Mansour *Advances in Difference Equations* 2013, **2013**:29
http://www.advancesindifferenceequations.com/content/2013/1/29

Advances in Difference Equations
a SpringerOpen Journal

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Generalizations of fractional q-Leibniz formulae and applications

Zeinab SI Mansour*

*Correspondence:
zsmansour@ksu.edu.sa
Department of Mathematics,
Faculty of Science, King Saud
University, Riyadh, Saudi Arabia

Abstract

In this paper we generalize the fractional q-Leibniz formula introduced by Agarwal in (Ganita 27(1-2):25-32, 1976) for the Riemann-Liouville fractional q-derivative. This extension is a q-version of a fractional Leibniz formula introduced by Osler in (SIAM J. Appl. Math. 18(3):658-674, 1970). We also introduce a generalization of the fractional q-Leibniz formula introduced by Purohit for the Weyl fractional q-difference operator in (Kyungpook Math. J. 50(4):473-482, 2010). Applications are included.

1 q-notions and notations

Let q be a positive number, $0 < q < 1$. In the following, we follow the notations and notions of q-hypergeometric functions, the q-gamma function $\Gamma_q(x)$, Jackson q-exponential functions $E_q(x)$, and the q-shifted factorial as in [1, 2]. By a q-geometric set A, we mean a set that satisfies if $x \in A$, then $qx \in A$. Let f be a function defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \neq 0. \tag{1}$$

The nth q-derivative, $D_q^n f$, can be represented by its values at the points $\{q^j x, j = 0, 1, \ldots, n\}$ through the identity

$$D_q^n f(x) = (-1)^n (1-q)^{-n} x^{-n} q^{-n(n-1)/2} \sum_{r=0}^{n} (-1)^r \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-1)/2} f\left(xq^{n-r}\right) \tag{2}$$

for every x in $A \setminus \{0\}$. After some straightforward manipulations, formula (2) can be written as

$$D_q^n f(x) = (1-q)^{-n} x^{-n} \sum_{r=0}^{n} q^r \frac{(q^{-n}; q)_r}{(q; q)_r} f\left(xq^r\right) \quad \text{for } x \in A \setminus \{0\}. \tag{3}$$

Moreover, formula (2) can be inverted through the relation

$$f\left(xq^n\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q (1-q)^k x^k q^{\binom{k}{2}} D_q^k f(x). \tag{4}$$

Formulas (2) and (4) are well known and follow easily by induction. Jackson [3] introduced an integral denoted by

$$\int_a^b f(x)\,d_qx$$

as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t)\,d_qt := \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt, \quad a,b\in\mathbb{C}, \tag{5}$$

where

$$\int_0^x f(t)\,d_qt := (1-q)\sum_{n=0}^{\infty} xq^n f\bigl(xq^n\bigr), \quad x\in\mathbb{C}, \tag{6}$$

provided that the series at the right-hand side of (6) converges at $x = a$ and b. In [4], Hahn defined the q-integration for a function f over $[0,\infty)$ and $[x,\infty)$, $x > 0$, by

$$\begin{aligned}
&\int_0^\infty f(t)\,d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n f\bigl(q^n\bigr),\\
&\int_x^\infty f(t)\,d_qt = (1-q)\sum_{n=1}^{\infty} xq^{-n}(1-q)f\bigl(xq^{-n}\bigr),
\end{aligned} \tag{7}$$

respectively, provided that the series converges absolutely. Al-Salam [5] defined a fractional q-integral operator $K_q^{-\alpha}$ by

$$\begin{aligned}
&K_q^{-\alpha}\phi(x) := \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_q(\alpha)}\int_x^\infty t^{\alpha-1}(x/t;q)_{\alpha-1}\phi\bigl(tq^{1-\alpha}\bigr)\,d_qt,\\
&K_q^0\phi(x) := \phi(x),
\end{aligned} \tag{8}$$

where $\alpha \neq -1,-2,\ldots$, as a generalization of the q-Cauchy formula

$$\begin{aligned}
K_q^{-n}\phi(x) &= \int_x^\infty \int_{x_{n-1}}^\infty \cdots \int_{x_1}^\infty \phi(t)\,d_qt\,d_qx_1\cdots d_qx_{n-1}\\
&= \frac{q^{-\frac{1}{2}n(n-1)}}{\Gamma_q(n)}\int_x^\infty t^{n-1}(x/t;q)_{n-1}\phi\bigl(tq^{1-n}\bigr)\,d_qt,
\end{aligned}$$

which he introduced in [6] for a positive integer n. Using (7), we can write (8) explicitly as

$$K_q^{-\alpha}\phi(x) = q^{-\frac{\alpha(\alpha+1)}{2}}x^\alpha(1-q)^\alpha\sum_{k=0}^{\infty}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}-\alpha\\ k\end{bmatrix}_q \phi\bigl(xq^{-\alpha-k}\bigr), \tag{9}$$

or in a more simple form

$$K_q^{-\alpha}\phi(x) = q^{-\frac{\alpha(\alpha+1)}{2}}x^\alpha(1-q)^\alpha\sum_{k=0}^{\infty}q^{-k\alpha}\frac{(q^\alpha;q)_k}{(q;q)_k}\phi\bigl(xq^{-\alpha-k}\bigr). \tag{10}$$

Using (2), we can prove

$$K_q^n \phi(x) = (-1)^n D_q^n \phi(x) \quad (n \in \mathbb{N}). \tag{11}$$

This paper is organized as follows. In Section 2, we mention in brief some known fractional and q-fractional Leibniz formulae. In Section 3, we generalize the fractional q-Leibniz formula of the Riemann-Liouville fractional q-derivative introduced by Agarwal in [7]. Finally, in Section 4, we extend the fractional q-Leibniz formula introduced by Purohit [8] for the q-Weyl derivatives of nonnegative integral orders to any real order.

2 Fractional and q-fractional Leibniz formulas

The Riemann-Liouville fractional q-integral operator is introduced by Al-Salam in [5] and later by Agarwal in [9] and defined by

$$I_q^\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t)\, d_q t, \quad \alpha \notin \{-1,-2,\ldots\}. \tag{12}$$

Using (6), (12) reduces to

$$I_q^\alpha f(x) = x^\alpha (1-q)^\alpha \sum_{n=0}^{\infty} q^n \frac{(q^\alpha;q)_n}{(q;q)_n} f\left(xq^n\right), \tag{13}$$

which is valid for all α. The Riemann-Liouville fractional q-derivative of order α, $\alpha > 0$, is defined by

$$D_q^\alpha = D_q^k I_q^{k-\alpha}, \quad k = \lceil \alpha \rceil.$$

For the definition of Caputo fractional q-derivatives, see [10]. See also [11] for more applications. Liouville [12] introduced the fractional Leibniz rule

$$I^\alpha \left\{f(x)g(x)\right\} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} f^{(k)}(x) I^{\alpha+k} g(x), \tag{14}$$

where

$$I^\alpha \left\{f(x)\right\} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

is the familiar Riemann-Liouville integral operator. While Liouville used Fourier expansions in obtaining (14), Grünwald [13] and Letnikov [14] obtained (14) by a different technique. Other extensions and proofs are in the work of Watanabe [15], Post [16], Bassam [17], and Gaer-Rubel [18]. In a series of papers [19–23], Osler introduced several generalizations of (14). For example, in [19] Osler introduced the fractional Leibniz formula

$$D_{g(z)}^\alpha \left\{u(z)v(z)\right\} = \sum_{-\infty}^{\infty} \binom{\alpha}{\gamma+k} D_{g(z)}^{\alpha-\gamma-k} u(z) D_{g(z)}^{\gamma+k} v(z), \tag{15}$$

which coincides with (14) when we set $g(z) = z$, $\gamma = 0$ and replace α with $-\alpha$ in (15). For an extensive study of the fractional calculus and its applications in physics and control theory, see [24–28]. There are two q-analogues of the fractional Leibniz rule (14). Al-Salam and Verma [29] introduced the fractional Leibniz formula

$$I_q^{\alpha}(UV)(z) = \sum_{m=0}^{\infty} \begin{bmatrix} -\alpha \\ m \end{bmatrix}_q D_q^m U\bigl(zq^{-\alpha-m}\bigr) I_q^{\alpha+m} V(z), \tag{16}$$

formally. An analytic proof of (16) is introduced in [10] where the following theorem is introduced.

Theorem 2.1 *Let $U(z)$ be an entire function with q-exponential growth of order k, $k < \ln q^{-1}$, and a finite type δ, $\delta \in \mathbb{R}$. Let V be a function that satisfies*

$$\sum_{j=0}^{\infty} q^j \bigl|V\bigl(zq^j\bigr)\bigr| < \infty \quad (z \in \mathbb{C}).$$

Then (16) *holds for $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.*

For the definition of the q-exponential growth, see [30]. In [7], Agarwal introduced the following fractional q-Leibniz formula.

Theorem 2.2 *Let U and V be two analytic functions which have power series representations at $z = 0$ with radii of convergence R_1 and R_2, respectively, and $R = \min\{R_1, R_2\}$. Then*

$$I_q^{\alpha}(UV)(z) = \sum_{n=0}^{\infty} \begin{bmatrix} -\alpha \\ n \end{bmatrix}_q D_q^n U(z) I_q^{\alpha+n} V\bigl(zq^n\bigr) \quad \bigl(|z| < R\bigr), \tag{17}$$

Proof See [7]. □

Recently, Purohit [31] used (17) to derive a number of summation formulae for the generalized basic hypergeometric functions. In the following section, we introduce a generalization of Agarwal's fractional q-Leibniz formula (17). Let $0 < R < \infty$ and $D_R := \{z \in \mathbb{C} : |z| < R\}$. In the following, we say that a function $f \in L_q^1(D_R)$ if

$$\sum_{j=0}^{\infty} q^j \bigl|f\bigl(zq^j\bigr)\bigr| < \infty \quad \text{for all } z \in D_R \setminus \{0\}.$$

In [8], Purohit derived a q-extension of the Leibniz rule for q-derivative via the Weyl q-derivative operator defined in (8). He proved that for a nonnegative integer α,

$$K_q^{\alpha}(UV)(z) = \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha}; q)_r}{(q; q)_r} K_q^{\alpha-r} U(z) K_{q,z}^r V\bigl(zq^{r-\alpha}\bigr), \tag{18}$$

where $U(z) = z^{-p_1} u(z)$, $V(z) = z^{-p_2} v(z)$, u and v are analytic functions having a power series expansion at $z = 0$ with radii of convergence ρ, $\rho > 0$, and $p_1, p_2 \geq 0$. Purohit established some summation formulae as an application of the fractional Leibniz formula (18) which can be represented as

$$K_q^{\alpha}(UV)(z) = \sum_{r=0}^{\alpha} \frac{(q^{-\alpha}; q)_r}{(q; q)_r} K_q^{\alpha-r} U(z) D_{q^{-1},z}^r \bigl\{V\bigl(zq^{\alpha}\bigr)\bigr\}, \tag{19}$$

where we used

$$D^r_{q^{-1},z} V\big(zq^\alpha\big) = (-1)^r q^{r(r+1)/2} K^r_{q,z} V\big(zq^{r-\alpha}\big).$$

3 A generalization for Agarwal's fractional q-Leibniz formula

In this section we introduce a q-analogue of the fractional Leibniz formula (15) when $g(z) = z$. The case $\gamma = 0$ of the derived fractional q-Leibniz formula is the fractional q-Leibniz formula (17) introduced by Agarwal [7].

Theorem 3.1 *Let G be a branch domain of the logarithmic function. Let a, b be complex numbers and R be a positive number. Let u and v be analytic functions in the disk D_R. Let U and V be defined in $G \cap D_R$ through the relations*

$$U(z) = z^a u(z), \qquad V(z) = z^b v(z). \tag{20}$$

If $V(\cdot)$ and $UV(\cdot)$ are in $L^1_q(D_R)$, then

$$\begin{aligned} &I^\alpha_q UV(z) \\ &\quad = z^\gamma \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\gamma)} \sum_{m=0}^\infty \begin{bmatrix} -\alpha+\gamma \\ m \end{bmatrix}_q D^m_q\big(\big(q^{\alpha-\gamma}\xi/z;q\big)_\gamma U(\xi)\big)\big|_{\xi=z}\big(I^{\alpha-\gamma+m}_q\big) V\big(zq^m\big), \end{aligned} \tag{21}$$

where $z \in G \cap D_R$, and $\alpha, \gamma \in \mathbb{R}$.

Remark 3.2 It is worthwhile to notice that if we set $\gamma = 0$ in (21), we obtain Agarwal's fractional Leibniz rule (17) with less restrictive conditions on the functions $U(z)$ and $V(z)$. Actually, the special case $\gamma = 0$ of Theorem 3.1 is an extension of the result given by Manocha and Sharma in [32].

Proof Since V, UV are in $L^1_q(D_R)$, then

$$z^a V \in L^1_q(D_R), \qquad \mathrm{Re}(b) > -1 \quad \text{and} \quad \mathrm{Re}(a+b) > -1.$$

From (13) we obtain

$$I^\alpha_q(UV)(z) = z^\alpha (1-q)^\alpha \sum_{n=0}^\infty q^n \frac{(q^\alpha;q)_n}{(q;q)_n} U\big(zq^n\big) V\big(zq^n\big). \tag{22}$$

Substituting with

$$\frac{(q^\alpha;q)_n}{(q;q)_n} = \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n} \big(q^\alpha;q\big)_{-\gamma} \big(q^{\alpha-\gamma+n};q\big)_\gamma$$

into (22), we obtain

$$I^\alpha_q(UV)(z) = z^\alpha (1-q)^\alpha \big(q^\alpha;q\big)_{-\gamma} \sum_{n=0}^\infty q^n \frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n} \big(q^{\alpha-\gamma+n};q\big)_\gamma U\big(zq^n\big) V\big(zq^n\big). \tag{23}$$

The existence of UV in the space $L_q^1(D_R)$ guarantees that the series in (22) or in (23) converges absolutely for all $z \in D_R \setminus \{0\}$. Replace x in (4) with ξ and then let

$$f(\xi) = \left(q^{\alpha-\gamma}\xi/z;q\right)_\gamma U(\xi).$$

Consequently,

$$\begin{aligned}&\left(q^{\alpha-\gamma+n};q\right)_\gamma U\left(zq^n\right)\\ &\quad= \sum_{k=0}^{n}(-1)^k(1-q)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(k-1)/2} z^k D_q^k\left(\left(q^{\alpha-\gamma}\xi/z;q\right)_\gamma U(\xi)\right)\Big|_{\xi=z}.\end{aligned} \tag{24}$$

Then substituting (24) into (23), we get

$$\begin{aligned}I_q^\alpha(UV)(z) &= z^\alpha(1-q)^\alpha\left(q^\alpha;q\right)_{-\gamma}\sum_{n=0}^{\infty} q^n\frac{(q^{\alpha-\gamma};q)_n}{(q;q)_n}V\left(zq^n\right)\\ &\quad\times \sum_{k=0}^{n}(-1)^k(1-q)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(k-1)/2} z^k D_q^k\left(\left(q^{\alpha-\gamma}\xi/z;q\right)_\gamma U(\xi)\right)\Big|_{\xi=z}.\end{aligned} \tag{25}$$

Using (2), we obtain

$$\begin{aligned}D_q^k\left(\left(q^{\alpha-\gamma}\frac{\xi}{z};q\right)_\gamma U(\xi)\right)\Big|_{\xi=k} &= (-1)^k(1-q)^{-k}z^{a-k}q^{-\frac{k(k-1)}{2}}\left(q^{\alpha-\gamma};q\right)_\gamma\\ &\quad\times\sum_{r=0}^{k} q^{\frac{r(r-1)}{2}}\begin{bmatrix} k\\ r\end{bmatrix}_q \frac{(q^\alpha;q)_{k-r}}{(q^{\alpha-\gamma};q)_{k-r}} q^{(k-r)a}u\left(zq^{k-r}\right).\end{aligned}$$

Therefore, since $u(z)$ is analytic in D_R, there exists $M > 0$ such that

$$\begin{aligned}&\left|D_q^k\left(\left(q^{\alpha-\gamma}\frac{\xi}{z};q\right)_\gamma U(\xi)\right)\Big|_{\xi=k}\right|\\ &\quad\le (1-q)^{-k}|z|^{\mathrm{Re}(a)-k}q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}}M\sum_{r=0}^{k} q^{\frac{r(r-1)}{2}}\begin{bmatrix} k\\ r\end{bmatrix}_q q^{-r\,\mathrm{Re}\,a}\\ &\quad= M(1-q)^{-k}|z|^{\mathrm{Re}(a)-k}q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}}\left(-q^{-\mathrm{Re}(a)};q\right)_k\\ &\quad\le M(1-q)^{-k}|z|^{\mathrm{Re}(a)-k}q^{k\,\mathrm{Re}\,a-\frac{k(k-1)}{2}}\left(-q^{-\mathrm{Re}(a)};q\right)_\infty.\end{aligned} \tag{26}$$

Consequently,

$$\begin{aligned}&\sum_{k=0}^{n}(1-q)^k\begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(k-1)/2}\left|z^kD_q^k\left(\left(q^{\alpha-\gamma}\frac{\xi}{z};q\right)_\gamma U(\xi)\right)\Big|_{\xi=z}\right|\\ &\quad\le M|z|^{\mathrm{Re}(a)}\frac{(-q^{\mathrm{Re}(a)};q)_\infty}{(q;q)_\infty^2}\frac{1-q^{(n+1)\,\mathrm{Re}(a)}}{1-q^{\mathrm{Re}(a)}}.\end{aligned} \tag{27}$$

Set $F(\xi) := (q^{\alpha-\gamma}\frac{\xi}{z};q)_\gamma U(\xi)$. Then substituting (27) into (25), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{q^n|(q^{\alpha-\gamma};q)_n|}{(q;q)_n}\bigl|V(zq^n)\bigr|\sum_{k=0}^{n}(1-q)^k\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-1)/2}\bigl|z^kD_q^kF(z)\bigr|\\&\quad\leq M|z|^{\operatorname{Re}(a)}\frac{(-q^{\alpha-\gamma};q)_\infty(-q^{-\operatorname{Re}(a)};q)_\infty}{(q;q)_\infty^3}\sum_{n=0}^{\infty}\frac{1-q^{(n+1)\operatorname{Re}(a)}}{1-q^{\operatorname{Re}(a)}}q^n\bigl|V(zq^n)\bigr|.\end{aligned}\tag{28}$$

The last series converges for all $z\in D_R\setminus\{0\}$ since $V,z^aV\in L_q^1(D_R)$. Consequently, the series in (28) is absolutely convergent, and we can interchange the order of summations in (25). This leads to

$$\begin{aligned}&I_q^\alpha(UV)(z)\\&\quad=z^\alpha(1-q)^\alpha\bigl(q^\alpha;q\bigr)_{1-\gamma}\sum_{k=0}^{\infty}(-1)^kq^{k(k-1)/2}z^k\frac{(1-q)^k}{(q;q)_k}D_q^kF(z)\sum_{n=k}^{\infty}q^n\frac{(q^{\alpha-\gamma};q)_n}{(q;q)_{n-k}}V\bigl(zq^n\bigr)\\&\quad=z^\alpha(1-q)^\alpha\bigl(q^\alpha;q\bigr)_{-\gamma}\sum_{k=0}^{\infty}(-1)^kq^{k(k+1)/2}z^k\frac{(1-q)^k}{(q;q)_k}\\&\qquad\times D_q^kF(z)\sum_{j=0}^{\infty}q^j\frac{(q^{\alpha-\gamma};q)_{j+k}}{(q;q)_j}V\bigl(zq^{j+k}\bigr).\end{aligned}\tag{29}$$

Since

$$\bigl(I_q^{\alpha-\gamma+k}V\bigr)\bigl(zq^k\bigr)=\bigl(zq^k\bigr)^{\alpha-\gamma+k}(1-q)^{\alpha-\gamma+k}\sum_{j=0}^{\infty}q^j\frac{(q^{\alpha-\gamma+k};q)_j}{(q;q)_j}V\bigl(zq^{j+k}\bigr)$$

and

$$\bigl(q^{\alpha-\gamma};q\bigr)_{j+k}=\bigl(q^{\alpha-\gamma};q\bigr)_k\bigl(q^{\alpha-\gamma+k};q\bigr)_j,$$

the substitution with the last two identities in (29) gives

$$\begin{aligned}&I_q^\alpha(UV)(z)\\&\quad=z^\gamma(1-q)^\gamma\bigl(q^\alpha;q\bigr)_{-\gamma}\sum_{k=0}^{\infty}(-1)^kq^{-k(k-1)/2+k(-\alpha+\gamma)}\frac{(q^{\alpha-\gamma};q)_k}{(q;q)_k}\\&\qquad\times\bigl(I_q^{\alpha-\gamma+k}V\bigr)\bigl(zq^k\bigr)D_q^k\biggl(\Bigl(q^{\alpha-\gamma}\frac{\xi}{z};q\Bigr)_\gamma U(\xi)\biggr)\bigg|_{\xi=z}\\&\quad=z^\gamma(1-q)^\gamma\bigl(q^\alpha;q\bigr)_{-\gamma}\sum_{k=0}^{\infty}\begin{bmatrix}-\alpha+\gamma\\k\end{bmatrix}_q\bigl(I_q^{\alpha-\gamma+k}V\bigr)\bigl(zq^k\bigr)D_q^k\biggl(\Bigl(q^{\alpha-\gamma}\frac{\xi}{z};q\Bigr)_\gamma U(\xi)\biggr)\bigg|_{\xi=z},\end{aligned}$$

and the theorem follows. □

Example 3.3 Let γ, λ, μ, and α be complex numbers satisfying

$$\operatorname{Re}(\lambda)>0,\qquad\operatorname{Re}(\lambda+\mu)>0,\quad\mu\notin\mathbb{N}_0\quad\text{and}\quad\operatorname{Re}(\alpha)>0.$$

Then

$$\begin{aligned}&\frac{\Gamma_q(\lambda+\mu)\Gamma_q(\alpha)\Gamma_q(\lambda+\alpha-\gamma)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda)\Gamma_q(\lambda+\mu+\alpha)}\\&\quad=\sum_{m=0}^{\infty}q^{m(\lambda+\mu)}\frac{(q^{\alpha-\gamma},q^{-\mu};q)_m}{(q,q^{\lambda+\alpha-\gamma};q)_m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr).\end{aligned}\tag{30}$$

Proof We prove the identity by using Theorem 3.1. Take $U(z)=z^{\mu}$ and $v(z)=z^{\lambda-1}$. Then

$$\begin{aligned}&D_q^m\biggl\{\biggl(q^{\alpha-\gamma}\frac{\xi}{z};q\biggr)_{\gamma}\xi^{\mu}\biggr\}\\&\quad=D_q^m\sum_{k=0}^{\infty}\frac{(q^{-\gamma};q)_k}{(q;q)_k}\biggl(\frac{q^{\alpha}}{z}\biggr)^k\xi^{\mu+k}\\&\quad=\sum_{k=0}^{\infty}\frac{(q^{-\gamma};q)_k}{(q;q)_k}\biggl(\frac{q^{\alpha}}{z}\biggr)^k\frac{\Gamma_q(\mu+k+1)}{\Gamma_q(\mu+k-m+1)}\xi^{\mu+k-m}.\end{aligned}$$

Hence,

$$\begin{aligned}&D_q^m\biggl\{\biggl(q^{\alpha-\gamma}\frac{\xi}{z};q\biggr)_{\gamma}\xi^{\mu}\biggr\}\bigg|_{\xi=z}\\&\quad=z^{\mu-m}\frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-m+1)}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr)\\&\quad=(-1)^m\frac{(q^{-\mu};q)_m}{(1-q)^m}q^{m\mu-\binom{m}{2}}z^{\mu-m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr)\end{aligned}\tag{31}$$

and

$$\begin{aligned}\bigl(I_q^{\alpha-\gamma+m}V\bigr)\bigl(zq^m\bigr)&=\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma+m)}\bigl(zq^m\bigr)^{\lambda+\alpha-\gamma+m-1}\\&=\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma)}\frac{(1-q)^m}{(q^{\lambda+\alpha-\gamma};q)_m}\bigl(zq^m\bigr)^{\lambda+\alpha-\gamma+m-1}.\end{aligned}\tag{32}$$

Then applying Theorem 3.1 gives

$$\begin{aligned}I_q^{\alpha}(UV)(z)&=z^{\mu+\lambda+\alpha-1}\frac{\Gamma_q(\alpha)\Gamma_q(\lambda)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda+\alpha-\gamma)}\\&\qquad\times\sum_{m=0}^{\infty}q^{m(\lambda+\mu)}\frac{(q^{\alpha-\gamma},q^{-\mu};q)_m}{(q,q^{\lambda+\alpha-\gamma};q)_m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr).\end{aligned}\tag{33}$$

On the other hand,

$$I_q^{\alpha}z^{\lambda+\mu-1}=\frac{\Gamma_q(\lambda+\mu)}{\Gamma_q(\lambda+\mu+\alpha)}z^{\lambda+\mu+\alpha-1}.\tag{34}$$

Equating (33) and (34) gives (30). □

Example 3.4 For complex numbers a, b, A, B, d, and D such that $\mathrm{Re}(b) > -1$, $\mathrm{Re}(B) > 0$, and $\mathrm{Re}(b+B) > 1$,

$$\begin{aligned}
&{}_2\phi_1\bigl(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)}z\bigr)\\
&\quad= (z;q)_{-a}\frac{\Gamma_q(d+D)\Gamma_q(B)}{\Gamma_q(b+B)\Gamma_q(d+B-b)}(1-q)^{B-D}\\
&\qquad\times\sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}}q^{mB}\frac{(q^{b-d};q)_m}{(q;q)_m(q^{d+D-b};q)_m}\\
&\qquad\times{}_3\phi_2\bigl(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr){}_2\phi_1\bigl(q^A, q^B; q^{d+B-b+m}; q, q^{-a-A+m}\bigr),
\end{aligned}\tag{35}$$

for $|zq^{-a-A}| < 1$.

Proof The previous identity follows by taking

$$U(z) = z^b(z;q)_{-a}, \qquad V(z) = z^{B-1}\bigl(zq^{-a};q\bigr)_{-A},$$

and applying Theorem 3.1 with

$$\alpha = d + D - b - B, \qquad \gamma = D - B.$$

Then using (3), we obtain

$$\begin{aligned}
D_q^m\left(\left(q^{d-b}\frac{\xi}{z};q\right)_{D-B}\xi^b(\xi;q)_{-a}\right)\bigg|_{\xi=z} &= (1-q)^{-m}z^{-m+b}(z;q)_{-a}\frac{(q^{d-b};q)_\infty}{(q^{d-b+D-B};q)_\infty}\\
&\quad\times{}_3\phi_2\bigl(q^{-m}, q^{d-b+D-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr).
\end{aligned}\tag{36}$$

In addition,

$$\begin{aligned}
\bigl(I_q^{\alpha-\gamma+m}V\bigr)\bigl(zq^m\bigr) &= z^{d-b+\beta-1+m}q^{m^2+(d-b+\beta-1)m}\frac{\Gamma_q(\beta)}{\Gamma_q(B+m+d-b)}\\
&\quad\times{}_2\phi_1\bigl(q^A, q^\beta; q^{m-b}; q, q^{m-a-A}\bigr)
\end{aligned}\tag{37}$$

and

$$\begin{bmatrix}\gamma-\alpha\\ m\end{bmatrix}_q = (-1)^m q^{(\gamma-\alpha)m}q^{\binom{-m}{2}}\frac{(q^{\alpha-\gamma};q)_m}{(q;q)_m}.\tag{38}$$

Substituting with (36)-(38) into (21), we obtain

$$\begin{aligned}
I_q^\alpha UV(z) &= z^{d+D-1}(1-q)^{D-B}(z;q)_{-a}\frac{\Gamma_q(\beta)}{\Gamma_q(B+d-b)}\\
&\quad\times\sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}}q^{mB}\frac{(q^{b-d};q)_m}{(q;q)_m(q^{d+D-b};q)_m}\\
&\quad\times{}_3\phi_2\bigl(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr)\\
&\quad\times{}_2\phi_1\bigl(q^A, q^B; q^{d+B-b+m}; q, q^{-a-A+m}\bigr).
\end{aligned}\tag{39}$$

On the other hand,

$$I_q^\alpha UV(z) = \frac{\Gamma_q(b+B)}{\Gamma_q(d+D)} z^{d+D-1} {}_2\phi_1\bigl(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)}\bigr). \tag{40}$$

Combining (39) and (40), we obtain (35). □

4 A q-extension of the Leibniz rule via Weyl-type of a q-derivative operator

In this section, we prove that the q-expansion in (18) can be derived for any $\alpha \in \mathbb{R}$. The proof we introduce is completely different from the one introduced by Purohit for non-negative integer values of α. We start with characterizing a sufficient class of functions for which $K_q^{-\alpha}$ exists for some α.

Definition 4.1 Let $\alpha \in \mathbb{C}$ and let f be a function defined on a q^{-1}-geometric set A. We say that f is of class $S_{q,\alpha}$ if there exists $\mu \in \mathbb{C}$, $\operatorname{Re}\mu > \operatorname{Re}\alpha$ such that

$$f\bigl(xq^{-n}\bigr) = O\bigl(q^{n\mu}\bigr) \quad \text{as } n \to \infty, x \in A.$$

Proposition 4.2 *If $\alpha \in \mathbb{Z}$, then $K_q^{-\alpha} f$ exists for any function f defined on $(0,\infty)$. If $\alpha \notin \mathbb{Z}$ and $f \in S_{q,\alpha}$, then $K_q^{-\alpha} f$ exists.*

Proof If $\alpha \in \mathbb{Z}$, then by (11), $K_q^{-\alpha} f$ exists for any functions f defined on a $(0,\infty)$. If $\alpha \notin \mathbb{Z}$ and $f \in S_{q,\alpha}$, then for each $x > 0$, there exists a constant $C > 0$, C depends on x and α, such that

$$\bigl|f\bigl(xq^{-n-\alpha}\bigr)\bigr| \le Cq^{n\mu}.$$

Applying the previous inequality in (10) gives

$$\begin{aligned}\bigl|K_q^{-\alpha} f(x)\bigr| &\le Cq^{-\alpha(\alpha+1)/2}|x|^\alpha (1-q)^\alpha \frac{(-q^{\operatorname{Re}\alpha}; q)_\infty}{(q;q)_\infty} \sum_{k=0}^\infty q^{k(\operatorname{Re}\mu - \operatorname{Re}\alpha)} \\ &\le Cq^{-\alpha(\alpha+1)/2}|x|^\alpha (1-q)^\alpha \frac{(-q^{\operatorname{Re}\alpha}; q)_\infty}{(1-q^{(\operatorname{Re}\mu - \operatorname{Re}\alpha)})(q;q)_\infty}.\end{aligned}$$ □

In the following, we define a sufficient class of functions $\mathfrak{S}_{q,\mu}$ for which $K_q^{-\alpha} f$ exists for all α.

Definition 4.3 Let f be a function defined on a q^{-1}-geometric set A. We say that f is in the class $\mathfrak{S}_{q,\mu}$ if there exist $\mu > 0$ and $\nu \in \mathbb{R}$ such that for each $x \in A$,

$$\bigl|f\bigl(xq^{-n}\bigr)\bigr| = O\bigl(q^{\mu n(n+\nu)}\bigr) \quad \text{as } n \to \infty.$$

It is clear that if $f \in \mathfrak{S}_{q,\mu}$, then $f \in S_{q,\alpha}$ for all α. The spaces $S_{q,\alpha}$ and $\mathfrak{S}_{q,\mu}$ are q-analogues of the spaces of fairly good functions and good functions, respectively, introduced by Lighthill [33, p.15], see also [34, Chapter VII].

Example 4.4 An example of a function in a class $S_{q,1/2}$ is any function of the form

$$\frac{P_n(x)}{(ax;q)_\infty} \quad \text{for all } n \in \mathbb{N}_0,$$

where $P_n(x)$ is a polynomial of degree n and a is a constant such that $axq^k \neq 1$ for all $k \in \mathbb{N}_0$.

The keynotes in proving the generalization of Purohit q-fractional Leibniz formula are two identities. The first one is

$$K_q^\alpha z^{-p} = q^{\alpha(1-\alpha)/2} q^{-\alpha p} z^{-\alpha-p} \frac{\Gamma_q(\alpha+p)}{\Gamma_q(p)}, \tag{41}$$

which holds for any $p \in \mathbb{R}$ when $\alpha \in \mathbb{N}$ or holds when $\alpha + p > 0$. The proof of (41) follows from (10) by replacing α with $-\alpha$, x with z, and setting $\phi(z) = z^{-p}$. The second identity follows from the formula (4) with q replaced with q^{-1} and x with z. That is,

$$\begin{aligned} f\bigl(zq^{-n}\bigr) &= \sum_{k=0}^{n} \bigl(q^{-1}-1\bigr)^k q^{-\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} z^k D_{q^{-1}}^k f(z) \\ &= \sum_{k=0}^{n} q^{k(k-1)/2} (1-q)^k q^{-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k D_{q^{-1}}^k f(z), \end{aligned} \tag{42}$$

where we use [1, Eq. (I.47)]

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2-nk}.$$

The identity in (41) leads to the following result.

Lemma 4.5 *Let p and α be such that $0 < \operatorname{Re} p < 1$ and $\alpha > \operatorname{Re} p$. Let G be the principal branch of the logarithmic function and let $D_R := \{z \in \mathbb{C} : |z| > R\} \cap G$. Assume that*

$$U(z) = z^p \sum_{j=0}^{\infty} a_j z^{-j}$$

is analytic on D_R. Let

$$\Omega_\alpha = \bigl\{z \in D_R : q^\alpha z \in D_R\bigr\}.$$

Then $K_q^\alpha U(z)$ exists for all $z \in \Omega_\alpha$ and is equal to

$$K_q^\alpha U(z) = q^{\alpha p} \frac{\Gamma_q(\alpha-p)}{\Gamma_q(-p)} z^{p-\alpha} \sum_{j=0}^{\infty} a_j \frac{(q^{-p};q)_j}{(q^{\alpha-p};q)_j} q^{-\alpha j} z^{-j}. \tag{43}$$

Proof From (10) we find that

$$K_q^\alpha U(z) = q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{k=0}^{\infty} q^{k(\alpha-p)} \frac{(q^{-\alpha};q)_k}{(q;q)_k} \sum_{j=0}^{\infty} a_j q^{(-\alpha+k)j}. \tag{44}$$

From the assumptions of the present lemma, we can easily deduce that the double series in (44) is absolutely convergent for all $z \in \Omega_\alpha$. Hence, we can interchange the order of summations in (44). This and the q-binomials theorem [1, Eq. (1.3.2)] give

$$\begin{aligned} K_q^\alpha U(z) &= q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{j=0}^{\infty} a_j q^{-\alpha j} z^{-j} \sum_{k=0}^{\infty} q^{k\alpha - kp + kj} \frac{(q^{-\alpha};q)_j}{(q;q)_j} \\ &= q^{\alpha(1-\alpha)/2} q^{\alpha p} z^{p-\alpha} (1-q)^{-\alpha} \sum_{j=0}^{\infty} a_j q^{-\alpha j} z^{-j} \frac{(q^{-p+j};q)_\infty}{(q^{\alpha-p+j};q)_\infty}. \end{aligned}$$

Simple manipulations give (43). □

Lemma 4.6 *Let p, α, G, U, D_R, and Ω_α be as in Lemma* 4.5. *Then*

$$\sum_{j=0}^{\infty} q^{\alpha j} \bigl| U\bigl(zq^{\alpha-j}\bigr) \bigr| < \infty \quad \forall z \in \Omega_\alpha.$$

Proof The proof is easy and is omitted. □

Theorem 4.7 *Let U and V be functions defined on a q^{-1}-geometric set A and let $\alpha \in \mathbb{R}$. Assume that $UV \in S_{q,\alpha}$ and $U \in S_{q,\mu}$, $\mu > \frac{1}{2}$. Then*

$$K_q^\alpha UV(z) = \sum_{m=0}^{\infty} \frac{(q^{-\alpha};q)_m}{(q;q)_m} K_q^{\alpha-m} U(z) D_{q^{-1},z}^m \bigl\{ V\bigl(zq^\alpha\bigr) \bigr\} \tag{45}$$

for all $z \in A$ and for all $\alpha \in \mathbb{R}$. If $\mu = 1/2$, then (45) *may not hold for all α on $\mathbb{R}$ but only for α in a subdomain of $\mathbb{R}$.*

Proof Let $z \in q^{-\alpha}A$ be arbitrary but fixed. Since $UV \in S_{q,\alpha}$, then

$$\sum_{k=0}^{\infty} q^{k\alpha} \bigl| U\bigl(zq^{\alpha-k}\bigr) V\bigl(zq^{\alpha-k}\bigr) \bigr| < \infty. \tag{46}$$

From (10),

$$\begin{aligned} &\bigl(K_q^\alpha UV\bigr)(z) \\ &\quad = q^{\alpha(1-\alpha)/2} (1-q)^{-\alpha} z^{-\alpha} \sum_{m=0}^{\infty} q^{\alpha m} \frac{(q^{-\alpha};q)_m}{(q;q)_m} U\bigl(zq^{\alpha-m}\bigr) V\bigl(zq^{\alpha-m}\bigr). \end{aligned}$$

Applying (42) with $f(z) = V(zq^\alpha)$ yields

$$\begin{aligned} \bigl(K_q^\alpha UV\bigr)(z) &= q^{\alpha(1-\alpha)/2} (1-q)^{-\alpha} z^{-\alpha} \sum_{m=0}^{\infty} q^{\alpha m} \frac{(q^{-\alpha};q)_m}{(q;q)_m} U\bigl(zq^{\alpha-m}\bigr) \\ &\quad \times \sum_{j=0}^{m} q^{j(1-j)/2} (1-q)^j q^{-mj} \begin{bmatrix} m \\ j \end{bmatrix}_q z^j D_{q^{-1},z}^j \bigl\{ V\bigl(zq^\alpha\bigr) \bigr\}. \end{aligned} \tag{47}$$

From the assumptions on the function U, there exists a constant $C_1 > 0$ and $\nu \in \mathbb{R}$ such that

$$\left|U\left(zq^{\alpha-m}\right)\right| \le C_1 q^{\mu m(m+\nu)}.$$

Using (2) with (q^{-1} instead of q), we obtain

$$z^j D^j_{q^{-1},z}\left\{V\left(zq^{\alpha}\right)\right\} \le C_2.$$

Consequently, the double series on (47) is bounded from above by

$$\begin{aligned}
&C_1C_2\frac{(-q^{-\operatorname{Re}\alpha};q)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty} q^{\operatorname{Re}\alpha m} q^{\mu m(m+\nu)} \sum_{j=0}^{m} q^{j(1-j)/2}\begin{bmatrix} m \\ j\end{bmatrix}_q q^{-mj} \\
&\quad\le C_1C_2\frac{(-q^{-\operatorname{Re}\alpha};q)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty} q^{\operatorname{Re}\alpha m} q^{\mu m(m+\nu)}\left(-q^{-m};q\right)_m \\
&\quad= C_1C_2\frac{(-q^{-\operatorname{Re}\alpha};q)_\infty(-q;q)_\infty}{(q;q)_\infty}\sum_{m=0}^{\infty} q^{\operatorname{Re}\alpha m} q^{\mu m(m+\nu)} q^{-m(m+1)/2},
\end{aligned} \tag{48}$$

where we applied the identity *cf.*, *e.g.*, [1, p.11],

$$(a;q)_n := \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k\end{bmatrix}_q q^{\frac{k(k-1)}{2}} a^k. \tag{49}$$

Now, it is clear that if $\mu > 1/2$, then the series on the most right-hand side of (48) is convergent for all $\alpha \in \mathbb{C}$. On the other hand, it is convergent only for $\operatorname{Re}\alpha > -\nu + \frac{1}{2}$ when $\mu = \frac{1}{2}$. Therefore, we can interchange the order of summation in the series on the right-hand side of (47). This gives

$$\begin{aligned}
\left(K_q^{\alpha} UV\right)(z) &= q^{\alpha(1-\alpha)/2}(1-q)^{-\alpha} z^{-\alpha} \\
&\quad\times \sum_{j=0}^{\infty} \frac{(q^{-\alpha};q)_j}{(q;q)_j} z^j (1-q)^j D^j_{q^{-1},z} V\left(zq^{\alpha}\right) \\
&\quad\times \sum_{r=0}^{\infty} q^{(\alpha-j)r}\frac{(q^{-\alpha+j};q)_r}{(q;q)_r} U\left(zq^{\alpha-j-r}\right).
\end{aligned} \tag{50}$$

But

$$\sum_{r=0}^{\infty} q^{(\alpha-j)r}\frac{(q^{-\alpha+j};q)_r}{(q;q)_r} U\left(zq^{\alpha-j-r}\right) = z^{\alpha-j}(1-q)^{\alpha-j} q^{(\alpha-j)(\alpha-j-1)/2} K_q^{\alpha-j} V(z).$$

Combining this latter identity with (50) yields the theorem. □

Example 4.8 Let γ, λ, μ, and α be complex numbers satisfying

$$\operatorname{Re}(\lambda) > 0, \qquad \operatorname{Re}(\lambda+\mu) > 0, \quad \mu \notin \mathbb{N}_0 \quad \text{and} \quad \operatorname{Re}(\alpha) > 0.$$

Then

$$\begin{aligned}&\frac{\Gamma_q(\lambda+\mu)\Gamma_q(\alpha)\Gamma_q(\lambda+\alpha-\gamma)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda)\Gamma_q(\lambda+\mu+\alpha)}\\&\quad=\sum_{m=0}^{\infty}q^{m(\lambda+\mu)}\frac{(q^{\alpha-\gamma},q^{-\mu};q)_m}{(q,q^{\lambda+\alpha-\gamma};q)_m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr).\end{aligned}\tag{51}$$

Proof We prove the identity by using Theorem 3.1. Take $U(z)=z^{\mu}$ and $v(z)=z^{\lambda-1}$. Then

$$\begin{aligned}&D_q^m\left(q^{\alpha-\gamma}\frac{\xi}{z};q\right)_{\gamma}\xi^{\mu}\\&\quad=D_q^m\sum_{k=0}^{\infty}\frac{(q^{-\gamma};q)_k}{(q;q)_k}\left(\frac{q^{\alpha}}{z}\right)^k\xi^{\mu+k}\\&\quad=\sum_{k=0}^{\infty}\frac{(q^{-\gamma};q)_k}{(q;q)_k}\left(\frac{q^{\alpha}}{z}\right)^k\frac{\Gamma_q(\mu+k+1)}{\Gamma_q(\mu+k-m+1)}\xi^{\mu+k-m}.\end{aligned}$$

Hence,

$$\begin{aligned}&D_q^m\left(q^{\alpha-\gamma}\frac{\xi}{z};q\right)_{\gamma}\xi^{\mu}\Big|_{\xi=z}\\&\quad=z^{\mu-m}\frac{\Gamma_q(\mu+1)}{\Gamma_q(\mu-m+1)}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr)\\&\quad=(-1)^m\frac{(q^{-\mu};q)_m}{(1-q)^m}q^{m\mu-\binom{m}{2}}z^{\mu-m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr)\end{aligned}\tag{52}$$

and

$$\begin{aligned}\bigl(I_q^{\alpha-\gamma+m}V\bigr)\bigl(zq^m\bigr)&=\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma+m)}\bigl(zq^m\bigr)^{\lambda+\alpha-\gamma+m-1}\\&=\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\alpha-\gamma)}\frac{(1-q)^m}{(q^{\lambda+\alpha-\gamma};q)_m}\bigl(zq^m\bigr)^{\lambda+\alpha-\gamma+m-1}.\end{aligned}\tag{53}$$

Then applying Theorem 3.1 gives

$$\begin{aligned}I_q^{\alpha}(UV)(z)&=z^{\mu+\lambda+\alpha-1}\frac{\Gamma_q(\alpha)\Gamma_q(\lambda)}{\Gamma_q(\alpha-\gamma)\Gamma_q(\lambda+\alpha-\gamma)}\\&\quad\times\sum_{m=0}^{\infty}q^{m(\lambda+\mu)}\frac{(q^{\alpha-\gamma},q^{-\mu};q)_m}{(q,q^{\lambda+\alpha-\gamma};q)_m}{}_2\phi_1\bigl(q^{-\gamma},q^{\mu+1};q^{\mu-m+1};q,q^{\alpha}\bigr).\end{aligned}\tag{54}$$

On the other hand,

$$I_q^{\alpha}z^{\lambda+\mu-1}=\frac{\Gamma_q(\lambda+\mu)}{\Gamma_q(\lambda+\mu+\alpha)}z^{\lambda+\mu+\alpha-1}.\tag{55}$$

Equating (54) and (55) gives (51). □

Example 4.9 For complex numbers a, b, A, B, d, and D such that $\mathrm{Re}(b) > -1$, $\mathrm{Re}(B) > 0$, and $\mathrm{Re}(b+B) > 1$,

$$\begin{aligned}
&{}_2\phi_1\bigl(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)}z\bigr) \\
&\quad = (z;q)_{-a}\frac{\Gamma_q(d+D)\Gamma_q(B)}{\Gamma_q(b+B)\Gamma_q(d+B-b)}(1-q)^{B-D} \\
&\qquad \times \sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}} q^{mB}\frac{(q^{b-d};q)_m}{(q;q)_m(q^{d+D-b};q)_m} \\
&\qquad \times {}_3\phi_2\bigl(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr){}_2\phi_1\bigl(q^A, q^B; q^{d+B-b+m}; q, q^{-a-A+m}\bigr)
\end{aligned} \tag{56}$$

for $|zq^{-a-A}| < 1$.

Proof The previous identity follows by taking

$$U(z) = z^b(z;q)_{-a}, \qquad V(z) = z^{B-1}\bigl(zq^{-a};q\bigr)_{-A}$$

and applying Theorem 3.1 with

$$\alpha = d + D - b - B, \qquad \gamma = D - B.$$

Then using (3), we obtain

$$\begin{aligned}
D_q^m\left(\left(q^{d-b}\frac{\xi}{z};q\right)_{D-B}\xi^b(\xi;q)_{-a}\right)\Bigg|_{\xi=z} &= (1-q)^{-m}z^{-m+b}(z;q)_{-a}\frac{(q^{d-b};q)_\infty}{(q^{d-b+D-B};q)_\infty} \\
&\quad \times {}_3\phi_2\bigl(q^{-m}, q^{d-b+D-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr).
\end{aligned} \tag{57}$$

In addition,

$$\begin{aligned}
\bigl(I_q^{\alpha-\gamma+m}V\bigr)\bigl(zq^m\bigr) &= z^{d-b+\beta-1+m}q^{m^2+(d-b+\beta-1)m}\frac{\Gamma_q(\beta)}{\Gamma_q(B+m+d-b)} \\
&\quad \times {}_2\phi_1\bigl(q^A, q^\beta; q^{m-b}; q, q^{m-a-A}\bigr)
\end{aligned} \tag{58}$$

and

$$\begin{bmatrix} \gamma-\alpha \\ m \end{bmatrix}_q = (-1)^m q^{(\gamma-\alpha)m}q^{\binom{-m}{2}}\frac{(q^{\alpha-\gamma};q)_m}{(q;q)_m}. \tag{59}$$

Substituting with (57)-(59) into (21), we obtain

$$\begin{aligned}
I_q^\alpha UV(z) &= z^{d+D-1}(1-q)^{D-B}(z;q)_{-a}\frac{\Gamma_q(\beta)}{\Gamma_q(B+d-b)} \\
&\quad \times \sum_{m=0}^{\infty}(-1)^m q^{\binom{m}{2}}q^{mB}\frac{(q^{b-d};q)_m}{(q;q)_m(q^{d+D-b};q)_m} \\
&\quad \times {}_3\phi_2\bigl(q^{-m}, q^{d+D-b-B}, zq^{-a}; q^{d-b}, z; q, q^{b+1}\bigr) \\
&\quad \times {}_2\phi_1\bigl(q^A, q^B; q^{d+B-b+m}; q, q^{-a-A+m}\bigr).
\end{aligned} \tag{60}$$

On the other hand,

$$I_q^\alpha UV(z) = \frac{\Gamma_q(b+B)}{\Gamma_q(d+D)} z^{d+D-1} {}_2\phi_1\bigl(q^{a+A}, q^{b+B}; q^{d+D}; q, q^{-(a+A)}\bigr). \tag{61}$$

Combining (60) and (61), we obtain (56). □

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This research is supported by NPST Program of King Saud University; project number 10-MAT1293-02.

Received: 5 September 2012 Accepted: 24 January 2013 Published: 11 February 2013

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doi:10.1186/1687-1847-2013-29
Cite this article as: Mansour: **Generalizations of fractional *q*-Leibniz formulae and applications.** *Advances in Difference Equations* 2013 **2013**:29.