4},1].\end{cases}$$

Corresponding to problem (1.1), we have $\alpha=\beta=\frac{5}{2}$,

$$\Delta_1=\begin{vmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{24} & \frac{5}{24}\end{vmatrix}=\frac{1}{24}\neq 0,\qquad \Delta_2=\begin{vmatrix}\frac{1}{8} & \frac{3}{8}\\ \frac{1}{96} & \frac{9}{32}\end{vmatrix}=\frac{1}{32}\neq 0.$$

Obviously, $\int_0^1 h_i(t)\,dt=1$, $\int_0^1 g_i(t)\,dt=1$, $i=1,2$. Thus, conditions (H_1) and (H_2) are satisfied. It is easy to get that $a_0=\frac{1}{8}$, $b_0=\frac{1}{32}$, $c_0=\frac{15}{32}$, $a_0'=\frac{1}{32}$, $b_0'=\frac{1}{10}$, $c_0'=\frac{40}{81}$. Take $M_1=8$, $L_1=1$, $\eta_1=\frac{1}{4}$, $M_2=20$, $L_2=4$, $\eta_2=\frac{1}{9}$. By a simple calculation, we can get that (H_3) is satisfied and the following inequations hold

$$\begin{aligned}&\int_0^1 f_1\bigl(t,x(t),y(t),D_{0^+}^{\alpha-1}x(t)\bigr)\int_t^1 h_1(s)\,ds\,dt\neq 0,\quad \text{if } \min_{t\in[\eta_1,1]}\bigl|D_{0^+}^{\alpha-1}x(t)\bigr|>L_1,\\ &\int_0^1 f_1\bigl(t,x(t),y(t),D_{0^+}^{\alpha-1}x(t)\bigr)\int_0^t h_2(s)\,ds\,dt\neq 0,\quad \text{if } \min_{t\in[\eta_1,1]}\bigl|x(t)\bigr|>M_1,\\ &\int_0^1 f_2\bigl(t,x(t),y(t),D_{0^+}^{\beta-1}y(t)\bigr)\int_t^1 g_1(s)\,ds\,dt\neq 0,\quad \text{if } \min_{t\in[\eta_2,1]}\bigl|D_{0^+}^{\beta-1}y(t)\bigr|>L_2,\end{aligned}$$

and

$$\int_0^1 f_2\bigl(t,x(t),y(t),D_{0^+}^{\beta-1}y(t)\bigr)\int_0^t g_2(s)\,ds\,dt\neq 0,\quad \text{if } \min_{t\in[\eta_2,1]}\bigl|y(t)\bigr|>M_2.$$

So, (H_4) holds. Set $k_1=1$, $k_2=20$, $l_1=4$, $l_2=140$. By a simple calculation, we can obtain that condition (H_5) is satisfied.

By Theorem 3.1, problem (4.1) has at least one solution.

Competing interests

The author declares that she has no competing interests.

Author's contributions

All results belong to WJ.

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