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Existence results for fractional differential inclusions with three-point fractional integral boundary conditions

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Abstract

This paper is concerned with fractional differential inclusions with three-point fractional integral boundary conditions. We consider the fractional differential inclusions under both convexity and nonconvexity conditions on the multivalued term. Some new existence results are obtained by using standard fixed point theorems. Two examples are given to illustrate the main results.

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1 Introduction

Fractional differential equations have recently gained much importance and attention due to the fact that they have been proved to be valuable tools in the modeling of many physical phenomena [1–3]. For some recent developments on the existence results of fractional differential equations, we can refer, for instance, to [4–17] and the references therein.

Differential inclusions arise in the mathematical modeling of certain problems in economics, optimal control, *etc.* and are widely studied by many authors, see [18, 19] and the references therein. For some recent works on differential inclusions of fractional order, we refer the reader to the references [4, 5, 20–29].

Motivated by the above papers, in this article, we study a new class of fractional boundary value problems, *i.e.*, the following fractional differential inclusions with three-point fractional integral boundary conditions:

$$\begin{cases} {}^cD^\alpha x(t) \in F(t, x(t), {}^cD^\beta x(t)), & t \in [0,1], 1 < \alpha \leq 2, 0 < \beta < 1, \\ x(0) = 0, \qquad aI^\gamma x(\eta) + bx(1) = c, & 0 < \eta < 1, \end{cases} \tag{1}$$

where ${}^cD^p$ denotes the Caputo fractional derivative of order p, I^q the Riemann-Liouville fractional integral of order q, $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multifunction and a, b, c are real constants with $a\eta^{1+\gamma} \neq -b\Gamma(\gamma + 2)$.

We remark that when $b = -1$, $c = 0$ and third variable of the function F in (1) vanishes, problem (1) reduces to a three-point fractional integral boundary value problem (see [17] with $F = f$ a given continuous function).

The rest of this paper is organized as follows. In Section 2 we present the notations, definitions and give some preliminary results that we need in the sequel, Section 3 is dedicated to the existence results of problem (1), in the final Section 4, two examples are given to illustrate the main results.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts that will be used in the remainder of this paper.

Let $(X, \|\cdot\|)$ be a normed space. We use the notations: $P(X) = \{Y \subseteq X : Y \neq \emptyset\}$, $P_{cl}(X) = \{Y \in P(X) : Y \text{ closed}\}$, $P_b(X) = \{Y \in P(X) : Y \text{ bounded}\}$, $P_{cp}(X) = \{Y \in P(X) : Y \text{ compact}\}$, $P_{cp,c}(X) = \{Y \in P(X) : Y \text{ compact, convex}\}$ and so on.

Let $A, B \in P_{cl}(X)$, the Pompeiu-Hausdorff distance of A, B is defined as

$$h(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\}.$$

A multivalued map $F : X \to P(X)$ is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$. F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in P_b(X)$. F is called upper semicontinuous on X if, for every $x \in X$, the set $F(x)$ is a nonempty closed subset of X, and for every open set O of X containing $F(x)$, there exists an open neighborhood U of x such that $F(U) \subseteq O$. Equivalently, F is upper semicontinuous if the set $\{x \in X : F(x) \subseteq O\}$ is open for any open set O of X. F is called lower semicontinuous if the set $\{x \in X : F(x) \cap O \neq \emptyset\}$ is open for each open set O in X. If a multivalued map F is completely continuous with nonempty compact values, then F is upper semicontinuous if and only if F has a closed graph, *i.e.*, if $x_n \to x_*$ and $y_n \to y_*$, then $y_n \in F(x_n)$ implies $y_* \in F(x_*)$ [30].

A multivalued map $F : [0,1] \to P_{cl}(X)$ is said to be measurable if, for every $x \in X$, the function $t \to d(x, F(t)) = \inf\{d(x,y) : y \in F(t)\}$ is a measurable function.

Definition 2.1 A multivalued map $F : X \to P_{cl}(X)$ is called

(1) γ-Lipschitz if there exists $\gamma > 0$ such that

$$h\big(F(x), F(y)\big) \le \gamma d(x,y) \quad \text{for each } x, y \in X.$$

(2) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 2.2 A multivalued map $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is said to be Carathéodory if:

(1) $t \to F(t,x,y)$ is measurable for each $x, y \in \mathbb{R}$;

(2) $x \to F(t,x,y)$ is upper semicontinuous for a.e. $t \in [0,1]$.

Further, a Carathéodory function F is said to be L^1-Carathéodory if

(3) for each $k > 0$, there exists $\varphi_k \in L^k([0,1], \mathbb{R}^+)$ such that

$$\big\|F(t,x,y)\big\| = \sup\big\{|v| : v \in F(t,x,y)\big\} \le \varphi_k(t)$$

for all $|x| \le k$, $|y| \le k$ and a.e. $t \in [0,1]$.

The following lemmas will be used in the sequel.

Lemma 2.1 (see [31]) *Let X be a Banach space. Let $G : [0,1] \times X \to P_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and P be a linear continuous map from $L^1([0,1],X)$ to $C([0,1],X)$, then the operator*

$$P \circ S_G : C\big([0,1],X\big) \to P_{cp,c}\big(C\big([0,1],X\big)\big), \qquad y \mapsto (P \circ S_G)(y) = P(S_{G,y})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Here the set of selections

$$S_{F,x} = \big\{v \in L^1\big([0,1],\mathbb{R}\big) : v(t) \in F\big(t,x(t)\big) \text{ for a.e. } t \in [0,1]\big\}.$$

Definition 2.3 ([32]) The Riemann-Liouville fractional integral of order q for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds, \quad q > 0,$$

provided the integral exists.

Definition 2.4 ([32]) For at least n-times differentiable function f, the Caputo derivative of order q is defined as

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)\, ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.2 ([16]) *Let $\alpha > 0$, then the differential equation*

$${}^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and

$$I^{\alpha\, c}D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

here $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$.

Lemma 2.3 *For any $y \in C([0,1],\mathbb{R})$, the unique solution of the three-point boundary value problem*

$$\begin{cases} {}^cD^\alpha x(t) = y(t), & t \in [0,1], 1 < \alpha \le 2, \\ x(0) = 0, & aI^\gamma x(\eta) + bx(1) = c, \quad 0 < \eta < 1, \end{cases} \tag{2}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + \frac{t(c - b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds)}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} - \frac{ta\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} y(s)\, ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}.$$

Proof For $1 < \alpha \leq 2$ and some constants $c_0, c_1 \in \mathbb{R}$, the general solution of the equation ${}^cD^\alpha x(t) = y(t)$ can be written as

$$x(t) = I^\alpha y(t) + c_0 + c_1 t. \tag{3}$$

From $x(0) = 0$, it follows that $c_0 = 0$. Using the integral boundary conditions of (2), we obtain

$$\left(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b\right)c_1 + aI^{\alpha+\gamma}y(\eta) + b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds = c.$$

Therefore, we have

$$c_1 = \frac{c - b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds - a\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}y(s)\,ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}.$$

Substituting the values of c_0, c_1, we obtain the result. This completes the proof. □

Let us define what we mean by a solution of problem (1).

Definition 2.5 A function $x \in AC^1([0,1],\mathbb{R})$ is a solution of problem (1) if it satisfies the boundary conditions in (1) and there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t, x(t), {}^cD^\beta x(t))$ a.e. on $t \in [0,1]$ and

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds + \frac{t(c - b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\,ds)}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} - \frac{ta\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}f(s)\,ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}.$$

Let $C([0,1],\mathbb{R})$ be the space of all continuous functions defined on $[0,1]$. Define the space $\mathcal{X} = \{x : x \text{ and } {}^cD^\beta x \in C([0,1],\mathbb{R}), 0 < \beta < 1\}$ endowed with the norm $\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|{}^cD^\beta x(t)|$. Obviously, $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Theorem 2.1 (Nonlinear alternative of Leray-Schauder type) *Let X be a Banach space, C be a closed convex subset of X, U be an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,c}(C)$ is an upper semicontinuous compact map. Then either* (1) *F has a fixed point in $\overline{U}$, or* (2) *there are $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

Theorem 2.2 (Covitz and Nadler) *Let (X,d) be a complete metric space. If $F : X \to P_{cl}(X)$ is a contraction, then F has a fixed point.*

3 Existence results

In this section, three existence results of problem (1) are presented. The first one concerns the convex valued case, and the others are related to the nonconvex valued case.

Now let us begin with the convex valued case.

Theorem 3.1 *Suppose that the following* (H1), (H2) *and* (H3) *are satisfied.*

(H1) *$F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map.*

(H2) *There exist* $m \in L^\infty([0,1],\mathbb{R}^+)$ *and* $\varphi : [0,\infty) \to (0,\infty)$ *continuous, nondecreasing such that*

$$\|F(t,x,y)\| = \sup\{|v| : v \in F(t,x,y)\} \le m(t)\bigl(\varphi(|x|) + \varphi(|y|)\bigr)$$

for $x, y \in \mathbb{R}$, $t \in [0,1]$.

(H3) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{O + \|m\|(\varphi(M) + \psi(M))Q} > 1, \tag{4}$$

where

$$\|m\| = \sup_{t\in[0,1]} |m(t)|, \qquad O = \frac{|c|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\left(1 + \frac{1}{\Gamma(2-\beta)}\right),$$

$$Q = \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{(\Gamma(2-\beta)+1)(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha+1)})}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\right).$$

Then boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof Consider the multivalued operator $N : \mathcal{X} \to P(\mathcal{X})$ defined as

$$N(x) = \{h \in \mathcal{X} : h = Sv, v \in S_{F,x}\} \tag{5}$$

with

$$S_{F,x} = \bigl\{v \in L^1([0,1],\mathbb{R}) : v(t) \in F\bigl(t, x(t), {}^cD^\beta x(t)\bigr) \text{ for a.e. } t \in [0,1]\bigr\},$$

$$(Sv)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds + \frac{t(c - b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} v(s)\,ds)}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} - \frac{ta\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} v(s)\,ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}.$$

Clearly, by Lemma 2.3, we know that the fixed points of N are solutions of problem (1). From (H1) and (H2), we have, for each $x \in \mathcal{X}$, that the set $S_{F,x}$ is nonempty [31]. Next we will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof is given in the following five steps.

Step 1: $N(x)$ *is convex valued*. Since F is convex valued, we know that $S_{F,x}$ is convex and therefore it is obvious that for each $x \in \mathcal{X}$, $N(x)$ is convex.

Step 2: N *maps bounded sets into bounded sets in* $\mathcal{X}$. Let

$$B_r = \{x \in \mathcal{X} : \|x\| \le r\}$$

be a bounded subset of $\mathcal{X}$. We need to prove that there exists a constant $k > 0$ such that for each $x \in B_r$, one has $\|h\| \le k$ for each $h \in N(x)$. Let $x \in B_r$ and $h \in N(x)$, then there exists $v \in S_{F,x}$ such that

$$h(t) = (Sv)(t) \quad \text{for } t \in [0,1].$$

By simple calculations, we have

$$\begin{aligned}
|h(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|\,ds + \frac{(|c| + |b|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|v(s)|\,ds)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad + \frac{|a|\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}|v(s)|\,ds}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\leq \frac{\|m\|(\varphi(r)+\psi(r))}{\Gamma(\alpha+1)} + \frac{|c|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad + \frac{\|m\|(\varphi(r)+\psi(r))(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha+1)})}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&|{}^cD^\beta h(t)| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|v(s)|\,ds + \frac{|c_1|}{\Gamma(2-\beta)} \\
&\quad\leq \frac{\|m\|(\varphi(r)+\psi(r))}{\Gamma(\alpha-\beta+1)} + \frac{|c|}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\qquad + \frac{\|m\|(\varphi(r)+\psi(r))(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha+1)})}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|h(t)| &\leq \frac{|c|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\left(1 + \frac{1}{\Gamma(2-\beta)}\right) + \|m\|\bigl(\varphi(r)+\psi(r)\bigr) \\
&\quad \times \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{(\Gamma(2-\beta)+1)(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha+1)})}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\right).
\end{aligned}$$

Hence, we obtain

$$\|h\| \leq O + \|m\|\bigl(\varphi(r)+\psi(r)\bigr)Q = k.$$

Step 3: *N maps bounded sets into equicontinuous sets in $\mathcal{X}$*. Let B_r be as in Step 2 and $0 \leq t_1 < t_2 \leq 1$. Then, for each $x \in B_r$ and $h \in N(x)$, there exists $v \in S_{F,x}$ such that $h(t) = (Sv)(t)$ for $t \in [0,1]$. Since

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\quad\leq \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds\right| + \left|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds\right| \\
&\qquad + \frac{|(t_2-t_1)(c - b\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds)|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} + \frac{|(t_2-t_1)a\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}v(s)\,ds|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\|m\|(\varphi(r)+\psi(r))(t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{\|m\|(\varphi(r)+\psi(r))|t_2^\alpha-(t_2-t_1)^\alpha-t_1^\alpha|}{\Gamma(\alpha+1)} \\
&\quad + \frac{|c|(t_2-t_1)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|} + \frac{\|m\|(\varphi(r)+\psi(r))(t_2-t_1)(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+\frac{|b|}{\Gamma(\alpha+1)})}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|{}^cD^\beta h(t_2)-{}^cD^\beta h(t_1)\bigr| \\
&\le \left|I^{\alpha-\beta}v(t_2)-I^{\alpha-\beta}v(t_1)+\frac{c_1t_2^{1-\beta}}{\Gamma(2-\beta)}-\frac{c_1t_1^{1-\beta}}{\Gamma(2-\beta)}\right| \\
&\le \frac{\|m\|(\varphi(r)+\psi(r))(t_2^{\alpha-\beta}-t_1^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)} + \frac{|c|(t_2^{1-\beta}-t_1^{1-\beta})}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|} \\
&\quad + \frac{\|m\|(\varphi(r)+\psi(r))(t_2^{1-\beta}-t_1^{1-\beta})(\frac{|a|\eta^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+\frac{|b|}{\Gamma(\alpha+1)})}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|},
\end{aligned}$$

we deduce that

$$\bigl\|h(t_2)-h(t_1)\bigr\| \to 0 \quad \text{as } t_2 \to t_1$$

independently of $x \in B_r$ and $h \in N(x)$.

Step 4: *N has a closed graph*. Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$, we need to show that $h_* \in N(x_*)$. Since $h_n \in N(x_n)$, there exists $v_n \in S_{F,x_n}$ such that $h_n(t) = (Sv_n)(t)$ for $t \in [0,1]$. We must prove that there exists $v_* \in S_{F,x_*}$ such that $h_*(t) = (Sv_*)(t)$ for $t \in [0,1]$.

Now, let us consider the continuous linear operator $P : L^1([0,1],\mathbb{R}) \to \mathcal{X}$

$$v \to P(v)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds - \frac{bt\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}v(s)\,ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b} - \frac{at\int_0^\eta\frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}v(s)\,ds}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b},$$

and denote

$$w(t) = \frac{ct}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b}.$$

Clearly, we have $Sv = Pv + w$ and

$$P(v_n)(t) = h_n(t) - w(t) \to h_*(t) - w(t) \quad \text{in } \mathcal{X}.$$

By the definition of P, we have

$$h_n - w \in P \circ S_F(x_n).$$

It follows from Lemma 2.1 that $P \circ S_F$ is a closed graph operator. Since $x_n \to x_*$, we have

$$h_*(t) - w(t) = P(v_*)(t)$$

for some $v_* \in S_{F,x_*}$. This implies that $h_* \in N(x_*)$.

Step 5: *A priori bounds for solutions*. Let $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Then there exists $v \in S_{F,x}$ such that $x(t) = \lambda(Sv)(t)$ for $t \in [0,1]$. By a similar discussion as in Step 2, we have

$$\left|x(t)\right| + \left|{}^cD^\beta x(t)\right| \leq O + \|m\|\left(\varphi\left(\|x\|\right) + \psi\left(\|x\|\right)\right)Q \quad \text{for } t \in [0,1].$$

Thus

$$\|x\| \leq O + \|m\|\left(\varphi\left(\|x\|\right) + \psi\left(\|x\|\right)\right)Q \quad \text{for } t \in [0,1].$$

By the assumption of (H3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \left\{x \in \mathcal{X} : \|x\| < M\right\}.$$

As a consequence of Steps 1-4, together with the Arzela-Ascoli theorem, we can obtain that $N : \overline{U} \to P_{cp,c}(\mathcal{X})$ is an upper semicontinuous and completely continuous map. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda N(x)$ for some $\lambda \in (0,1)$. Hence, by Theorem 2.1, we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of problem (1). This is the end of the proof. □

Next we study the case when F is not necessarily convex valued.

Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\Sigma \otimes \mathcal{B}_{\mathbb{R}}$ measurable if A belongs to the σ-algebra generated by all sets of the form $J \times D$, where J is Lebesgue measurable in $[0,1]$ and D is a Borel set of $\mathbb{R}$. A subset A of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $J \subseteq [0,1]$ Lebesgue measurable, then $u\chi_J + v\chi_{[0,1]-J} \in A$, where χ stands for the characteristic function.

Theorem 3.2 *Let* (H2) *and* (H3) *hold and assume*:

(H4) $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that*: (1) $(t,x,y) \to F(t,x,y)$ *is* $\Sigma \otimes \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ *measurable*; (2) *the map* $(x,y) \to F(t,x,y)$ *is lower semicontinuous for a.e.* $t \in [0,1]$.

Then problem (1) *has at least one solution on* $[0,1]$.

Proof From (H2), (H4) and Lemma 4.4 of [27], the map

$$\mathcal{F} : \mathcal{X} \to P\left(L^1\left([0,1],\mathbb{R}\right)\right), \qquad x \to \mathcal{F}(x) = S_{F,x} \tag{6}$$

is lower semicontinuous and has nonempty closed and decomposable values. Then, from a selection theorem due to Bressan and Colombo [33], there exists a continuous function $f : \mathcal{X} \to L^1([0,1],\mathbb{R})$ such that for all $x \in \mathcal{X}$, $f(x)(t) \in F(t, x(t), {}^cD^\beta x(t))$ a.e. $t \in [0,1]$. Now consider the problem

$${}^cD^\alpha x(t) = f(x)(t), \quad t \in [0,1] \tag{7}$$

with the boundary conditions in (2). Note that if $x \in \mathcal{X}$ is a solution of problem (7), then x is a solution to problem (1).

Problem (7) is then reformulated as a fixed point problem for the operator $\bar{N} : \mathcal{X} \to \mathcal{X}$ defined by

$$\bar{N}(x)(t) = \left(Sf(x)\right)(t).$$

It can easily be shown that $\bar{N}$ is continuous and completely continuous and satisfies all conditions of the Leray-Schauder nonlinear alternative for single-valued maps [34]. By a discussion similar to the one in Theorem 3.1, Theorem 3.2 follows. □

Theorem 3.3 *We assume that*:

(H5) $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is such that*: (1) *the map* $t \to F(t,x,y)$ *is measurable for all* $x, y \in \mathbb{R}$; (2) *there exists* $m \in L^{\infty}([0,1],\mathbb{R}^+)$ *such that for a.e.* $t \in [0,1]$ *and all* $x_1, x_2, y_1, y_2 \in \mathbb{R}$,

$$h\bigl(F(t,x_1,y_1), F(t,x_2,y_2)\bigr) \le m(t)\bigl(|x_1 - x_2| + |y_1 - y_2|\bigr),$$

and

$$\frac{\|m\|}{\Gamma(\alpha+1)} + \frac{\|m\|}{\Gamma(\alpha-\beta+1)} + \frac{(\frac{\|m\||b|}{\Gamma(\alpha+1)} + \frac{\|m\||a|}{\Gamma(\alpha+\gamma+1)})(1+\Gamma(2-\beta))}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} < 1, \tag{8}$$

then problem (1) *has at least one solution on* $[0,1]$.

Proof From (H5), for each $x \in \mathcal{X}$, the multivalued map $t \to F(t, x(t), {}^cD^{\beta}x(t))$ is measurable and closed valued. Hence it has measurable selection (Theorem 2.2.1 [30]) and the set $S_{F,x}$ is nonempty. Let N be defined in (5). We will show that N satisfies the requirements of Theorem 2.2.

Step 1: For each $x \in \mathcal{X}$, $N(x) \in P_{cl}(\mathcal{X})$. Let $h_n \in N(x)$ be such that $h_n \to h$ in $\mathcal{X}$. Then $h \in \mathcal{X}$ and there exists $v_n \in S_{F,x}$ such that

$$h_n(t) = (Sv_n)(t), \quad t \in [0,1].$$

By (H5), the sequence v_n is integrable bounded. Since F has compact values, we may pass to a subsequence if necessary to get that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$h_n(t) \to h(t) = (Sv)(t).$$

This implies that $h \in N(x)$ and $N(x)$ is closed.

Step 2: There exists $0 < \lambda < 1$ such that

$$h\bigl(N(x), N(y)\bigr) \le \lambda \|x - y\| \quad \text{for all } x, y \in \mathcal{X}.$$

Let $x, y \in \mathcal{X}$ and $h_1 \in N(y)$, then there exists $v_1 \in S_{F,y}$ such that

$$h_1(t) = (Sv_1)(t), \quad t \in [0,1].$$

From (H5), we know that

$$h\bigl(F\bigl(t, x(t), {}^cD^{\beta}x(t)\bigr), F\bigl(t, y(t), {}^cD^{\beta}y(t)\bigr)\bigr) \le m(t)\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|{}^cD^{\beta}x(t) - {}^cD^{\beta}y(t)\bigr|\bigr).$$

Hence, for a.e. $t \in [0,1]$, there exists $u \in F(t, x(t), {}^cD^\beta x(t))$ such that

$$\bigl|v_1(t) - u\bigr| \le m(t)\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|{}^cD^\beta x(t) - {}^cD^\beta y(t)\bigr|\bigr). \tag{9}$$

Consider the multivalued map $V : [0,1] \to P(\mathbb{R})$ given by

$$V(t) = \bigl\{u \in \mathbb{R} : \bigl|v_1(t) - u\bigr| \le m(t)\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|{}^cD^\beta x(t) - {}^cD^\beta y(t)\bigr|\bigr)\bigr\}.$$

Since $v_1(t)$, $\alpha(t) = m(t)(|x(t) - y(t)| + |{}^cD^\beta x(t) - {}^cD^\beta y(t)|)$ are measurable, Theorem III.41 in [35] implies that V is measurable. It follows from (H5) that the map $t \to F(t, x(t))$ is measurable. Hence, by (9) and Proposition 2.1.43 in [30], the multivalued map $t \to V(t) \cap F(t, x(t), {}^cD^\beta x(t))$ with nonempty closed values is measurable. Therefore, we can find $v_2(t) \in F(t, x(t), {}^cD^\beta x(t))$ and

$$\bigl|v_1(t) - v_2(t)\bigr| \le m(t)\bigl(\bigl|x(t) - y(t)\bigr| + \bigl|{}^cD^\beta x(t) - {}^cD^\beta y(t)\bigr|\bigr) \quad \text{for a.e. } t \in [0,1].$$

Let $h_2(t) = (Sv_2)(t)$, *i.e.*, $h_2 \in N(x)$. Since

$$\begin{aligned}
&\bigl|h_1(t) - h_2(t)\bigr| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|v_1(s) - v_2(s)\bigr|\,ds + \frac{(|b|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|v_1(s) - v_2(s)|\,ds)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\qquad+ \frac{(|a|\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}|v_1(s) - v_2(s)|\,ds)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad\le \biggl(\frac{\|m\|}{\Gamma(\alpha+1)} + \frac{\|m\||b|}{\Gamma(\alpha+1)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} + \frac{\|m\||a|}{\Gamma(\alpha+\gamma+1)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\biggr)\|x - y\|
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|{}^cD^\beta h_1(t) - {}^cD^\beta h_2(t)\bigr| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|v_1(s) - v_2(s)\bigr|\,ds + \frac{(|b|\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|v_1(s) - v_2(s)|\,ds)}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\qquad+ \frac{(|a|\int_0^\eta \frac{(\eta-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}|v_1(s) - v_2(s)|\,ds)}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad\le \biggl(\frac{\|m\|}{\Gamma(\alpha-\beta+1)} + \frac{\frac{\|m\||b|}{\Gamma(\alpha+1)} + \frac{\|m\||a|}{\Gamma(\alpha+\gamma+1)}}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\biggr)\|x - y\|,
\end{aligned}$$

we obtain that

$$\begin{aligned}
&\bigl\|h_1(t) - h_2(t)\bigr\| \\
&\quad\le \biggl(\frac{\|m\|}{\Gamma(\alpha+1)} + \frac{\|m\|}{\Gamma(\alpha-\beta+1)} + \frac{\frac{\|m\||b|}{\Gamma(\alpha+1)}(1 + \Gamma(2-\beta))}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\qquad+ \frac{\frac{\|m\||a|}{\Gamma(\alpha+\gamma+1)}(1 + \Gamma(2-\beta))}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\biggr)\|x - y\|.
\end{aligned}$$

Define

$$\lambda = \frac{\|m\|}{\Gamma(\alpha+1)} + \frac{\|m\|}{\Gamma(\alpha-\beta+1)} + \frac{(\frac{\|m\||b|}{\Gamma(\alpha+1)} + \frac{\|m\||a|}{\Gamma(\alpha+\gamma+1)})(1+\Gamma(2-\beta))}{\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}.$$

By using an analogous relation obtained by interchanging the roles of x and y, we get

$$h\big(N(x), N(y)\big) \le \lambda\|x-y\|.$$

Therefore from condition (8), Theorem 2.2 implies that N has a fixed point which is a solution of problem (1). This completes the proof. □

4 Examples

In this section, we give two examples to illustrate the results.

Example 1 Consider the following three-point fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) \in F(t, x(t), {}^cD^{\frac{3}{4}}x(t)), & t \in [0,1], \\ x(0) = 0, \qquad I^{\frac{4}{3}}x(\frac{1}{2}) + \frac{1}{3}x(1) = -1, \end{cases} \tag{10}$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{3}{4}$, $\gamma = \frac{4}{3}$, $\eta = \frac{1}{2}$, $a = 1$, $b = \frac{1}{3}$, $c = -1$ and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map given by

$$F(t,x,y) = \left\{ v \in \mathbb{R} : e^{-|x|} - \frac{|y|}{1+y^2} + \sin t + t^2 \le v \le 4 + \frac{|x|}{1+x^2} + \sin y + 6t^3 \right\}.$$

In the context of this problem, we have

$$\big\|F(t,x,y)\big\| = \sup\big\{|v| : v \in F(t,x,y)\big\} \le 6 + 6t^2 \le 12 \quad \text{for } t \in [0,1], x, y \in \mathbb{R}.$$

It is clear that F is convex compact valued and is of Carathéodory type. Let $m(t) \equiv 1$ and $\varphi(|x|) \equiv 4$, $\psi(|y|) \equiv 8$, we get that for $t \in [0,1]$, $x, y \in \mathbb{R}$,

$$\big\|F(t,x,y)\big\| = \sup\big\{|v| : v \in F(t,x,y)\big\} \le m(t)\big(\varphi\big(|x|\big) + \varphi\big(|y|\big)\big).$$

As for condition (4), since $O + \|m\|(\varphi(|x|) + \psi(|y|))Q = O + 12Q$ (see O, Q in (H3)) is a constant, we can choose M large enough so that

$$\frac{M}{O + \|m\|(\varphi(|M|) + \psi(|M|))Q} > 1.$$

Thus, by the conclusion of Theorem 3.1, boundary value problem (10) has at least one solution on $[0,1]$.

Example 2 Consider the following three-point fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{7}{4}}x(t) \in F(t, x(t), {}^cD^{\frac{1}{2}}x(t)), & t \in [0,1], \\ x(0) = 0, \qquad I^{\frac{4}{3}}x(\frac{1}{3}) + \frac{2}{3}x(1) = -\frac{1}{2}, \end{cases} \tag{11}$$

where $\alpha = \frac{7}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{4}{3}$, $\eta = \frac{1}{3}$, $a = 1$, $b = \frac{2}{3}$, $c = -\frac{1}{2}$,

$$F(t,x,y) = \left[-l_1(t) - \frac{\sin x}{(4+t)^2} - 3, -\frac{1}{10}\right] \cup \left[0, \frac{|y|}{16(1+|y|)} + l_2(t)\right],$$

and $l_1, l_2 \in L^1([0,1], \mathbb{R}^+)$.

From the data given above, we have

$$\sup\bigl\{|v| : v \in F(t,x,y)\bigr\} \le 4 + \frac{1}{(4+t)^2} + l_1(t) + l_2(t), \quad t \in [0,1], x, y \in \mathbb{R},$$

$$h\bigl(F(t,x_1,y_1), F(t,x_2,y_2)\bigr) \le \frac{1}{16}\bigl(|x_1 - x_2| + |y_1 - y_2|\bigr).$$

Let $m = \frac{1}{16}$, we can get

$$\frac{1}{16}\left(\frac{1}{\Gamma(\frac{11}{4})} + \frac{1}{\Gamma(\frac{9}{4})} + \frac{\frac{\frac{2}{3}}{\Gamma(\frac{11}{4})}(1+\Gamma(\frac{3}{2}))}{\Gamma(\frac{3}{2})|\frac{\frac{1}{3}^{\frac{7}{3}}}{\Gamma(\frac{10}{3})} + \frac{2}{3}|} + \frac{\frac{1}{\Gamma(\frac{49}{12})}(1+\Gamma(\frac{3}{2}))}{\Gamma(\frac{3}{2})|\frac{\frac{1}{3}^{\frac{7}{3}}}{\Gamma(\frac{10}{3})} + \frac{2}{3}|}\right) \approx 0.2841 < 1.$$

Hence it follows from Theorem 3.3 that problem (11) has at least one solution on $[0,1]$.

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author carried out the proofs of the theorems and approved the final manuscript.

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