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Normalized polynomials and their multiplication formulas

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Abstract
The aim of this paper is to prove multiplication formulas of the normalized polynomials by using the umbral algebra and umbral calculus methods. Our polynomials are related to the Hermite-type polynomials.

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1 Introduction

In this paper, we use the following notations:

$$\mathbb{N} := \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

$$\delta_{n,k} = \begin{cases} 0 & \text{if } n \neq k, \\ 1 & \text{if } n = k, \end{cases}$$

$$(n)_k = n(n-1)\cdots(n-k+1).$$

Here, we first give some remarks on the normalized polynomials.

Firstly, we introduce some notations which are related to the earlier works by (among others) Carlitz [1, 2], Bodin [3], Roman [4, pp.1-125]. We recall from the work of Bodin [3]: Let p be a prime number and $n \geq 1$. For $q = p^n$, we denote by $\mathbb{F}_q$ the finite field having q elements. $\mathbb{F}_q^*$ denotes the multiplicative group of non-zero elements of $\mathbb{F}_q$.

Let $f(x,y) \in \mathbb{F}_q[x,y]$ be a polynomial of degree exactly d

$$f(x,y) = \alpha_0 x^d + \alpha_1 x^d y + \alpha_2 x^{d-2} y^2 + \cdots + \alpha_d y^d + \text{terms of lower degree.}$$

f is said to be *normalized* if the first non-zero term in the sequence $(\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_d)$ is equal to 1. Any polynomial g can be written

$$g(x,y) = cf(x,y),$$

where f is a normalized polynomial and $c \in \mathbb{F}_q^*$ (*cf.* [3]).

We recall the work of Carlitz [2, p.60]: Let k be a fixed integer > 1 and let $\alpha_{k_1}, \ldots, \alpha_{k_k}$ denote (complex) numbers such that

$$\alpha_{k_1} + \cdots + \alpha_{k_k} = 1.$$

Let $|\lambda_k| \neq 1$ or 0 and let $\beta_{k_1}, \ldots, \beta_{k_k}$ be distinct numbers. Then consider the functional equation

$$\sum_{r=1}^{k} \alpha_{k_r} f_m(x + \beta_{k_r}) = \lambda_k^{-m} f_m(\lambda_k x), \tag{1}$$

where $f_m(x)$ denotes a normalized polynomial of degree m (that is, a polynomial with the highest coefficient 1). Here $f_m(x)$ is completely determined by (1); moreover, $f_m(x)$ form an Appell set of polynomials (*cf.* [2]).

Theorem 1.1 *Let k be a fixed integer > 1 and let $\alpha_{k_1}, \ldots, \alpha_{k_k}$ be complex numbers such that*

$$\alpha_{k_1} + \cdots + \alpha_{k_k} = 1.$$

Let $|\lambda_k| \neq 1$ or 0 and let $\beta_{k_1}, \ldots, \beta_{k_k}$ be distinct numbers. Then equation (1) *is satisfied by a unique set of normalized polynomials $\{f_m(x)\}$ which form an Appell set* (*cf.* [2]).

Every Appell set satisfies an equation of the form (1) (*cf.* [2]).

If $f_n(x)$ is a normalized polynomial, then it satisfies the following formula:

$$f_n(yx) = y^{n-1} \sum_{j=0}^{y-1} f_n\left(x + \frac{j}{y}\right). \tag{2}$$

If y is an even positive integer, some normalized polynomials satisfy the following equation (*cf.* [1]):

$$g_{n-1}(yx) = -\frac{2y^{n-1}}{n} \sum_{j=0}^{y-1} (-1)^j f_n\left(x + \frac{j}{y}\right), \tag{3}$$

where $g_{n-1}(x)$ and $f_n(x)$ denote the normalized polynomials of degree $n-1$ and n, respectively.

We give some Hermite base polynomials of higher order, which are defined as follows (*cf.* [5] and [6]):

$$\left(\frac{t}{e^t - 1}\right)^a e^{xt - \frac{vt^2}{2}} = \sum_{n=0}^{\infty} \mathcal{B}_{H,n}^{(a)}(x, v) \frac{t^n}{n!} \quad \left(|t| < 2\pi\right),$$

where $\mathcal{B}_{H,n}^{(a)}(x, v)$ denotes Hermite base Bernoulli polynomials of higher order,

$$\left(\frac{2}{e^t + 1}\right)^a e^{xt - \frac{vt^2}{2}} = \sum_{n=0}^{\infty} \mathcal{E}_{H,n}^{(a)}(x, v) \frac{t^n}{n!} \quad \left(|t| < \pi\right),$$

where $\mathcal{E}_{H,n}^{(a)}(x,\nu)$ denotes Hermite base Euler polynomials of higher order and

$$\left(\frac{2t}{e^t+1}\right)^a e^{xt-\frac{\nu t^2}{2}} = \sum_{n=0}^{\infty} \mathcal{G}_{H,n}^{(a)}(x,\nu)\frac{t^n}{n!} \quad \bigl(|t|<\pi\bigr),$$

where $\mathcal{G}_{H,n}^{(a)}(x,\nu)$ denotes Hermite base Genocchi polynomials of higher order.

The proof of polynomials which satisfied (2) was given in various ways. In this paper, we study normalized polynomials which are defined above by using the umbral algebra and umbral calculus methods. We also recall from the work of Roman [4] the following.

Let P be the algebra of polynomials in the single variable x over the field complex numbers. Let P^* be the vector space of all linear functionals on P. Let

$$\langle L \mid p(x)\rangle$$

be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denote the algebra of formal power series in the variable t over $\mathbb{C}$. The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$$

defines a linear functional on P by setting

$$\langle f(t) \mid x^n\rangle = a_n$$

for all $n \geq 0$. In a special case,

$$\langle t^k \mid x^n\rangle = n!\delta_{n,k}.$$

This kind of algebra is called an umbral algebra (*cf.* [4]). Any power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k$$

is a linear operator on P defined by

$$f(t)x^n = \sum_{k=0}^{\infty} \binom{n}{k} a_k x^{n-k}.$$

Here, each $f(t) \in \mathfrak{F}$ plays three roles in the umbral calculus: *a formal power series*, *a linear functional* and *a linear operator*. For example, let $p(x) \in P$ and

$$f(t) = e^{yt}.$$

As a linear functional, e^{yt} satisfies the following property:

$$\langle e^{yt} \mid p(x)\rangle = p(y). \tag{4}$$

As a linear operator, e^{yt} satisfies the following property:

$$e^{yt}p(x) = p(x+y). \tag{5}$$

Let $f(t)$, $g(t)$ be in $\mathfrak{F}$, then

$$\langle f(t)g(t) \mid p(x)\rangle = \langle f(t) \mid g(t)p(x)\rangle$$

for all polynomials $p(x)$. The order $o(f(t))$ of a power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $f(t) = 0$, $o(f(t)) = +\infty$. A series $f(t)$ for which

$$o(f(t)) = 1$$

is called a delta series. A series $f(t)$ for which

$$o(f(t)) = 0$$

is called an invertible series (for details, see [4]).

Theorem 1.2 [4, p.20, Theorem 2.3.6] *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k \mid s_n(x)\rangle = n!\delta_{n,k} \tag{6}$$

for all $n, k \geq 0$.

The sequence $s_n(x)$ in (6) is the Sheffer polynomials for a pair $(g(t), f(t))$. The Sheffer polynomials for a pair $(g(t), t)$ are the Appell polynomials or Appell sequences for $g(t)$ (*cf.* [4]).

The Appell polynomials are defined by means of the following generating function (*cf.* [4]):

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{7}$$

The Appell polynomials satisfy the following relations:

$$s_n(x) = g(t)^{-1} x^n, \tag{8}$$

the derivative formula

$$t s_n(x) = s_n'(x) = n s_{n-1}(x) \tag{9}$$

and

$$\frac{1}{t} s_n(x) = \frac{1}{n+1} s_{n+1}(x), \tag{10}$$

the multiplication formula

$$s_n(\alpha x) = \alpha^n \frac{g(t)}{g(\frac{t}{\alpha})} s_n(x), \tag{11}$$

where $\alpha \neq 0$.

In the next section, we need the following generalized multinomial identity.

Lemma 1.3 (Generalized multinomial identity [7, p.41, Equation (12m)]) *If $x_1, x_2, \ldots, x_m$ are commuting elements of a ring ($\Leftrightarrow x_i x_j = x_j x_i$, $1 \leq i < j \leq m$), then we have for all real or complex variables α*:

$$(x_1 + x_2 + \cdots + x_m)^\alpha = \sum_{v_1, v_2, \ldots, v_m \geq 0} \binom{\alpha}{v_1, v_2, \ldots, v_m} x_1^{v_1} x_2^{v_2} \cdots x_m^{v_m},$$

the last summation takes places over all positive or zero integers $v_i \geq 0$, where

$$\binom{\alpha}{v_1, v_2, \ldots, v_m} := \frac{(\alpha)_{v_1 + v_2 + \cdots + v_m}}{v_1! v_2! \cdots v_m!}$$

are called generalized multinomial coefficients defined by [7, p.27, Equation (10c″)], *where $b \in \mathbb{N}$ and $(x)_0 = 1$.*

2 Some identities of normalized polynomials

In this section, we derive some identities and properties related to Hermite base normalized polynomials.

If we set

$$g(t) = \left(\frac{e^t - 1}{t} \right)^a e^{\frac{vt^2}{2}} \tag{12}$$

in (8), we obtain the following lemma.

Lemma 2.1 *Let $n \in \mathbb{N}_0$. The following relationship holds true*:

$$\mathcal{B}_{H,n}^{(a)}(x, v) = \left(\frac{t}{e^t - 1} \right)^a e^{-\frac{vt^2}{2}} x^n. \tag{13}$$

If we set

$$g(t) = \left(\frac{e^t + 1}{2} \right)^a e^{\frac{vt^2}{2}} \tag{14}$$

in (8), we obtain the following lemma.

Lemma 2.2 *Let $n \in \mathbb{N}_0$. The following relationship holds true*:

$$\mathcal{E}_{H,n}^{(a)}(x, v) = \left(\frac{2}{e^t + 1} \right)^a e^{-\frac{vt^2}{2}} x^n. \tag{15}$$

If we set

$$g(t) = \left(\frac{e^t + 1}{2t}\right)^a e^{\frac{vt^2}{2}} \tag{16}$$

in (8), we obtain the following lemma.

Lemma 2.3 *Let $n \in \mathbb{N}_0$. The following relationship holds true*:

$$\mathcal{G}_{H,n}^{(a)}(x,v) = \left(\frac{2t}{e^t + 1}\right)^a e^{-\frac{vt^2}{2}} x^n. \tag{17}$$

3 Multiplication formulas for normalized polynomials

In this section, we study the Hermite base normalized polynomials. The Hermite base Bernoulli-type polynomials satisfy equation (2), which is given by the following theorem.

Theorem 3.1 *Let $m \in \mathbb{N}$. The following multiplication formula of the $\mathcal{B}_{H,n}^{(a)}(x,v)$ polynomials holds true*:

$$\mathcal{B}_{H,n}^{(a)}(mx,v) = m^{n-a} \sum_{u_1,\dots,u_{m-1} \geq 0} \binom{a}{u_1,\dots,u_{m-1}} \mathcal{B}_{H,n}^{(a)}\left(x + \frac{r}{m}, \frac{v}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

Proof By using (12) in (11), we get

$$\mathcal{B}_{H,n}^{(a)}(mx,v) = m^{n-a} \frac{(e^t - 1)^a}{(e^{\frac{t}{m}} - 1)^a} e^{\frac{vt^2}{2}} e^{-\frac{vt^2}{2m^2}} \mathcal{B}_{H,n}^{(a)}(x,v).$$

Using (13), we have

$$\mathcal{B}_{H,n}^{(a)}(mx,v) = m^{n-a} \frac{(e^t - 1)^a}{(e^{\frac{t}{m}} - 1)^a} \mathcal{B}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right).$$

After some calculations, we obtain

$$\mathcal{B}_{H,n}^{(a)}(mx,v) = m^{n-a} \left(\sum_{k=0}^{m-1} e^{\frac{kt}{m}}\right)^a \mathcal{B}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right).$$

By using Lemma 1.3 in the above equation, we get

$$\mathcal{B}_{H,n}^{(a)}(mx,v) = m^{n-a} \sum_{u_1,\dots,u_{m-1} \geq 0} \binom{a}{u_1,\dots,u_{m-1}} e^{\frac{rt}{m}} \mathcal{B}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

From (5), we arrive at the desired result. $\square$

The Hermite base Euler-type polynomials and the Hermite base Genocchi-type polynomials satisfy equation (2) for all $m \in \mathbb{N}$. But for m being odd, these polynomials in studies normalize condition in (2). For even m, these polynomials satisfy (3).

Theorem 3.2 *Let $m \in \mathbb{N}$. The following multiplication formula of the $\mathcal{E}_{H,n}^{(a)}(x,v)$ polynomials holds true*:

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = m^n \sum_{u_1,\ldots,u_{m-1}\geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r \mathcal{E}_{H,n}^{(a)}\left(x+\frac{r}{m},\frac{v}{m^2}\right)$$

when m is odd,

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = \frac{m^n 2^a}{(n+1)_a} \sum_{u_1,\ldots,u_{m-1}\geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r \mathcal{B}_{H,n+a}^{(a)}\left(x+\frac{r}{m},\frac{v}{m^2}\right)$$

when m is even, where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

Proof Let m be odd.

From (11), (14) and (15), we obtain

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = m^n \left(\frac{e^t+1}{e^{\frac{t}{m}}+1}\right)^a \mathcal{E}_{H,n}^{(a)}\left(x,\frac{v}{m^2}\right).$$

After some calculations, we get

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = m^n \left(\sum_{k=0}^{m-1}(-1)^k e^{\frac{kt}{m}}\right)^a \mathcal{E}_{H,n}^{(a)}\left(x,\frac{v}{m^2}\right).$$

By using Lemma 1.3, we obtain

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = m^n \sum_{u_1,\ldots,u_{m-1}\geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r e^{\frac{rt}{m}} \mathcal{E}_{H,n}^{(a)}\left(x,\frac{v}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

From (5), we arrive at the desired result.

Let m be even.

From (11), (14) and (15), we obtain

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = m^n \left(\frac{e^t+1}{e^{\frac{t}{m}}+1}\right)^a e^{\frac{vt^2}{2}} e^{-\frac{vt^2}{2m^2}} \left(\frac{2}{e^t+1}\right)^a e^{-\frac{vt^2}{2}} x^n.$$

From (13), we get

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = 2^a m^n \left(\frac{e^t-1}{e^{\frac{t}{m}}+1}\right)^a \frac{1}{t^a} \mathcal{B}_{H,n}^{(a)}\left(x,\frac{v}{m^2}\right).$$

After some calculations, we have

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = 2^a m^n \left(\sum_{k=0}^{m-1} (-1)^k e^{\frac{kt}{m}} \right)^a \frac{1}{t^a} \mathcal{B}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right).$$

By using Lemma 1.3, we obtain

$$\mathcal{E}_{H,n}^{(a)}(mx,v) = 2^a m^n \sum_{u_1,\ldots,u_{m-1} \geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r e^{\frac{rt}{m}} \frac{1}{t^a} \mathcal{B}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

From (5) and (10), we arrive at the desired result. □

Theorem 3.3 *Let $m \in \mathbb{N}$. The following multiplication formula of the $\mathcal{G}_{H,n}^{(a)}(x,v)$ polynomials holds true*:

$$\mathcal{G}_{H,n}^{(a)}(mx,v) = m^{n-a} \sum_{u_1,\ldots,u_{m-1} \geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r \mathcal{G}_{H,n}^{(a)}\left(x + \frac{r}{m}, \frac{v}{m^2}\right)$$

when m is odd,

$$\mathcal{G}_{H,n}^{(a)}(mx,v) = m^n 2^a \sum_{u_1,\ldots,u_{m-1} \geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r \mathcal{B}_{H,n}^{(a)}\left(x + \frac{r}{m}, \frac{v}{m^2}\right)$$

when m is even, where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

Proof Let m be odd.

From (11), (16) and (17), we obtain

$$\mathcal{G}_{H,n}^{(a)}(mx,v) = m^{n-a} \left(\frac{e^t + 1}{e^{\frac{t}{m}} + 1} \right)^a \mathcal{G}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right).$$

After some calculations, we get

$$\mathcal{G}_{H,n}^{(a)}(mx,v) = m^{n-a} \left(\sum_{k=0}^{m-1} (-1)^k e^{\frac{kt}{m}} \right)^a \mathcal{G}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right).$$

By using Lemma 1.3, we obtain

$$\mathcal{G}_{H,n}^{(a)}(mx,v) = m^{n-a} \sum_{u_1,\ldots,u_{m-1} \geq 0} \binom{a}{u_1,\ldots,u_{m-1}} (-1)^r e^{\frac{rt}{m}} \mathcal{G}_{H,n}^{(a)}\left(x, \frac{v}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

From (5), we arrive at the desired result.

Let m be even.

From (11), (16) and (17), we obtain

$$\mathcal{G}_{H,n}^{(a)}(mx,\nu) = m^{n-a}\left(\frac{e^t+1}{e^{\frac{t}{m}}+1}\right)^a e^{\frac{\nu t^2}{2}} e^{-\frac{\nu t^2}{2m^2}} \left(\frac{2t}{e^t+1}\right)^a e^{-\frac{\nu t^2}{2}} x^n.$$

Using (13), we get

$$\mathcal{G}_{H,n}^{(a)}(mx,\nu) = 2^a m^{n-a}\left(\frac{e^t-1}{e^{\frac{t}{m}}+1}\right)^a \mathcal{B}_{H,n}^{(a)}\left(x,\frac{\nu}{m^2}\right).$$

After some calculations, we have

$$\mathcal{G}_{H,n}^{(a)}(mx,\nu) = 2^a m^{n-a}\left(\sum_{k=0}^{m-1}(-1)^k e^{\frac{kt}{m}}\right)^a \mathcal{B}_{H,n}^{(a)}\left(x,\frac{\nu}{m^2}\right).$$

By using Lemma 1.3, we obtain

$$\mathcal{G}_{H,n}^{(a)}(mx,\nu) = 2^a m^{n-a}\sum_{u_1,\dots,u_{m-1}\geq 0}\binom{a}{u_1,\dots,u_{m-1}}(-1)^r e^{\frac{rt}{m}} \mathcal{B}_{H,n}^{(a)}\left(x,\frac{\nu}{m^2}\right),$$

where

$$r = u_1 + 2u_2 + \cdots + (m-1)u_{m-1}.$$

From (5), we arrive at the desired result. □

Remark 3.4 By substituting $a = 1$ and $\nu = 0$ into Theorem 3.1, Theorem 3.2 and Theorem 3.3, one can obtain multiplication formulas for the Bernoulli, Euler and Genocchi polynomials (cf. [1, 2, 4–16]).

Remark 3.5 The proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3 are also given by the generating functions method and may be other.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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References

1. Carlitz, L: A note on the multiplication formulas for the Bernoulli and Euler polynomials. Proc. Am. Math. Soc. **4**, 184-188 (1953)
2. Carlitz, L: The multiplication formulas for the Bernoulli and Euler polynomials. Math. Mag. **27**, 59-64 (1953)
3. Bodin, A: Number of irreducible polynomials in several variables over finite fields. Am. Math. Mon. **115**, 653-660 (2008). arXiv:0706.0157v2 [math.AC]
4. Roman, S: The Umbral Calculus. Dover, New York (2005)
5. Dere, R, Simsek, Y: Bernoulli type polynomials on Umbral Algebra. arXiv:1110.1484v1 [math.CA]
6. Dere, R, Simsek, Y: Applications of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. **22**, 433-438 (2012)
7. Comtet, L: Advanced Combinatorics: the Art of Finite and Infinite Expansions (Translated from the French by J. M. Nienhuys). Reidel, Dordrecht (1974)
8. Dattoli, G, Migliorati, M, Srivastava, HM: Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials. Math. Comput. Model. **45**, 1033-1041 (2007)
9. Dere, R, Simsek, Y: Genocchi polynomials associated with the Umbral algebra. Appl. Math. Comput. **218**, 756-761 (2011)
10. Luo, Q-M: The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order. Integral Transforms Spec. Funct. **20**, 377-391 (2009)
11. Milne-Thomson, LM: Two classes of generalized polynomials. Proc. Lond. Math. Soc. **s2-35**, 514-522 (1933)
12. Simsek, Y: Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their applications. http://arxiv.org/pdf/1111.3848v2.pdf
13. Srivastava, HM, Kurt, B, Simsek, Y: Some families of Genocchi type polynomials and their interpolation functions. Integral Transforms Spec. Funct. (2012), *iFirst*, 1-20
14. Srivastava, HM: Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials. Appl. Math. Inf. Sci. **5**, 390-444 (2011)
15. Srivastava, HM, Kim, T, Simsek, Y: q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series. Russ. J. Math. Phys. **12**, 241-268 (2005)
16. Srivastava, HM, Choi, J: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)

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