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# Solutions for a fractional difference boundary value problem

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**Abstract**

Using a variational approach and critical point theory, we investigate the existence of solutions for a fractional difference boundary value problem.

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**Keywords:** fractional difference boundary value problem; variational approach; critical point theory; solution

## 1 Introduction

In this work, using variational methods and critical point theory, we study the fractional difference boundary value problem

$$\begin{cases} {}_T\Delta_{t-1}^{\nu}({}_t\Delta_{\nu-1}^{\nu}x(t)) = f(x(t+\nu-1)), & t \in [0,T]_{\mathbb{N}_0}, \\ x(\nu-2) = [{}_t\Delta_{\nu-1}^{\nu}x(t)]_{t=T} = 0, \end{cases} \tag{1.1}$$

where $\nu \in (0,1)$, ${}_t\Delta_{\nu-1}^{\nu}$ and ${}_T\Delta_t^{\nu}$ are, respectively, the left fractional difference and the right fractional difference operators, $t \in [0,T]_{\mathbb{N}_0} := \{0,1,2,\dots,T\}$, and $f:\mathbb{R}\to\mathbb{R}$ is continuous.

Fractional calculus has a long history, and there is renewed interest in the study of both fractional calculus and fractional difference equations. In [1, 2], the authors discussed properties of the generalized falling function, a corresponding power rule for fractional delta-operators and the commutativity of fractional sums. A number of papers have appeared which build the theoretical foundations of discrete fractional calculus (for more details, we refer the reader to [3–8] and the references therein).

Atici and Eloe [3] considered the existence of positive solutions for the following two-point boundary value problem for a nonlinear finite fractional difference equation:

$$\begin{cases} -\Delta^{\nu}y(t) = f(t+\nu-1, y(t+\nu-1)), & t = 1,2,\dots,b+1, \\ y(\nu-2) = 0, \qquad y(\nu+b+1) = 0. \end{cases} \tag{1.2}$$

In [4], the authors used the mountain pass theorem, a linking theorem, and Clark's theorem to establish the existence of multiple solutions for a fractional difference boundary value problem with a parameter. Under some suitable assumptions, they obtained some results which ensure the existence of a precise interval of parameters for which the problem admits multiple solutions. We note that there are many papers in the literature [9–18] which discuss discrete problems via variational and critical point theory.

In [9], Tian and Henderson studied the $2n$th order nonlinear difference equation

$$\Delta^n\big(r(t-n)\Delta^n x(t-n)\big)+f\big(t,x(t)\big)=0,\quad t\in\mathbb{Z},\tag{1.3}$$

and established some existence results for anti-periodic solutions under various assumptions on the nonlinearity. In [10], Ye and Tang considered the second-order discrete Hamiltonian system

$$\Delta^2 u(t-1)+d(t)\big|u(t)\big|^{\mu-2}u(t)+\nabla H\big(t,u(t)\big)=0,\quad \forall t\in\mathbb{Z},$$

and obtained an existence theorem for a nonzero $T$-periodic solution.

In the literature on discrete problem via critical point theory, the authors are interested in the existence of at least one solution or infinitely many solutions. The existence of a unique solution is not usually studied. In this paper, using Browder's theorem, first we present a uniqueness result in Section 3. Then a linking theorem is used to establish existence. Finally, assuming an Ambrosetti-Rabinowitz type condition, we show that problem (1.1) has many solutions if the nonlinearity is odd.

## 2 Preliminaries

For convenience, throughout this paper, we arrange $\sum_{i=j}^{m} x(i)=0$ for $m<j$. We present some definitions and lemmas for discrete fractional operators.

For any integer $\beta$, let $\mathbb{N}_\beta := \{\beta,\beta+1,\beta+2,\ldots\}$ and $t^{(\nu)} := \Gamma(t+1)/\Gamma(t+1-\nu)$, where $t$ and $\nu$ are determined by (1.1). We also appeal to the convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{(\nu)}=0$.

**Definition 2.1** (see [2, 3]) The $\nu$th fractional sum of $f$ for $\nu>0$ is defined by

$$\triangle_a^{-\nu} f(t)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{(\nu-1)}f(s)\quad\text{for } t\in\mathbb{N}_{a-\nu}.\tag{2.1}$$

We also define the $\nu$th fractional difference for $\nu>0$ by $\triangle^\nu f(t) := \triangle^N\triangle^{\nu-N}f(t)$, where $t\in\mathbb{N}_{a+N-\nu}$ and $N\in\mathbb{N}$ is chosen so that $0\le N-1<\nu\le N$.

**Definition 2.2** (see [2, 3]) Let $f$ be any real-valued function and $\nu\in(0,1)$. The left discrete fractional difference and the right discrete fractional difference operators are, respectively, defined as

$$\begin{aligned}
{}_t\triangle_a^\nu f(t) &= \triangle_t\triangle_a^{-(1-\nu)}f(t)\\
&=\frac{1}{\Gamma(1-\nu)}\triangle\sum_{s=a}^{t+\nu-1}(t-s-1)^{(-\nu)}f(s),\quad t\equiv a-\nu+1(\operatorname{mod}1),\\
{}_b\triangle_t^\nu f(t) &= -\triangle_b\triangle_t^{-(1-\nu)}f(t)\\
&=\frac{1}{\Gamma(1-\nu)}(-\triangle)\sum_{s=t+1-\nu}^{b}(s-t-1)^{(-\nu)}f(s),\quad t\equiv b+\nu-1(\operatorname{mod}1).
\end{aligned}\tag{2.2}$$

**Definition 2.3** Suppose that $X$ is a Banach space and $I : X \to \mathbb{R}$ is a functional defined on $X$. For given $x, y \in X$, assume that

$$\lim_{\varepsilon \to 0} \frac{I(x + \varepsilon y) - I(x)}{\varepsilon} \tag{2.3}$$

exists. Then $I$ is Gateaux differentiable at $x$, the limit in (2.3) is called the Gateaux differential of $I$ at $x$ in direction $y$, and is denoted by $(I'(x), y)$, *i.e.*,

$$\bigl(I'(x), y\bigr) = \lim_{\varepsilon \to 0} \frac{I(x + \varepsilon y) - I(x)}{\varepsilon}. \tag{2.4}$$

**Definition 2.4** (see [19, p.303]) Let $X$ be a reflexive real Banach space and $X^*$ its dual. The operator $L : X \to X^*$ is said to be demicontinuous if $L$ maps strongly convergent sequences in $X$ to weakly convergent sequences in $X^*$.

**Lemma 2.5** (Browder theorem, see [19, Theorem 5.3.22]) *Let $X$ be a reflexive real Banach space. Moreover, let $L : X \to X^*$ be an operator satisfying the conditions*

(i) *$L$ is bounded and demicontinuous,*
(ii) *$L$ is coercive, i.e., $\lim_{\|x\| \to \infty} \frac{(L(x), x)}{\|x\|} = +\infty$,*
(iii) *$L$ is monotone on the space $X$, i.e., for all $x, y \in X$, we have*

$$\bigl(L(x) - L(y), x - y\bigr) \geq 0. \tag{2.5}$$

*Then the equation $L(x) = f^*$ has at least one solution $x \in X$ for every $f^* \in X^*$. If, moreover, the inequality* (2.5) *is strict for all $x, y \in X$, $x \neq y$, then the equation $L(x) = f^*$ has precisely one solution $x \in X$ for all $f^* \in X^*$.*

**Definition 2.6** (see [20–22]) Let $X$ be a real Banach space, $I \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say that $I$ satisfies the $(\mathrm{PS})_c$ condition if any sequence $\{x_n\} \subset X$ such that $I(x_n) \to c$ and $I'(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

**Lemma 2.7** (Linking theorem, Rabinowitz, see [20–22]) *Let $X = Y \oplus Z$ be a Banach space with $Z$ closed in $X$ and* $\dim Y < \infty$. *Let $\rho > r > 0$, and let $z \in Z$ be such that $\|z\| = r$. Define*

$$\mathcal{M} := \bigl\{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\bigr\}, \qquad \mathcal{N} := \bigl\{u \in Z : \|u\| = r\bigr\},$$
$$\mathcal{M}_0 := \bigl\{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0, \text{ or } \|u\| \leq \rho \text{ and } \lambda = 0\bigr\}.$$

*Let $I \in C^1(X, \mathbb{R})$ be such that*

$$b := \inf_{u \in \mathcal{N}} I(u) > a := \max_{u \in \mathcal{M}_0} I(u).$$

*If $I$ satisfies the* $(\mathrm{PS})_c$ *condition with*

$$c := \inf_{\gamma \in \Gamma} \max_{u \in \mathcal{M}} I\bigl(\gamma(u)\bigr), \quad \text{where } \Gamma := \bigl\{\gamma \in C(\mathcal{M}, X) : \gamma|_{\mathcal{M}_0} = \mathrm{id}\bigr\},$$

*then $c$ is a critical point of $I$.*

**Definition 2.8** (see [20–22]) Let $X$ be a real Banach space and $I \in C^1(X,\mathbb{R})$. We say that $I$ satisfies the Cerami condition ((C) condition for short) if any sequence $\{x_n\} \subset X$ such that $I(x_n)$ is bounded and $(1 + \|x_n\|)\|I'(x_n)\| \to 0$ as $n \to \infty$, there exists a subsequence of $\{x_n\}$ which is convergent in $X$.

**Lemma 2.9** (Mountain pass theorem, see [20–22]) *Let $X$ be a real Banach space, and let $I \in C^1(X,\mathbb{R})$ satisfy the* (C) *condition. If $I(\theta) = 0$ and the following conditions hold*:

(i) *there are two positive constants $\rho$, $\eta$ and a closed linear subspace $X_1$ of $X$ such that codim $X_1 = l$ and $I|_{X_1 \cap \partial B_\rho} \geq \eta$, where $B_\rho$ is an open ball of radius $\rho$ with center $\theta$*;

(ii) *there is a subspace $X_2$ with $\dim X_2 = m$, $m > l$, such that*

$$I(x) \to -\infty \quad \textit{as } \|x\| \to \infty, x \in X_2.$$

*Then $I$ possesses at least $m - l$ distinct pairs of nontrivial critical points.*

In what follows, we establish the variational framework for (1.1). Let

$$X := \{x = (x(\nu - 1), x(\nu), \ldots, x(\nu + T - 1))^+ : x(\nu + i - 1) \in \mathbb{R}, i = 0, 1, \ldots, T\}. \tag{2.6}$$

Then $X$ is the $T + 1$-dimensional Hilbert space with the usual inner product and the usual norm

$$(x, z) = \sum_{t=\nu-1}^{T+\nu-1} x(t)z(t), \qquad \|x\| = \left(\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^2\right)^{\frac{1}{2}}, \quad x, z \in X. \tag{2.7}$$

For $\alpha > 1$, we define the $\alpha$-norm on $X$: $\|x\|_\alpha = (\sum_{t=\nu-1}^{T+\nu-1} |x(t)|^\alpha)^{\frac{1}{\alpha}}$. Since $\dim X < \infty$, we see that there exist $c_{1\alpha} > 0$, $c_{2\alpha} > 0$ such that

$$c_{1\alpha}\|x\| \leq \|x\|_\alpha \leq c_{2\alpha}\|x\| \tag{2.8}$$

for all $x$ belonging to $X$ (or its subspace).

In view of [4, (3.4)], we can define an energy functional on $X$ by

$$I(x) = \frac{1}{2}\sum_{t=-1}^{T} ({}_t\Delta_{\nu-1}^{\nu} x(t))^2 - \sum_{t=-1}^{T} F(x(t + \nu - 1)), \quad x \in X, \tag{2.9}$$

where

$$F(x(t + \nu - 1)) = \int_0^{x(t+\nu-1)} f(s)\,\mathrm{d}s,$$

$$x(\nu - 2) = 0, \qquad [{}_t\Delta_{\nu-1}^{\nu} x(t)]_{t=T} = \frac{-\nu}{\Gamma(1 - \nu)} \sum_{s=\nu-1}^{T+\nu} (T + s - 1)^{(-\nu-1)} x(s) = 0.$$

Clearly, $I(\theta) = 0$. Let

$$\begin{aligned} E := \{\chi = {} & (x(\nu - 2), x(\nu - 1), \ldots, x(\nu + T))^+ \\ & \in \mathbb{R}^{T+3} : x(\nu - 2) = 0, [{}_t\Delta_{\nu-1}^{\nu} x(t)]_{t=T} = 0\}. \end{aligned} \tag{2.10}$$

Then, from the boundary conditions of (1.1), it is easy to see that $E$ is isomorphic to $X$. In the following, when we say $x \in X$, we always imply that $x$ can be extended to $\chi \in E$ if it is necessary. Now we claim that if $x = (x(\nu-1), x(\nu), \dots, x(\nu+T-1))^{+} \in X$ is a critical point of $I$, then $\chi = (x(\nu-2), x(\nu-1), \dots, x(\nu+T))^{+} \in E$ is precisely a solution of (1.1). Indeed, since $I$ can be viewed as a continuously differentiable functional defined on the finite dimensional Hilbert space $X$, the Fréchet derivative $I'(x)$ is zero if and only if $\partial I(x)/\partial x(i) = 0$ for all $i = \nu-1, \nu, \dots, \nu+T-1$. From the relation between the Fréchet derivative and the Gateaux derivative, we obtain

$$\begin{aligned}
\bigl(I'(x), y\bigr) = \lim_{\varepsilon\to 0}\frac{I(x+\varepsilon y) - I(x)}{\varepsilon} &= \frac{1}{\varepsilon}\Biggl[\frac{1}{2}\sum_{t=-1}^{T}\bigl[\bigl({}_t\triangle_{\nu-1}^{\nu}\bigl(x(t)+\varepsilon y(t)\bigr)\bigr)^2 - \bigl({}_t\triangle_{\nu-1}^{\nu}x(t)\bigr)^2\bigr] \\
&\quad - \sum_{t=-1}^{T}\bigl[F\bigl((x+\varepsilon y)(t+\nu-1)\bigr) - F\bigl(x(t+\nu-1)\bigr)\bigr]\Biggr] \\
&= \sum_{t=-1}^{T}{}_t\triangle_{\nu-1}^{\nu}x(t)\,{}_t\triangle_{\nu-1}^{\nu}y(t) - \sum_{t=-1}^{T}f\bigl(x(t+\nu-1)\bigr)y(t+\nu-1).
\end{aligned} \tag{2.11}$$

Therefore, in order to obtain the existence of solutions for (1.1), we only need to study the existence of critical points of the energy functional $I$ on $X$.

Next, noting Definition 2.2, for $t \in [-1, T]_{\mathbb{N}_{-1}}$, we let

$${}_t\triangle_{\nu-1}^{\nu}x(t) = \triangle\frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{t-(1-\nu)}(t-s-1)^{(-\nu)}x(s) := \triangle z(t+\nu-1). \tag{2.12}$$

Then we have

$$\begin{aligned}
z(\nu-2) &= 0, \\
z(\nu-1) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{0-(1-\nu)}(-s-1)^{(-\nu)}x(s) = x(\nu-1), \\
z(\nu) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{1-(1-\nu)}(1-s-1)^{(-\nu)}x(s) = (1-\nu)x(\nu-1) + x(\nu), \\
z(\nu+1) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{2-(1-\nu)}(2-s-1)^{(-\nu)}x(s) \\
&= \frac{(2-\nu)(1-\nu)}{2!}x(\nu-1) + (1-\nu)x(\nu) + x(\nu+1), \\
&\vdots \\
z(\nu+T-1) &= \frac{1}{\Gamma(1-\nu)}\sum_{s=\nu-1}^{T-(1-\nu)}(T-s-1)^{(-\nu)}x(s) \\
&= \frac{(T-\nu)(T-1-\nu)\cdots(1-\nu)}{T!}x(\nu-1) \\
&\quad + \frac{(T-1-\nu)(T-2-\nu)\cdots(1-\nu)}{(T-1)!}x(\nu) + \cdots \\
&\quad + (1-\nu)x(\nu+T-2) + x(\nu+T-1),
\end{aligned}$$

*i.e.*, $z = Bx$, where $z = (z(\nu-1), z(\nu), \dots, z(\nu+T-1))^+$, $x = (x(\nu-1), x(\nu), \dots, x(\nu+T-1))^+$,

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-\nu & 1 & 0 & \cdots & 0 \\ \frac{(2-\nu)(1-\nu)}{2!} & 1-\nu & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(T-\nu)(T-1-\nu)\cdots(1-\nu)}{T!} & \frac{(T-1-\nu)(T-2-\nu)\cdots(1-\nu)}{(T-1)!} & \cdots & \cdots & 1 \end{pmatrix}_{(T+1)\times(T+1)}.$$

Clearly, $(B^{-1})^+B^{-1}$ is a positive definite matrix. All the eigenvalues of $(B^{-1})^+B^{-1}$ are positive. Let $\lambda_{\min}$ and $\lambda_{\max}$ denote respectively the minimum and the maximum eigenvalues of $(B^{-1})^+B^{-1}$. Since $x = B^{-1}z$, we have

$$\lambda_{\min}\|z\|^2 \le \|x\|^2 = \left(z^+\left(B^{-1}\right)^+, B^{-1}z\right) \le \lambda_{\max}\|z\|^2. \tag{2.13}$$

Next, let

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{(T+1)\times(T+1)}.$$

By direct verification, we see that $A$ is a positive definite matrix. Let $\eta_1, \dots, \eta_{T+1}$ be the orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_{T+1}$ of $A$, where $0 < \lambda_1 < \cdots < \lambda_{T+1}$. Clearly, $X := \operatorname{span}\{\eta_1, \dots, \eta_{T+1}\}$. Let $Y := \operatorname{span}\{\eta_1, \dots, \eta_l\}$, $Z := \operatorname{span}\{\eta_{l+1}, \dots, \eta_{T+1}\}$, for $l \in [1, T+1]_{\mathbb{N}_0}$. Then $X = Y \oplus Z$.

## 3 Main results

Now we state our main results and give their proof. For convenience, we list assumptions on $f$ and $F$:

(H1) $f(\theta) \not\equiv 0$, and there is a constant $c$ such that $|f(x) - f(y)| \le c|x-y|$, $\forall x, y \in X$.

(H2) There exist a constant $d > 0$ and $\mu \in (1,2)$ such that $\limsup_{|x|\to 0} \frac{F(x)}{|x|^\mu} < d$.

(H3) $\frac{\lambda_l}{2\lambda_{\min}} x^2 \le \int_0^x f(s)\,\mathrm{d}s$ for all $x \in \mathbb{R}$.

(H4) There exist $\alpha > 2$ and $R > 0$ such that

$$0 < \alpha \int_0^x f(s)\,\mathrm{d}s \le x f(x) \quad \text{for } |x| > R.$$

(H5) There is a constant $\alpha > 2$ such that $\liminf_{|x|\to\infty} \frac{F(x)}{|x|^\alpha} > 0$.

(H6) There is $\gamma > 1$ such that

$$\liminf_{|x|\to\infty} \frac{f(x)x - 2F(x)}{|x|^\gamma} > 0.$$

(H7) $f(-x) + f(x) = 0$, $\forall x \in \mathbb{R}$.

**Theorem 3.1** *Let* (H1) *hold. Then* (1.1) *has precisely one solution for* $c \in (0, \frac{\lambda_1}{\lambda_{\max}})$.

*Proof* We shall apply Lemma 2.5 to prove the result. From (2.11), we define the operator

$$\bigl(L(x), y\bigr) = \sum_{t=-1}^{T} {}_t\triangle_{\nu-1}^{\nu} x(t) {}_t\triangle_{\nu-1}^{\nu} y(t) - \sum_{t=-1}^{T} f\bigl(x(t+\nu-1)\bigr) y(t+\nu-1), \quad \forall x, y \in X. \tag{3.1}$$

Clearly, if for all $y \in X$, there exists $x_0 \in X$ such that $(L(x_0), y) = 0$, then $x_0$ is a solution of (1.1). Let

$$\bigl(L_1(x), y\bigr) = \sum_{t=-1}^{T} {}_t\triangle_{\nu-1}^{\nu} x(t) {}_t\triangle_{\nu-1}^{\nu} y(t),$$

$$\bigl(L_2(x), y\bigr) = \sum_{t=-1}^{T} f\bigl(x(t+\nu-1)\bigr) y(t+\nu-1), \quad \forall x, y \in X.$$

We sketch the properties of $L_1$ and $L_2$. It is clear that $L_1$ is a linear operator, and furthermore, $L_1$ is bounded. Indeed, the Cauchy-Schwarz inequality enables us to obtain, notice (2.12) and (2.13),

$$\begin{aligned} \bigl|\bigl(L_1(x), y\bigr)\bigr| &\le \sum_{t=-1}^{T} \bigl|{}_t\triangle_{\nu-1}^{\nu} x(t)\bigr| \bigl|{}_t\triangle_{\nu-1}^{\nu} y(t)\bigr| \\ &\le \left(\sum_{t=-1}^{T} \bigl|{}_t\triangle_{\nu-1}^{\nu} x(t)\bigr|^2\right)^{\frac{1}{2}} \left(\sum_{t=-1}^{T} \bigl|{}_t\triangle_{\nu-1}^{\nu} y(t)\bigr|^2\right)^{\frac{1}{2}} \\ &\le \frac{\lambda_{T+1}}{\lambda_{\min}} \|x\| \|y\| < \infty, \quad x, y \in X. \end{aligned} \tag{3.2}$$

Consequently, $L_1$ is continuous on $X$. Next, we show that $L_2$ is bounded and continuous. Let $y = \theta$ in (H1) and $|f(\theta)| = c_1 > 0$. Then we have from (H1)

$$\bigl|f(x)\bigr| \le c|x| + c_1, \quad \forall x \in X. \tag{3.3}$$

From (3.3), the definition of $L_2$, and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \bigl|\bigl(L_2(x), y\bigr)\bigr| &\le \sum_{t=0}^{T} \bigl|f\bigl(x(t+\nu-1)\bigr)\bigr| \bigl|y(t+\nu-1)\bigr| \\ &\le \sum_{t=0}^{T} \bigl(c\bigl|x(t+\nu-1)\bigr| + c_1\bigr) \bigl|y(t+\nu-1)\bigr| \\ &\le c\|x\| \|y\| + c_1\sqrt{T+1}\|y\| < \infty, \quad \forall x, y \in X, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \bigl|\bigl(L_2(x_1) - L_2(x_2), y\bigr)\bigr| &\le \sum_{t=0}^{T} \bigl|f\bigl(x_1(t+\nu-1)\bigr) - f\bigl(x_2(t+\nu-1)\bigr)\bigr| \bigl|y(t+\nu-1)\bigr| \\ &\le \sum_{t=0}^{T} c\bigl|x_1(t+\nu-1) - x_2(t+\nu-1)\bigr| \bigl|y(t+\nu-1)\bigr| \\ &\le c\|x_1 - x_2\| \|y\|, \quad \forall x, y \in X. \end{aligned} \tag{3.5}$$

Therefore, $L_2$ is bounded and continuous, as required. Hence, $L$ is bounded and continuous, so demicontinuous.

From (3.3), notice (2.12) and (2.13), we see

$$\begin{aligned}
\bigl(L(x),x\bigr) &= \sum_{t=-1}^{T}\bigl({}_t\triangle_{\nu-1}^{\nu}x(t)\bigr)^2 - \sum_{t=-1}^{T} f\bigl(x(t+\nu-1)\bigr)x(t+\nu-1) \\
&= \sum_{t=-1}^{T-1}\bigl(\triangle z(t+\nu-1)\bigr)^2 - \sum_{t=0}^{T} f\bigl(x(t+\nu-1)\bigr)x(t+\nu-1) \\
&\geq \frac{\lambda_1}{\lambda_{\max}}\|x\|^2 - \sum_{t=0}^{T}\bigl(c\bigl|x(t+\nu-1)\bigr| + c_1\bigr)\bigl|x(t+\nu-1)\bigr| \\
&\geq \left(\frac{\lambda_1}{\lambda_{\max}} - c\right)\|x\|^2 - c_1\sqrt{T+1}\|x\|.
\end{aligned} \tag{3.6}$$

Therefore, $\lim_{\|x\|\to\infty}\frac{(L(x),x)}{\|x\|} = +\infty$, *i.e.*, $L$ is coercive on $X$.

Finally, we prove that $L$ is strictly monotone. Indeed, from (H1), we have

$$\begin{aligned}
\bigl(L(x)-L(y),x-y\bigr) &\geq \sum_{t=-1}^{T}\bigl({}_t\triangle_{\nu-1}^{\nu}x(t) - {}_t\triangle_{\nu-1}^{\nu}y(t)\bigr)^2 \\
&\quad - \sum_{t=0}^{T}\bigl|f\bigl(x(t+\nu-1)\bigr) - f\bigl(y(t+\nu-1)\bigr)\bigr|\bigl|x(t+\nu-1) - y(t+\nu-1)\bigr| \\
&\geq \frac{\lambda_1}{\lambda_{\max}}\|x-y\|^2 - c\|x-y\|^2 > 0, \quad \text{for } x,y\in X \text{ and } x\neq y.
\end{aligned} \tag{3.7}$$

All the conditions of Lemma 2.5 are satisfied, as claimed. Hence, (1.1) has precisely one solution. This completes the proof. □

**Theorem 3.2** *Let* (H2)-(H4) *hold. Then* (1.1) *has at least one solution.*

*Proof* From (H2), there exists $\delta > 0$ with

$$F(x) \leq d|x|^{\mu} \quad \text{for } |x| \leq \delta. \tag{3.8}$$

Thus, for $x \in Z$ with $\|x\| \leq \delta$, it follows from the Hölder inequality that

$$\begin{aligned}
I(x) &= \frac{1}{2}\sum_{t=-1}^{T}\bigl({}_t\triangle_{\nu-1}^{\nu}x(t)\bigr)^2 - \sum_{t=-1}^{T} F\bigl(x(t+\nu-1)\bigr) \\
&= \frac{1}{2}\sum_{t=-1}^{T-1}\bigl(\Delta z(t+\nu-1)\bigr)^2 - \sum_{t=0}^{T} F\bigl(x(t+\nu-1)\bigr) \\
&\geq \frac{\lambda_{l+1}}{2}\|z\|^2 - \sum_{t=0}^{T} d\bigl|x(t+\nu-1)\bigr|^{\mu} \\
&\geq \frac{\lambda_{l+1}}{2\lambda_{\max}}\|x\|^2 - d\left(\sum_{t=0}^{T}\bigl|x(t+\nu-1)\bigr|^2\right)^{\frac{\mu}{2}}\left(\sum_{t=0}^{T}1\right)^{1-\frac{\mu}{2}} \\
&= \frac{\lambda_{l+1}}{2\lambda_{\max}}\|x\|^2 - d(T+1)^{1-\frac{\mu}{2}}\|x\|^{\mu}.
\end{aligned} \tag{3.9}$$

By virtue of the inequality $1 < \mu < 2$, there exists $r > 0$ such that

$$b := \inf_{\|x\|=r, x\in Z} I(x) > 0. \tag{3.10}$$

From (H3), for $x \in Y$, we see

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right)^2 - \sum_{t=-1}^{T}F\left(x(t+\nu-1)\right) \\ &= \frac{1}{2}\sum_{t=-1}^{T-1}\left(\Delta z(t+\nu-1)\right)^2 - \sum_{t=0}^{T}F\left(x(t+\nu-1)\right) \\ &\le \frac{\lambda_l}{2\lambda_{\min}}\|x\|^2 - \sum_{t=0}^{T}F\left(x(t+\nu-1)\right) \\ &= \sum_{t=0}^{T}\left[\frac{\lambda_l}{2\lambda_{\min}}x^2(t+\nu-1) - F\left(x(t+\nu-1)\right)\right] \le 0. \end{aligned} \tag{3.11}$$

From (H4), we see that there exist $c_2, c_3 > 0$ such that

$$\int_0^x f(s)\,\mathrm{d}s \ge c_2|x|^\alpha - c_3, \quad \forall x \in \mathbb{R}. \tag{3.12}$$

Hence, for $x \in X$, we find

$$\begin{aligned} I(x) &= \frac{1}{2}\sum_{t=-1}^{T}\left({}_t\Delta_{\nu-1}^{\nu}x(t)\right)^2 - \sum_{t=-1}^{T}F\left(x(t+\nu-1)\right) \\ &= \frac{1}{2}\sum_{t=-1}^{T-1}\left(\Delta z(t+\nu-1)\right)^2 - \sum_{t=0}^{T}F\left(x(t+\nu-1)\right) \\ &\le \frac{\lambda_{T+1}}{2\lambda_{\min}}\|x\|^2 - c_2\sum_{t=0}^{T}\left|x(t+\nu-1)\right|^\alpha + c_3(T+1). \end{aligned} \tag{3.13}$$

Set $z := r\frac{\eta_{l+1}}{\|\eta_{l+1}\|}$ with $r > 0$ is given in (3.10). For $Y \oplus \mathbb{R}_z \subset X$, (2.8) holds true. This, together with (3.13), implies

$$I(x) \le \frac{\lambda_{T+1}}{2\lambda_{\min}}\|x\|^2 - c_2 c_{1\alpha}^{\alpha}\|x\|^\alpha + c_3(T+1).$$

Since $\alpha > 2$, we obtain

$$\lim_{\|x\|\to\infty, x\in Y\oplus\mathbb{R}_z} I(x) = -\infty. \tag{3.14}$$

Define

$$\mathcal{M} := \left\{x = y + \lambda z : \|x\| \le \rho, \lambda \ge 0, y \in Y\right\},$$

$$\mathcal{M}_0 := \left\{x = y + \lambda z : y \in Y, \|x\| = \rho \text{ and } \lambda \ge 0, \text{ or } \|x\| \le \rho \text{ and } \lambda = 0\right\}.$$

Since $z \in Z$ and then $I(z) \geq b > 0$, (3.11) and (3.14) guarantee that there is $\rho > r$ such that

$$a := \max_{x \in \mathcal{M}_0} I(x) \leq 0.$$

It remains to prove that $I$ satisfies the $(\mathrm{PS})_c$ condition. This will be the case if we show that any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$d := \sup_n I(x_n) < \infty, \qquad I'(x_n) \to 0,$$

contains a convergent subsequence. Note that $\dim X < \infty$, so we only need to show the boundedness of $\{x_n\}_{n=1}^{\infty}$. Take $\beta > 0$ such that $\beta^{-1} \in (2, \alpha)$ for $n$ large enough, and (H4), (3.12) and (2.8) enable us to obtain

$$\begin{aligned}
d + \|x_n\| &\geq I(x_n) - \frac{1}{\beta}\bigl(I'(x_n), x_n\bigr) \\
&= \left(\frac{1}{2} - \frac{1}{\beta}\right) \sum_{t=-1}^{T} \bigl({}_t\triangle_{\nu-1}^{\nu} x_n(t)\bigr)^2 \\
&\quad + \sum_{t=-1}^{T} \left(\frac{1}{\beta} f\bigl(x_n(t+\nu-1)\bigr) x_n(t+\nu-1) - F\bigl(x_n(t+\nu-1)\bigr)\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\beta}\right) \frac{\lambda_1}{\lambda_{\max}} \|x_n\|^2 + \sum_{t=0}^{T} \left(\frac{\alpha}{\beta} - 1\right) F\bigl(x_n(t+\nu-1)\bigr) \\
&\geq \left(\frac{1}{2} - \frac{1}{\beta}\right) \frac{\lambda_1}{\lambda_{\max}} \|x_n\|^2 + \sum_{t=0}^{T} \left(\frac{\alpha}{\beta} - 1\right) \bigl(c_2 \bigl|x_n(t+\nu-1)\bigr|^{\alpha} - c_3\bigr) \\
&\geq \left(\frac{1}{2} - \frac{1}{\beta}\right) \frac{\lambda_1}{\lambda_{\max}} \|x_n\|^2 + \left(\frac{\alpha}{\beta} - 1\right) c_2 c_{1\alpha}^{\alpha} \|x_n\|^{\alpha} - \left(\frac{\alpha}{\beta} - 1\right)(T+1) c_3.
\end{aligned}$$

Since $\alpha > 2$ and $(\frac{\alpha}{\beta} - 1) > 0$, we see that $\{x_n\}_{n=1}^{\infty}$ is bounded.

Thus the functional $I$ satisfies all the conditions of Lemma 2.7, and then $I$ has a critical point, and (1.1) has at least one solution. This completes the proof. □

**Theorem 3.3** *Let* (H2), (H5)-(H7) *hold. Then* (1.1) *has at least* $m - l$ *solutions.*

*Proof* We shall utilize Lemma 2.9 to prove the result. If $X_1 = Z = \operatorname{span}\{\eta_{l+1}, \ldots, \eta_{T+1}\}$, we see codim $X_1 = l$. From (H2), noting (3.9), we can take $\rho = (dd_1 \lambda_{\max} \lambda_{l+1}^{-1} (T+1)^{1-\frac{\mu}{2}})^{\frac{1}{2-\mu}}$ so that $\rho \leq \delta$, where $d_1 > 2$. Therefore,

$$\begin{aligned}
I_{X_1 \cap \partial B_\rho} &\geq \bigl(dd_1 \lambda_{\max} \lambda_{l+1}^{-1} (T+1)^{1-\frac{\mu}{2}}\bigr)^{\frac{\mu}{2-\mu}} \left[\frac{\lambda_{l+1}}{2\lambda_{\max}} dd_1 \lambda_{\max} \lambda_{l+1}^{-1} (T+1)^{1-\frac{\mu}{2}} - d(T+1)^{1-\frac{\mu}{2}}\right] \\
&= \left(\frac{d_1}{2} - 1\right) d \bigl(dd_1 \lambda_{\max} \lambda_{l+1}^{-1} (T+1)^{1-\frac{\mu}{2}}\bigr)^{\frac{\mu}{2-\mu}} (T+1)^{1-\frac{\mu}{2}}.
\end{aligned}$$

Thus (i) of Lemma 2.9 holds true.

Choose $X_2 := \operatorname{span}\{\eta_1, \ldots, \eta_m\}$, where $m > l$, and $\dim X_2 = m$. From (H5), we see that there exist $c_4, c_5 > 0$ such that

$$F(x) \geq c_4 |x|^{\alpha} - c_5, \quad \forall x \in \mathbb{R}.$$

Therefore, from (3.13) and (2.8), we arrive at

$$\begin{aligned}-I(x) &\geq -\frac{\lambda_m}{2\lambda_{\min}}\|x\|^2 + c_4\sum_{t=0}^{T}\bigl|x(t+\nu-1)\bigr|^{\alpha} - c_5(T+1)\\ &\geq -\frac{\lambda_m}{2\lambda_{\min}}\|x\|^2 + c_4 c_{1\alpha}^{\alpha}\|x\|^{\alpha} - c_5(T+1).\end{aligned}$$

Since $\alpha > 2$, $I(x) \to -\infty$ as $\|x\| \to \infty$, $x \in X_2$. Thus (ii) of Lemma 2.9 holds true.

Finally, we prove that $I$ satisfies the (C) condition. Let $\{x_n\} \subset X$ be such that for some $M_1 > 0$,

$$\bigl|I(x_n)\bigr| \leq M_1, \qquad \bigl(1+\|x_n\|\bigr)\bigl\|I'(x_n)\bigr\| \to 0 \quad \text{as } n \to \infty.$$

We claim that $\|x_n\|$ is bounded. Otherwise, suppose that $\|x_n\| \to \infty$ as $n \to \infty$. It is easy to see that for any $n \in \mathbb{N}$, there exists $M_2$ such that

$$2I(x_n) - \bigl(I'(x_n), x_n\bigr) \leq M_2.$$

On the other hand, from (H6), there exist $c_6, c_7 > 0$ such that

$$f(x)x - 2F(x) \geq c_6|x|^{\gamma} - c_7, \quad \forall x \in \mathbb{R}.$$

Consequently, from (2.8),

$$\begin{aligned}2I(x_n) - \bigl(I'(x_n), x_n\bigr) &= \sum_{t=0}^{T}\bigl[f\bigl(x_n(t+\nu-1)\bigr)x_n(t+\nu-1) - 2F\bigl(x_n(t+\nu-1)\bigr)\bigr]\\ &\geq \sum_{t=0}^{T}\bigl(c_6\bigl|x_n(t+\nu-1)\bigr|^{\gamma} - c_7\bigr) \geq c_6 c_{1\gamma}^{\gamma}\|x_n\|^{\gamma} - c_7(T+1).\end{aligned}$$

Let $n \to \infty$, and we get a contradiction.

It is easy to see that $I$ is even and $I(\theta) = 0$. Thus all the conditions of Lemma 2.9 are satisfied, and (1.1) has at least $m - l$ solutions. The proof is complete. □

**Examples**

1. Let $f(x) = \eta_1 x + \eta_2$, where $\eta_1 \in (0, \frac{\lambda_1}{\lambda_{\max}})$ and $\eta_2 \neq 0$. Clearly, (H1) holds.
2. Let $f(x) = \frac{\lambda_l}{\lambda_{\min}}(x + x^3)e^{x^2} + 4x^3$. Then $F(x) = \frac{\lambda_l}{2\lambda_{\min}}x^2 e^{x^2} + x^4$. Thus, (H2) and (H3) hold automatically. For $R \geq 1$, $\alpha = 4$, we see

   $$0 < \frac{2\lambda_l}{\lambda_{\min}}x^2 e^{x^2} + 4x^4 \leq xf(x) = \frac{\lambda_l}{\lambda_{\min}}\bigl(x^2 + x^4\bigr)e^{x^2} + 4x^4 \quad \text{for all } |x| > 1.$$

   Therefore, (H4) holds.
3. Let $f(x) = 2x + 4x^3$. Then $F(x) = x^2 + x^4$ and (H2), (H7) hold. Choose $\alpha = 4$, $\gamma = 4$, and we see

   $$\liminf_{|x|\to\infty}\frac{x^2 + x^4}{|x|^4} = 1 > 0, \qquad \liminf_{|x|\to\infty}\frac{f(x)x - 2F(x)}{|x|^4} = \liminf_{|x|\to\infty}\frac{2x^4}{|x|^4} = 2 > 0.$$

   Hence (H5) and (H6) hold.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
WD and JX carried out the main results of this article and drafted the manuscript. DO directed the study and helped with the inspection. All the authors read and approved the final manuscript.

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