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Some symmetry identities for the Apostol-type polynomials related to multiple alternating sums

Veli Kurt*

*Correspondence: vkurt@akdeniz.edu.tr
Department of Mathematics, Faculty of Science, Akdeniz University, Campus, Antalya, 07058, Turkey

Abstract

In recent years, symmetry properties of the Bernoulli polynomials and the Euler polynomials have been studied by a large group of mathematicians (He and Wang in Discrete Dyn. Nat. Soc. 2012:927953, 2012, Kim *et al.* in J. Differ. Equ. Appl. 14:1267-1277, 2008; Abstr. Appl. Anal. 2008, doi:11.1155/2008/914347, Yang *et al.* in Discrete Math. 308:550-554, 2008; J. Math. Res. Expo. 30:457-464, 2010). Luo (Integral Transforms Spec. Funct. 20:377-391, 2009), introduced the lambda-multiple power sum and proved the multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order. Ozarslan (Comput. Math. Appl. 2011:2452-2462, 2011), Lu and Srivastava (Comput. Math. Appl. 2011, doi:10.1016/j.2011.09.010.2011) gave some symmetry identities relations for the Apostol-Bernoulli and Apostol-Euler polynomials.

In this work, we prove some symmetry identities for the Apostol-Bernoulli and Apostol-Euler polynomials related to multiple alternating sums.

AMS Subject Classification: 11F20; 11B68; 11M35; 11M41

Keywords: Bernoulli polynomials; Euler polynomials; Apostol-Bernoulli polynomials; Apostol-Euler polynomials; symmetry relation; power sums; alternating sums

1 Introduction, definitions and notations

The generalized Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$ and the generalized Euler polynomials $\mathcal{E}_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{N}_0$, each of degree n as well as in α, are defined respectively by the following generating functions [1–3]:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} \quad \left(|t| < 2\pi, 1^{\alpha} := 1\right), \tag{1}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} \quad \left(|t| < \pi, 1^{\alpha} := 1\right). \tag{2}$$

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$ and the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}_0$ are defined respectively by the following generating functions [3]:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} \quad \bigl(|z| < 2\pi \text{ when } \lambda = 1, |z| < |\ln\lambda| \text{ when } \lambda \neq 1\bigr), \tag{3}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} e^{xt} \quad \bigl(|z| < \pi \text{ when } \lambda = 1, |z| < \bigl|\ln(-\lambda)\bigr| \text{ when } \lambda \neq 1\bigr). \tag{4}$$

Recently, Garg *et al.* in [4] introduced the following generalization of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$:

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}$$

($\mu \in \mathbb{C}$, a, $\upsilon \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $\rho,\sigma \in \mathbb{R}^+$, $\rho < \sigma$ when $s,z \in \mathbb{C}(|z| < 1)$; $\rho = \sigma$ and $\Re(s - \mu + \nu) > 0$ when $|z| = 1$). It is obvious that

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_{\mu}^{*}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s} \tag{5}$$

(for details on this subject, see [3–5]).

The multiple power sums and the λ-multiple alternating sums are defined by Luo [6] as follows:

$$S_k^{(l)}(m;\lambda) = \sum_{\substack{0\le\nu_1\le\cdots\le\nu_m\le l\\ \nu_1+\nu_2+\cdots+\nu_m}} \binom{l}{\nu_1,\nu_2,\ldots,\nu_m} \lambda^{\nu_1+2\nu_2+\cdots+m\nu_m}(\nu_1+2\nu_2+\cdots+m\nu_m)^k, \tag{6}$$

$$T_k^{(l)}(m;\lambda) = (-1)^l \sum_{\substack{0\le\nu_1\le\cdots\le\nu_m\le l\\ \nu_1+\nu_2+\cdots+\nu_m}} \binom{l}{\nu_1,\nu_2,\ldots,\nu_m} (-\lambda)^{\nu_1+2\nu_2+\cdots+m\nu_m} \times (\nu_1+2\nu_2+\cdots+m\nu_m)^k. \tag{7}$$

From (6) and (7), we have

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^l = \lambda^{(-l)} \sum_{n=0}^{\infty} \left\{\sum_{p=0}^{n} \binom{n}{p} (-l)^{n-p} S^{(l)}(m;\lambda)\right\} \frac{t^n}{n!} \tag{8}$$

and

$$\left(\frac{1+(-1)^{m+1}(\lambda e^t)^m}{1+\lambda e^t}\right)^l = \lambda^{(-l)} \sum_{n=0}^{\infty} \left\{\sum_{p=0}^{n} \binom{n}{p} (-l)^{n-p} T^{(l)}(m;\lambda)\right\} \frac{t^n}{n!} \tag{9}$$

(see [6]).

From (8) and (9), for $l = 1$, we have respectively

$$\frac{1-\lambda^m e^{mt}}{1-\lambda e^t} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} S_k^{(l)}(m;\lambda)\right) \frac{t^n}{n!}, \tag{10}$$

$$\frac{1+(-1)^{m+1}(\lambda e^t)^m}{1+\lambda e^t} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} T_k^{(l)}(m;\lambda)\right) \frac{t^n}{n!}. \tag{11}$$

Symmetry property and some recurrence relations of the Bernoulli polynomials, Euler polynomials, Apostol-Bernoulli polynomials and Apostol-Euler polynomials have been investigated by a lot of mathematicians [1–24]. Firstly, Yang [22] proved symmetry relation for Bernoulli polynomials. Wang *et al.* in [1, 20, 21] gave some symmetry relations for the Apostol-Bernoulli polynomials. Kim in [8, 10, 11, 14, 15] proved symmetric identities for the Bernoulli polynomials and Euler polynomials. Luo in [6, 17] gave multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials. Also, he defined λ-power sums. Srivastava *et al.* [2, 3, 5] proved some theorems and relations for these polynomials. They proved some symmetry identities for these polynomials.

In this work, we give some symmetry identities for the Apostol-type polynomials related to multiple alternating sums.

2 Symmetry identities for the Apostol-Bernoulli polynomials

We will prove the following theorem for the Apostol-Euler polynomials, which are symmetric in a and b.

Theorem 2.1 *There is the following relation between Apostol-Bernoulli polynomials and the Hurwitz-Lerch zeta function* $\Phi^*(z,s,a)$:

$$\begin{aligned}
&\sum_{s=0}^{n}\binom{n}{s}b^{s+1-\alpha}a^{n-s}(-1)^{\alpha-1}\sum_{k=0}^{s}\binom{s}{k}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}\lambda^{-b} \\
&\qquad\times S_p\bigl(a;\lambda^b\bigr)\mathcal{B}_k^{(\alpha-1)}\bigl(ay;\lambda^b\bigr)\Phi_\mu^*(\lambda,s-n,bx) \\
&\quad=\sum_{s=0}^{n}\binom{n}{s}a^{s+1-\alpha}b^{n-s}(-1)^{\alpha-1}\sum_{k=0}^{s}\binom{s}{k}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}\lambda^{-a} \\
&\qquad\times S_p\bigl(b;\lambda^a\bigr)\mathcal{B}_k^{(\alpha-1)}\bigl(by;\lambda^a\bigr)\Phi_\mu^*(\lambda,s-n,ax).
\end{aligned} \tag{12}$$

Proof Let $f(t)=\frac{t^{\alpha-1}e^{abxt}(1-\lambda^{ab}e^{abt})e^{abyt}}{(1-\lambda^a e^{at})^\alpha(1-\lambda^b e^{bt})^\alpha}$. Then

$$f(t)=\frac{1}{b^{\alpha-1}}\frac{e^{abxt}}{(1-\lambda^a e^{at})^\alpha}\left(\frac{1-\lambda^{ab}e^{abt}}{1-\lambda^b e^{bt}}\right)\left(\frac{bt}{1-\lambda^b e^{bt}}\right)^{\alpha-1}e^{abyt}.$$

From (3) and (10), we write

$$\begin{aligned}
f(t)={}&\frac{(-1)^{\alpha-1}}{b^{\alpha-1}}\sum_{\beta=0}^{\infty}\binom{\beta+\alpha-1}{\beta}\lambda^{\alpha\beta}e^{at(\beta+bx)}\frac{1}{\lambda^b}\sum_{r=0}^{\infty}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}S_p\bigl(a;\lambda^b\bigr)\frac{b^r t^r}{r!} \\
&\times\sum_{k=0}^{\infty}\mathcal{B}_k^{(\alpha-1)}\bigl(ay;\lambda^b\bigr)\frac{b^k t^k}{k!},
\end{aligned}$$

where $|\log\lambda+t|<\min(\frac{2\pi}{a},\frac{2\pi}{b})$. After the Cauchy product, we have

$$\begin{aligned}
={}&\sum_{n=0}^{\infty}\Biggl\{\sum_{s=0}^{n}\binom{n}{s}b^{s+1-\alpha}a^{n-s}(-1)^{\alpha-1}\sum_{k=0}^{s}\binom{s}{k}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}\lambda^{-b} \\
&\times S_p\bigl(a;\lambda^b\bigr)\mathcal{B}_k^{(\alpha-1)}\bigl(ay;\lambda^b\bigr)\sum_{\beta=0}^{\infty}\binom{\beta+\alpha-1}{\beta}\frac{\lambda^{\alpha\beta}}{(\beta+bx)^{s-n}}\Biggr\}\frac{t^n}{n!}.
\end{aligned}$$

In a similar manner,

$$f(t) = \frac{t^{\alpha-1}e^{abyt}(1-\lambda^{ab}e^{abt})e^{abxt}}{(1-\lambda^{b}e^{bt})^{\alpha}(1-\lambda^{a}e^{at})^{\alpha}}$$
$$= \frac{1}{a^{\alpha-1}}\frac{e^{abxt}}{(1-\lambda^{b}e^{bt})^{\alpha}}\left(\frac{1-\lambda^{ab}e^{abt}}{1-\lambda^{a}e^{at}}\right)\left(\frac{at}{1-\lambda^{a}e^{at}}\right)^{\alpha-1}e^{abyt}.$$

From (3) and (10), we write

$$f(t) = \frac{(-1)^{\alpha-1}}{a^{\alpha-1}}\sum_{\beta=0}^{\infty}\binom{\beta+\alpha-1}{\beta}\lambda^{\alpha\beta}e^{bt(\beta+ax)}\frac{1}{\lambda^{a}}\sum_{r=0}^{\infty}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}S_p\big(b;\lambda^a\big)\frac{a^r t^r}{r!}$$
$$\times\sum_{k=0}^{\infty}\mathcal{B}_k^{(\alpha-1)}\big(by;\lambda^a\big)\frac{a^k t^k}{k!}.$$

Since $|\log\lambda + t| < \min(\frac{2\pi}{a}, \frac{2\pi}{b})$, after the Cauchy product, we have

$$= \sum_{n=0}^{\infty}\left\{\sum_{s=0}^{n}\binom{n}{s}a^{s+1-\alpha}b^{n-s}(-1)^{\alpha-1}\sum_{k=0}^{s}\binom{s}{k}\sum_{p=0}^{r}\binom{r}{p}(-1)^{r-p}\lambda^{-a}\right.$$
$$\left.\times S_p\big(b;\lambda^a\big)\mathcal{B}_k^{(\alpha-1)}\big(by;\lambda^a\big)\sum_{\beta=0}^{\infty}\binom{\beta+\alpha-1}{\beta}\frac{\lambda^{b\beta}}{(\beta+ay)^{s-n}}\right\}\frac{t^n}{n!}.$$

Compressing to coefficients $\frac{t^n}{n!}$ and by using (5), we prove the theorem. □

Theorem 2.2 *For all $a, b, m \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have the following symmetry identity*:

$$b^{m+1}a\sum_{r=0}^{n}\binom{n}{r}\mathcal{B}_{n-r}^{(m+1)}\big(bx;\lambda^a\big)a^{n-r}\lambda^{-mb}$$
$$\times\sum_{k=0}^{r}\binom{r}{k}\sum_{p=0}^{k}\binom{k}{p}(-l)^{k-p}S_p^{(m)}\big(a;\lambda^b\big)\mathcal{B}_{r-k}\big(ay;\lambda^b\big)b^r$$
$$= a^{m+1}b\sum_{r=0}^{n}\binom{n}{r}\mathcal{B}_{n-r}^{(m+1)}\big(ay;\lambda^b\big)b^{n-r}\lambda^{-ma}$$
$$\times\sum_{k=0}^{r}\binom{r}{k}\sum_{p=0}^{k}\binom{k}{p}(-l)^{k-p}S_p^{(m)}\big(b;\lambda^a\big)\mathcal{B}_{r-k}\big(bx;\lambda^a\big)a^r. \tag{13}$$

Proof Let $h(t) = \frac{t^{2m+1}e^{abxt}(1-\lambda^{ab}e^{abt})^m e^{abyt}}{(1-\lambda^a e^{at})^{m+1}(1-\lambda^b e^{bt})^{m+1}}$. Then

$$h(t) = \frac{1}{a^{m+1}b}\left(\frac{at}{1-\lambda^a e^{at}}\right)^{m+1}e^{abxt}\left(\frac{1-\lambda^{ab}e^{abt}}{1-\lambda^b e^{bt}}\right)^{m}\left(\frac{bt}{1-\lambda^b e^{bt}}\right)e^{abyt}.$$

From (3) and (8), we have

$$h(t) = \frac{(-1)^m}{a^{m+1}b}\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r}\mathcal{B}_{n-r}^{(m+1)}\big(bx;\lambda^a\big)a^{n-r}\lambda^{-mb}\sum_{k=0}^{r}\binom{r}{k}\right.$$
$$\left.\times\sum_{p=0}^{k}\binom{k}{p}(-l)^{k-p}S_p^{(m)}\big(a;\lambda^b\big)\mathcal{B}_{r-k}\big(ay;\lambda^b\big)b^r\right\}\frac{t^n}{n!}.$$

In a similar manner,

$$h(t)=\frac{(-1)^m}{b^{m+1}a}\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r}\mathcal{B}_{n-r}^{(m+1)}\left(ay;\lambda^b\right)b^{n-r}\lambda^{-ma}\sum_{k=0}^{r}\binom{r}{k}\right.$$
$$\left.\times\sum_{p=0}^{k}\binom{k}{p}(-l)^{k-p}S_p^{(m)}\left(b;\lambda^a\right)\mathcal{B}_{r-k}\left(ax;\lambda^a\right)a^r\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we proved the theorem. □

Corollary 2.3 *We put* $a=b=\lambda=1$ *in* (13). *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k^{(m+1)}(y)\mathcal{B}_{n-k}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k^{(m+1)}(x)\mathcal{B}_{n-k}(y).$$

3 Some symmetry identities for the Apostol-Euler polynomials

Theorem 3.1 *Let a and b be positive integers with the same parity. Then*

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(bx;\lambda^a\right)\lambda^{-b}a^kb^{n-k}\sum_{p=0}^{n-k}\binom{n-k}{p}(-1)^{n-k-p}T_{n-k}\left(a;\lambda^b\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(ax;\lambda^b\right)\lambda^{-a}b^ka^{n-k}\sum_{p=0}^{n-k}\binom{n-k}{p}(-1)^{n-k-p}T_{n-k}\left(b;\lambda^a\right). \tag{14}$$

Proof Let $h(t)=\frac{2e^{abxt}}{\lambda^ae^{at}+1}\frac{1+(-1)^{a+1}(\lambda^be^{bt})^a}{\lambda^be^{bt}+1}$. From (4) and (9) for $l=1$, we have

$$h(t)=\sum_{k=0}^{\infty}\mathcal{E}_k\left(bx;\lambda^a\right)\frac{a^kt^k}{k!}\frac{1}{\lambda^a}\sum_{l=0}^{\infty}\sum_{p=0}^{l}\binom{l}{p}(-1)^{l-p}T_p\left(a;\lambda^b\right)\frac{b^lt^l}{l!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(bx;\lambda^a\right)\lambda^{-b}a^kb^{n-k}\sum_{p=0}^{n-k}\binom{n-k}{p}(-1)^{n-k-p}T_{n-k}\left(a;\lambda^b\right)\right)\frac{t^n}{n!}.$$

Since $(-1)^{a+1}=(-1)^{b+1}$, the expression for $h(t)=\frac{2e^{abxt}}{\lambda^be^{bt}+1}\frac{1+(-1)^{b+1}(\lambda^ae^{at})^b}{\lambda^ae^{at}+1}$ is symmetric in a and b. Therefore, we obtain the following power series for $h(t)$ by symmetry:

$$h(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(ax;\lambda^b\right)\lambda^{-a}b^ka^{n-k}\sum_{p=0}^{n-k}\binom{n-k}{p}(-1)^{n-k-p}T_{n-k}\left(b;\lambda^a\right)\right)\frac{t^n}{n!}.$$

Equating the coefficient of $\frac{t^n}{n!}$ in the two expressions for $h(t)$ gives us the desired result. □

Theorem 3.2 *Let a and b be positive integers with the same parity. Then*

$$\sum_{s=0}^{n}\binom{n}{s}\mathcal{E}_{n-s}^{(\alpha+1)}\left(bx;\lambda^a\right)\lambda^{(-b\alpha)}a^{n-s}\sum_{k=0}^{s}\binom{s}{k}\sum_{p=0}^{k}\binom{k}{p}(-\alpha)^{k-p}T_k^{(\alpha)}\left(a;\lambda^b\right)\mathcal{E}_{s-k}\left(ay;\lambda^b\right)b^s$$

$$= \sum_{s=0}^{n} \binom{n}{s} \mathcal{E}_{n-s}^{(\alpha+1)}(ay;\lambda^b)\lambda^{(-a\alpha)} b^{n-s}$$

$$\times \sum_{k=0}^{s} \binom{s}{k} \sum_{p=0}^{k} \binom{k}{p} (-\alpha)^{k-p} T_k^{(\alpha)}(b;\lambda^a)\mathcal{E}_{s-k}(bx;\lambda^b)a^s. \tag{15}$$

Proof Let $k(t) = \frac{2^{\alpha+2}e^{abxt}(1+(-1)^{a+1}(\lambda^{ab}e^{abt}))^\alpha}{(\lambda^a e^{at}+1)^{\alpha+1}(\lambda^b e^{bt}+1)^{\alpha+1}} e^{abyt}$. From (4) and (9), we write

$$k(t) = \left(\frac{2}{\lambda^a e^{at}+1}\right)^{(\alpha+1)} e^{abxt} \left(\frac{1+(-1)^{a+1}(\lambda^{ab}e^{abt})}{\lambda^b e^{bt}+1}\right)^{\alpha} \left(\frac{2}{\lambda^b e^{bt}+1}\right) e^{abyt}$$

$$= \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha+1)}(bx;\lambda^a)\frac{a^n t^n}{n!} \frac{1}{\lambda^{b\alpha}} \sum_{n=0}^{\infty}\sum_{n=0}^{p} \binom{p}{n} (-\alpha)^{n-p} T_p^{(\alpha)}(a;\lambda^b)\frac{b^n t^n}{n!}$$

$$\times \sum_{n=0}^{\infty} \mathcal{E}_n(ay;\lambda^b)\frac{b^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{n} \binom{n}{s} \mathcal{E}_{n-s}^{(\alpha+1)}(bx;\lambda^a)\lambda^{(-b\alpha)} a^{n-s}\right.$$

$$\left. \times \sum_{k=0}^{s} \binom{s}{k} \sum_{p=0}^{k} \binom{k}{p} (-\alpha)^{k-p} T_k^{(\alpha)}(a;\lambda^b)\mathcal{E}_{s-k}(ay;\lambda^b) b^s \right\} \frac{t^n}{n!}.$$

Since $(-1)^{a+1} = (-1)^{b+1}$, the expression for $h(t)$ is symmetric in a and b.

In a similar manner, we have

$$k(t) = \sum_{n=0}^{\infty} \left\{ \sum_{s=0}^{n} \binom{n}{s} \mathcal{E}_{n-s}^{(\alpha+1)}(ay;\lambda^b)\lambda^{(-a\alpha)} b^{n-s}\right.$$

$$\left. \times \sum_{k=0}^{s} \binom{s}{k} \sum_{p=0}^{k} \binom{k}{p} (-\alpha)^{k-p} T_k^{(\alpha)}(b;\lambda^a)\mathcal{E}_{s-k}(bx;\lambda^b) a^s \right\} \frac{t^n}{n!}.$$

Equating the coefficient of $\frac{t^n}{n!}$ in the two expressions for $k(t)$ gives us the desired result. □

Theorem 3.3 *Let p, l, a, b and n be positive integers and a, b be of the same parity. Then*

$$\sum_{l=0}^{n} \binom{n}{l} \mathcal{B}_{n-l}(n;\lambda^a) a^{n-l}\lambda^{-a} \sum_{p=0}^{l} \binom{l}{p} (-1)^{l-p} T_p(b;\lambda^a) a^l$$

$$= 2^{n-l} a^n \left(\mathcal{B}_n\left(\frac{n}{2};\lambda^{2a}\right) + \frac{(-1)^{b+1}\lambda^{ab}}{2} \mathcal{B}_n\left(\frac{b+n}{2};\lambda^n\right)\right). \tag{16}$$

Proof Let $k(t) = \frac{ate^{ant}}{\lambda^a e^{at}-1}\frac{1+(-1)^b(\lambda^b e^{bt})^a}{\lambda^a e^{at}+1}$. From (3) and (10), we have

$$k(t) = \frac{ate^{ant}}{\lambda^a e^{at}-1}\frac{1+(-1)^{b+1}(\lambda^b e^{bt})^a}{\lambda^a e^{at}+1}$$

$$= \sum_{n=0}^{\infty} \mathcal{B}_n(n;\lambda^a)\frac{a^n t^n}{n!}\lambda^{-\alpha} \sum_{n=0}^{\infty}\sum_{n=0}^{p} \binom{p}{n} (-1)^{n-p} T_p(b;\lambda^a)\frac{a^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{B}_{n-l}\big(n;\lambda^{a}\big)a^{n-l}\lambda^{-a}\sum_{p=0}^{l}\binom{l}{p}(-1)^{l-p}T_{p}\big(b;\lambda^{a}\big)a^{l}\right)\frac{t^{n}}{n!}.$$

On the other hand, we write the function $k(t)$ as

$$\begin{aligned} k(t) &= \frac{1}{2}\frac{2ae^{\frac{n}{2}(2at)}}{\lambda^{2a}e^{2at}-1} + \frac{(-1)^{b+1}\lambda^{ab}2ate^{2at(\frac{n+b}{2})}}{2(\lambda^{2a}e^{2at}-1)} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{B}_{n}\left(\frac{n}{2};\lambda^{2a}\right)2^{n}\frac{a^{n}t^{n}}{n!} + \frac{(-1)^{b+1}\lambda^{ab}}{2}\sum_{n=0}^{\infty}\mathcal{B}_{n}\left(\frac{b+n}{2};\lambda^{2a}\right)2^{n}\frac{a^{n}t^{n}}{n!} \\ &= \sum_{n=0}^{\infty}\left(2^{n-l}a^{n}\left(\mathcal{B}_{n}\left(\frac{n}{2};\lambda^{2a}\right) + \frac{(-1)^{b+1}\lambda^{ab}}{2}\mathcal{B}_{n}\left(\frac{b+n}{2};\lambda^{n}\right)\right)\right)\frac{t^{n}}{n!}. \end{aligned}$$

Equating the coefficient of $\frac{t^n}{n!}$, we obtain (16). □

Competing interests
The author declares that they have no competing interests.

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