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Apostol-Euler polynomials arising from umbral calculus

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Abstract

In this paper, by using the orthogonality type as defined in the umbral calculus, we derive an explicit formula for several well-known polynomials as a linear combination of the Apostol-Euler polynomials.

MSC: 05A40

Keywords: Bernoulli polynomial; Bessel polynomial; Euler polynomial; Frobenius-Euler polynomial; umbral calculus

1 Introduction

Let Π_n be the set of all polynomials in a single variable x over the complex field $\mathbb{C}$ of degree at most n. Clearly, Π_n is a $(n+1)$-dimensional vector space over $\mathbb{C}$. Define

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\} \tag{1.1}$$

to be the algebra of formal power series in a single variable t. As is known, $\langle L|p(x)\rangle$ denotes the action of a linear functional $L \in \mathcal{H}$ on a polynomial $p(x)$, and we remind that the vector space on Π_n is defined by

$$\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$$

for any $c, c' \in \mathbb{C}$ and $L, L' \in \mathcal{H}$ (see [1–4]). The formal power series in variable t define a linear functional on Π_n by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \geq 0 \text{ (see [1–4])}. \tag{1.2}$$

By (1.1) and (1.2), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad \text{for all } n, k \geq 0 \text{ (see [1–4])}, \tag{1.3}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{k\geq 0}\langle L|x^k\rangle \frac{t^k}{k!}$ with $L \in \mathcal{H}$. From (1.3), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphic from Π_n onto $\mathcal{H}$. Henceforth, $\mathcal{H}$ is thought of as a set of both formal power series and linear functionals. We call $\mathcal{H}$ *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

Let $f(t) \in \mathcal{H}$. The smallest integer k for which the coefficient of t^k does not vanish is called the *order* of $f(t)$ and is denoted by $O(f(t))$ (see [1–4]). If $O(f(t)) = 1$, $O(f(t)) = 0$, then $f(t)$ is called a *delta*, an *invertible* series, respectively. For given two power series $f(t), g(t) \in \mathcal{H}$ such that $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $S_n(x)$ of polynomials with $\langle g(t)(f(t))^k | S_n(x)\rangle = n!\delta_{n,k}$ (this condition sometimes is called *orthogonality type*) for all $n, k \geq 0$. The sequence $S_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$ which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [1–4]).

For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have

$$\langle e^{yt} | p(x)\rangle = p(y), \qquad \langle f(t)g(t) | p(x)\rangle = \langle f(t) | g(t)p(x)\rangle, \tag{1.4}$$

and

$$f(t) = \sum_{k\geq 0} \langle f(t) | x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k\geq 0} \langle t^k | p(x)\rangle \frac{x^k}{k!} \tag{1.5}$$

(see [1–4]). From (1.5), we derive

$$\langle t^k | p(x)\rangle = p^{(k)}(0), \qquad \langle 1 | p^{(k)}(x)\rangle = p^{(k)}(0), \tag{1.6}$$

where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k\geq 0} S_k(y) \frac{t^k}{k!}, \tag{1.7}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [1–6]).

For $\lambda \in \mathbb{C}$ with $\lambda \neq -1$, the *Apostol-Euler polynomials* (see [7–10]) are defined by the generating function to be

$$\frac{2}{\lambda e^t + 1} e^{xt} = \sum_{k\geq 0} E_k(x|\lambda) \frac{t^k}{k!}. \tag{1.8}$$

In particular, $x = 0$, $E_n(0|\lambda) = E_n(\lambda)$ is called the *nth Apostol-Euler number*. From (1.8), we can derive

$$E_n(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(\lambda) x^k. \tag{1.9}$$

By (1.9), we have $\frac{d}{dx} E_n(x|\lambda) = n E_{n-1}(x|\lambda)$. Also, from (1.8) we have

$$\frac{2}{\lambda e^t + 1} = e^{E(\lambda)t} = \sum_{n\geq 0} E_n(\lambda) \frac{t^n}{n!} \tag{1.10}$$

with the usual convention about replacing $E^n(\lambda)$ by $E_n(\lambda)$. By (1.10), we get

$$2 = e^{E(\lambda)t}\left(\lambda e^t + 1\right) = \lambda e^{(E(\lambda)+1)t} + e^{E(\lambda)t} = \sum_{n\geq 0} \left(\lambda\left(E(\lambda) + 1\right)^n + E_n(\lambda)\right) \frac{t^n}{n!}.$$

Thus, by comparing the coefficients of the both sides, we have

$$\lambda\big(E(\lambda)+1\big)^n + E_n(\lambda) = 2\delta_{n,0}. \tag{1.11}$$

As is well known, the *Bernoulli polynomial* (see [11–14]) is also defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}. \tag{1.12}$$

In the special case, $x=0$, $B_n(0)=B_n$ is called the *nth Bernoulli number*. By (1.12), we get

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k. \tag{1.13}$$

From (1.12), we note that

$$\frac{t}{e^t-1} = e^{Bt} = \sum_{n\geq 0} B_n\frac{t^n}{n!} \tag{1.14}$$

with the usual convention about replacing B^n by B_n. By (1.13) and (1.14), we get

$$t = e^{Bt}\big(e^t-1\big) = e^{(B+1)t} - e^{Bt} = \sum_{n\geq 0}\big((B+1)^n - B_n\big)\frac{t^n}{n!},$$

which implies

$$B_n(1) - B_n = (B+1)^n - B_n = \delta_{n,1}, \qquad B_0 = 1. \tag{1.15}$$

Euler polynomials (see [4, 11, 13, 15]) are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{k\geq 0} E_k(x)\frac{t^k}{k!}. \tag{1.16}$$

In the special case, $x=0$, $E_n(0)=E_n$ is called the *nth Euler number*. By (1.16), we get

$$\frac{2}{e^t+1} = e^{Et} = \sum_{n\geq 0} E_n\frac{t^n}{n!} \tag{1.17}$$

with the usual convention about replacing E^n by E_n. By (1.16) and (1.17), we get

$$2 = e^{Et}\big(e^t+1\big) = e^{(E+1)t} + e^{Et} = \sum_{n\geq 0}\big((E+1)^n + E_n\big)\frac{t^n}{n!},$$

which implies

$$E_n(1) + E_n = (E+1)^n + E_n = 2\delta_{n,0}. \tag{1.18}$$

For $\lambda \in \mathbb{C}$ with $\lambda \neq -1$, the *Frobenius-Euler* (see [11, 16–19]) polynomials are defined by

$$\frac{1+\lambda}{e^t+\lambda}e^{xt} = \sum_{k\geq 0} F_k(x|-\lambda)\frac{t^k}{k!}. \tag{1.19}$$

In the special case, $x=0$, $F_n(0|-\lambda) = F_n(-\lambda)$ is called the *nth Frobenius-Euler number* (see [17]). By (1.19), we get

$$\frac{1+\lambda}{e^t+\lambda} = e^{Ft} = \sum_{n\geq 0} F_n(-\lambda)\frac{t^n}{n!} \tag{1.20}$$

with the usual convention about replacing $F^n(-\lambda)$ by $F_n(-\lambda)$ (see [17]). By (1.19) and (1.20), we get

$$1+\lambda = e^{F(-\lambda)t}\left(e^t+\lambda\right) = e^{(F(-\lambda)+1)t} + \lambda e^{F(-\lambda)t} = \sum_{n\geq 0}\left(\left(F(-\lambda)+1\right)^n + \lambda F_n(-\lambda)\right)\frac{t^n}{n!},$$

which implies

$$\lambda F_n(-\lambda) + F_n(1|-\lambda) = \lambda F_n(-\lambda) + \left(F(-\lambda)+1\right)^n = (1+\lambda)\delta_{n,0}. \tag{1.21}$$

In the next section, we present our main theorem and its applications. More precisely, by using the orthogonality type, we write any polynomial in Π_n as a linear combination of the Apostol-Euler polynomials. Several applications related to Bernoulli, Euler and Frobenius-Euler polynomials are derived.

2 Main results and applications

Note that the set of the polynomials $E_0(x|\lambda), E_1(x|\lambda), \dots, E_n(x|\lambda)$ is a good basis for Π_n. Thus, for $p(x) \in \Pi_n$, there exist constants $c_0, c_1, \dots, c_n$ such that $p(x) = \sum_{k=0}^n c_k E_k(x|\lambda)$. Since $E_n(x|\lambda) \sim ((1+\lambda e^t)/2, t)$ (see (1.7) and (1.8)), we have

$$\left\langle \frac{1+\lambda e^t}{2}t^k \middle| E_n(x|\lambda)\right\rangle = n!\delta_{n,k},$$

which gives

$$\left\langle \frac{1+\lambda e^t}{2}t^k \middle| p(x)\right\rangle = \sum_{\ell=0}^n c_\ell \left\langle \frac{1+\lambda e^t}{2}t^k \middle| E_\ell(x|\lambda)\right\rangle = \sum_{\ell=0}^n c_\ell \ell!\delta_{\ell,k} = k!c_k.$$

Hence, we can state the following result.

Theorem 2.1 *For all $p(x) \in \Pi_n$, there exist constants $c_0, c_1, \dots, c_n$ such that $p(x) = \sum_{k=0}^n c_k E_k(x|\lambda)$, where*

$$c_k = \frac{1}{2k!}\left\langle\left(1+\lambda e^t\right)t^k | p(x)\right\rangle.$$

Now, we present several applications for our theorem. As a first application, let us take $p(x) = x^n$ with $n \geq 0$. By Theorem 2.1, we have $x^n = \sum_{k=0}^{n} c_k E_k(x|\lambda)$, where

$$c_k = \frac{1}{2k!}\langle (1+\lambda e^t)t^k | x^n\rangle = \frac{1}{2}\binom{n}{k}\langle 1+\lambda e^t | x^{n-k}\rangle = \frac{1}{2}\binom{n}{k}(\delta_{n-k,0}+\lambda),$$

which implies the following identity.

Corollary 2.2 *For all* $n \geq 0$,

$$x^n = \frac{1}{2}E_n(x|\lambda) + \frac{\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}E_k(x|\lambda).$$

Let $p(x) = B_n(x) \in \Pi_n$, then by Theorem 2.1 we have that $B_n(x) = \sum_{k=0}^{n} c_k E_k(x|\lambda)$, where

$$\begin{aligned} c_k &= \frac{1}{2k!}\langle (1+\lambda e^t)t^k | B_n(x)\rangle = \frac{1}{2}\binom{n}{k}\langle 1+\lambda e^t | B_{n-k}(x)\rangle \\ &= \frac{1}{2}\binom{n}{k}\bigl(B_{n-k}+\lambda B_{n-k}(1)\bigr), \end{aligned}$$

which, by (1.15), implies the following identity.

Corollary 2.3 *For all* $n \geq 2$,

$$B_n(x) = \frac{(\lambda-1)n}{4}E_{n-1}(x|\lambda) + \frac{1+\lambda}{2}\sum_{k=0,k\neq n-1}^{n}\binom{n}{k}B_{n-k}E_k(x|\lambda).$$

Let $p(x) = E_n(x)$, then by Theorem 2.1 we have that $E_n(x) = \sum_{k=0}^{n} c_k E_k(x|\lambda)$, where

$$\begin{aligned} c_k &= \frac{1}{2k!}\langle (1+\lambda e^t)t^k | E_n(x)\rangle = \frac{1}{2}\binom{n}{k}\langle 1+\lambda e^t | E_{n-k}(x)\rangle \\ &= \frac{1}{2}\binom{n}{k}\bigl(E_{n-k}+\lambda E_{n-k}(1)\bigr), \end{aligned}$$

which, by (1.18), implies the following identity.

Corollary 2.4 *For all* $n \geq 0$,

$$E_n(x) = \frac{1+\lambda}{2}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}E_k(x|\lambda).$$

For another application, let $p(x) = F_n(x|-\lambda)$, then by Theorem 2.1 we have that $F_n(x|-\lambda) = \sum_{k=0}^{n} c_k E_k(x|\lambda)$, where

$$\begin{aligned} c_k &= \frac{1}{2k!}\langle (1+\lambda e^t)t^k | F_n(x|-\lambda)\rangle = \frac{1}{2}\binom{n}{k}\langle 1+\lambda e^t | F_{n-k}(x|-\lambda)\rangle \\ &= \frac{1}{2}\binom{n}{k}\bigl(F_{n-k}(-\lambda)+\lambda F_{n-k}(1|-\lambda)\bigr), \end{aligned}$$

which, by (1.21), implies the following identity.

Corollary 2.5 *For all $n \geq 1$,*

$$F_n(x|-\lambda) = \frac{1+\lambda}{2} E_n(x|\lambda) + \frac{1-\lambda^2}{2} \sum_{k=0}^{n-1} \binom{n}{k} F_{n-k}(-\lambda) E_k(x|\lambda).$$

Again, let $p(x) = y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \frac{x^k}{2^k}$ be the nth *Bessel polynomial* (which is the solution of the following differential equation $x^2 f''(x) + 2(x+1)f' + n(n+1)f = 0$, where $f'(x)$ denotes the derivative of $f(x)$, see [3, 4]). Then, by Theorem 2.1, we can write $y_n(x) = \sum_{k=0}^{n} c_k E_k(x|\lambda)$, where

$$\begin{aligned} c_k &= \frac{1}{2k!} \sum_{\ell=0}^{n} \frac{(n+\ell)!}{(n-\ell)!\ell!2^\ell} \langle 1 + \lambda e^t | t^k x^\ell \rangle \\ &= \frac{1}{2} \sum_{\ell=k}^{n} \frac{(n+\ell)!}{(n-\ell)!\ell!2^\ell} \binom{\ell}{k} \langle 1 + \lambda e^t | x^{\ell-k} \rangle \\ &= \frac{1}{2} \sum_{\ell=k}^{n} \frac{(n+\ell)!}{(n-\ell)!\ell!2^\ell} \binom{\ell}{k} (\delta_{n-k,0} + \lambda) \\ &= \frac{k!}{2^{k+1}} \binom{n}{k} \binom{n+k}{k} + \lambda \sum_{\ell=k}^{n} \frac{k!}{2^{\ell+1}} \binom{\ell}{k} \binom{n}{\ell} \binom{n+\ell}{\ell}, \end{aligned}$$

which implies the following identity.

Corollary 2.6 *For all $n \geq 1$,*

$$y_n(x) = \sum_{k=0}^{n} \frac{k!}{2^{k+1}} \binom{n}{k} \binom{n+k}{k} E_k(x|\lambda) + \lambda \sum_{k=0}^{n} \sum_{\ell=k}^{n} \frac{k!}{2^{\ell+1}} \binom{\ell}{k} \binom{n}{\ell} \binom{n+\ell}{\ell} E_k(x|\lambda).$$

We end by noting that if we substitute $\lambda = 0$ in any of our corollaries, then we get the well-known value of the polynomial $p(x)$. For instance, by setting $\lambda = 0$, the last corollary gives that $y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \frac{x^k}{2^k}$, as expected.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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