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Some identities related to Dedekind sums and the second-order linear recurrence polynomials

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Abstract

In this paper, we use the elementary method and the reciprocity theorem of Dedekind sums to study the computational problem of one kind Dedekind sums, and give two interesting computational formulae related to Dedekind sums and the second-order linear recurrence polynomials.

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1 Introduction

For any positive integer x, we define the generalized Lucas polynomial $L_n(x)$ as follows: $L_0(x) = 2$, $L_1(x) = x$, and $L_{n+1}(x) = xL_n(x) + L_{n-1}(x)$ for all $n \geq 1$.

It is clear that this polynomial is a second-order linear recurrence polynomial, it is satisfying the computational formula:

$$L_n(x) = \left(\frac{x+\sqrt{x^2+4}}{2}\right)^n + \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n.$$

$L_n(1) = L_n$ is the well-known Lucas sequence; $L_n(2) = Q_n$ is the Lucas-Pell sequence. About the properties of this sequence and related contents, some authors had studied them, and obtained many interesting results, see [1–3]. In this paper, we use the elementary method and the reciprocity theorem of Dedekind sums to study the computational problem of one kind Dedekind sums, and obtain some interesting identities related to Dedekind sums and the second-order linear recurrence polynomials. For convenience, we first give the definition of the Dedekind sums $S(h,q)$ as follows:

For a positive integer q and integer h with $(q,h) = 1$, the classical Dedekind sum $S(h,q)$ is defined by

$$S(h,q) = \sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

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This sum describes the behaviour of the logarithm of eta-function (see [4]) under modular transformations. About its other properties and applications, see [5–8]. For example, Carlitz [5] proved the reciprocity theorem

$$S(h,q)+S(q,h)=\frac{h^2+q^2+1}{12hq}-\frac{1}{4}, \tag{1}$$

where $h>0$, $q>0$ and $(h,q)=1$.

In this paper, we shall give an exact computational formula for $S(L_n(x),L_{n+1}(x))$. That is, we shall prove the following two theorems.

Theorem 1 *For any integers $n\geq 0$ and odd number $x\geq 1$, we have the computational formula*

$$S\bigl(L_{2n}(x),L_{2n+1}(x)\bigr)=\frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_{2n}(x)}{L_{2n+1}(x)}+\frac{x^3-6x^2+5x-24}{24(x^2+4)}.$$

Theorem 2 *For any integers $n\geq 0$ and odd number $x\geq 1$, we have the computational formula*

$$S\bigl(L_{2n-1}(x),L_{2n}(x)\bigr)=\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{L_{2n-1}(x)}{L_{2n}(x)}+\frac{x}{24}\cdot\frac{x^2+3}{x^2+4}.$$

From the theorems, we may immediately deduce the following two corollaries.

Corollary 1 *For any positive integer n, we have the identities*

$$S(L_{n-1},L_n)=\begin{cases}\frac{1}{10}\cdot\frac{L_{2m}}{L_{2m+1}}-\frac{1}{5}, & \textit{if } n=2m+1;\\ \frac{1}{15}\cdot\frac{L_{2m-1}}{L_{2m}}+\frac{1}{30}, & \textit{if } n=2m.\end{cases}$$

Corollary 2 *For any odd number $x\geq 1$, we have the limits*

$$\lim_{n\to\infty}S\bigl(L_{2n-1}(x),L_{2n}(x)\bigr)=\frac{1}{24}\cdot\frac{x^2+3}{\sqrt{x^2+4}}\quad \textit{and}$$
$$\lim_{n\to\infty}S\bigl(L_{2n}(x),L_{2n+1}(x)\bigr)=\frac{1}{24}\cdot\frac{x^2+5-6\cdot\sqrt{x^2+4}}{\sqrt{x^2+4}}.$$

In our theorems, x must be a positive odd number. If x is an even number, then $(L_n(x),L_{n+1}(x))=\cdots=(2,x)=2$. This time, the situation is more complex, it is very difficult for us to give an exact computational formula for $S(L_n(x),L_{n+1}(x))$.

2 Proof of the theorems

In this section, we shall prove our theorems directly. First, we prove Theorem 1. It is clear that for any positive integer n and odd number x, we have $(L_n(x),L_{n+1}(x))=(L_{n-1}(x),L_n(x))=\cdots=(L_0(x),L_1(x))=(2,x)=1$. So, by reciprocity theorem (1), we have

$$\begin{aligned}&S\bigl(L_{2n}(x),L_{2n+1}(x)\bigr)+S\bigl(L_{2n+1}(x),L_{2n}(x)\bigr)\\&\quad=\frac{1}{12}\left[\frac{L_{2n+1}(x)}{L_{2n}(x)}+\frac{L_{2n}(x)}{L_{2n+1}(x)}+\frac{1}{L_{2n+1}(x)L_{2n}(x)}\right]-\frac{1}{4}\end{aligned}$$

$$= \frac{1}{12}\left[\frac{xL_{2n}(x)+L_{2n-1}(x)}{L_{2n}(x)} + \frac{L_{2n}(x)}{L_{2n+1}(x)} + \frac{1}{L_{2n+1}(x)L_{2n}(x)}\right] - \frac{1}{4}$$

$$= \frac{1}{12}\left[\frac{L_{2n}(x)}{L_{2n+1}(x)} + \frac{L_{2n-1}(x)}{L_{2n}(x)} + \frac{1}{L_{2n+1}(x)L_{2n}(x)}\right] + \frac{x}{12} - \frac{1}{4}. \tag{2}$$

Similarly, we also have the identity

$$S\bigl(L_{2n-1}(x), L_{2n}(x)\bigr) + S\bigl(L_{2n}(x), L_{2n-1}(x)\bigr)$$

$$= \frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n}(x)} + \frac{L_{2n-2}(x)}{L_{2n-1}(x)} + \frac{1}{L_{2n}(x)L_{2n-1}(x)}\right] + \frac{x}{12} - \frac{1}{4}. \tag{3}$$

Note that $S(L_{2n}(x), L_{2n-1}(x)) = S(L_{2n-2}(x), L_{2n-1}(x))$ and $S(L_{2n+1}(x), L_{2n}(x)) = S(L_{2n-1}(x), L_{2n}(x))$, from identities (2) and (3), we have

$$S\bigl(L_{2n}(x), L_{2n+1}(x)\bigr) - S\bigl(L_{2n-2}(x), L_{2n-1}(x)\bigr)$$

$$= \frac{1}{12}\left[\frac{L_{2n}(x)}{L_{2n+1}(x)} - \frac{L_{2n-2}(x)}{L_{2n-1}(x)} + \frac{1}{L_{2n+1}(x)L_{2n}(x)} - \frac{1}{L_{2n}(x)L_{2n-1}(x)}\right]$$

$$= \frac{1}{12}\left[\frac{L_{2n}(x)}{L_{2n+1}(x)} - \frac{L_{2n-2}(x)}{L_{2n-1}(x)} - \frac{x}{L_{2n+1}(x)L_{2n-1}(x)}\right]$$

$$= \frac{1}{12}\left[\frac{L_{2n}(x)}{L_{2n+1}(x)} - \frac{L_{2n-2}(x)}{L_{2n-1}(x)} + \frac{x}{x^3+4x}\left(\frac{L_{2n}(x)}{L_{2n+1}(x)} - \frac{L_{2n-2}(x)}{L_{2n-1}(x)}\right)\right]$$

$$= \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\left[\frac{L_{2n}(x)}{L_{2n+1}(x)} - \frac{L_{2n-2}(x)}{L_{2n-1}(x)}\right]. \tag{4}$$

From (4), we may immediately deduce the recurrence formula

$$S\bigl(L_{2n}(x), L_{2n+1}(x)\bigr) - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_{2n}(x)}{L_{2n+1}(x)}$$

$$= S\bigl(L_{2n-2}(x), L_{2n-1}(x)\bigr) - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_{2n-2}(x)}{L_{2n-1}(x)}. \tag{5}$$

Using (5), repeatedly, and note that formula (1) and $S(x,2) = 0$, we have

$$S\bigl(L_{2n}(x), L_{2n+1}(x)\bigr) - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_{2n}(x)}{L_{2n+1}(x)}$$

$$= \cdots$$

$$= S\bigl(L_0(x), L_1(x)\bigr) - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_0(x)}{L_1(x)} = S(2,x) - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{2}{x}$$

$$= \frac{x^2+5}{24x} - \frac{1}{4} - \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{2}{x} = \frac{x^3-6x^2+5x-24}{24(x^2+4)}$$

or

$$S\bigl(L_{2n}(x), L_{2n+1}(x)\bigr) = \frac{1}{12}\cdot\frac{x^2+5}{x^2+4}\cdot\frac{L_{2n}(x)}{L_{2n+1}(x)} + \frac{x^3-6x^2+5x-24}{24(x^2+4)}.$$

This proves Theorem 1.

Now, we prove Theorem 2. From the method of proving (2), we have

$$\begin{aligned}
&S\big(L_{2n-1}(x),L_{2n}(x)\big)+S\big(L_{2n}(x),L_{2n-1}(x)\big)\\
&\quad=\frac{1}{12}\left[\frac{L_{2n}(x)}{L_{2n-1}(x)}+\frac{L_{2n-1}(x)}{L_{2n}(x)}+\frac{1}{L_{2n}(x)L_{2n-1}(x)}\right]-\frac{1}{4}\\
&\quad=\frac{1}{12}\left[\frac{xL_{2n-1}(x)+L_{2n-2}(x)}{L_{2n-1}(x)}+\frac{L_{2n-1}(x)}{L_{2n}(x)}+\frac{1}{L_{2n}(x)L_{2n-1}(x)}\right]-\frac{1}{4}\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n}(x)}+\frac{L_{2n-2}(x)}{L_{2n-1}(x)}+\frac{1}{L_{2n}(x)L_{2n-1}(x)}\right]+\frac{x}{12}-\frac{1}{4}
\end{aligned}\tag{6}$$

and

$$\begin{aligned}
&S\big(L_{2n-2}(x),L_{2n-1}(x)\big)+S\big(L_{2n-1}(x),L_{2n-2}(x)\big)\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n-2}(x)}+\frac{L_{2n-2}(x)}{L_{2n-1}(x)}+\frac{1}{L_{2n-1}(x)L_{2n-2}(x)}\right]-\frac{1}{4}\\
&\quad=\frac{1}{12}\left[\frac{xL_{2n-2}(x)+L_{2n-3}(x)}{L_{2n-2}(x)}+\frac{L_{2n-2}(x)}{L_{2n-1}(x)}+\frac{1}{L_{2n-1}(x)L_{2n-2}(x)}\right]-\frac{1}{4}\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-2}(x)}{L_{2n-1}(x)}+\frac{L_{2n-3}(x)}{L_{2n-2}(x)}+\frac{1}{L_{2n-1}(x)L_{2n-2}(x)}\right]+\frac{x}{12}-\frac{1}{4}.
\end{aligned}\tag{7}$$

Note that $S(L_{2n}(x),L_{2n-1}(x))=S(L_{2n-2}(x),L_{2n-1}(x))$, $S(x^2+2,x)=S(2,x)$, $S(x,2)=0$, from (1), (6) and (7), we have

$$\begin{aligned}
&S\big(L_{2n-1}(x),L_{2n}(x)\big)-S\big(L_{2n-1}(x),L_{2n-2}(x)\big)\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n}(x)}-\frac{L_{2n-3}(x)}{L_{2n-2}(x)}+\frac{1}{L_{2n}(x)L_{2n-1}(x)}-\frac{1}{L_{2n-1}(x)L_{2n-2}(x)}\right]\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n}(x)}-\frac{L_{2n-3}(x)}{L_{2n-2}(x)}-\frac{x}{L_{2n}(x)L_{2n-2}(x)}\right]\\
&\quad=\frac{1}{12}\left[\frac{L_{2n-1}(x)}{L_{2n}(x)}-\frac{L_{2n-3}(x)}{L_{2n-2}(x)}-\frac{1}{x^2+4}\left(\frac{L_{2n-1}(x)}{L_{2n}(x)}-\frac{L_{2n-3}(x)}{L_{2n-2}(x)}\right)\right]\\
&\quad=\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\left(\frac{L_{2n-1}(x)}{L_{2n}(x)}-\frac{L_{2n-3}(x)}{L_{2n-2}(x)}\right)
\end{aligned}$$

or

$$\begin{aligned}
&S\big(L_{2n-1}(x),L_{2n}(x)\big)-\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{L_{2n-1}(x)}{L_{2n}(x)}\\
&\quad=S\big(L_{2n-3}(x),L_{2n-2}(x)\big)-\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{L_{2n-3}(x)}{L_{2n-2}(x)}\\
&\quad=\cdots\\
&\quad=S\big(x,x^2+2\big)-\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{x}{x^2+2}\\
&\quad=S\big(x,x^2+2\big)+S\big(x^2+2,x\big)-S(2,x)-S(x,2)-\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{x}{x^2+2}\\
&\quad=\frac{x^2+(x^2+2)^2+1}{12x(x^2+2)}-\frac{x^2+5}{24x}-\frac{1}{12}\cdot\frac{x^2+3}{x^2+4}\cdot\frac{x}{x^2+2}=\frac{x}{24}\cdot\frac{x^2+3}{x^2+4}
\end{aligned}$$

or

$$S\big(L_{2n-1}(x), L_{2n}(x)\big) = \frac{1}{12} \cdot \frac{x^2+3}{x^2+4} \cdot \frac{L_{2n-1}(x)}{L_{2n}(x)} + \frac{x}{24} \cdot \frac{x^2+3}{x^2+4}.$$

This completes the proof of Theorem 2.

Note that the definition of $L_n(x)$, $x > 1$, and the limit

$$\lim_{n\to+\infty} \frac{L_{2n}(x)}{L_{2n+1}(x)} = \lim_{n\to+\infty} \frac{L_{2n-1}(x)}{L_{2n}(x)} = \frac{2}{x+\sqrt{x^2+4}} = \frac{-x+\sqrt{x^2+4}}{2},$$

from our theorems, we may immediately deduce Corollary 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JL carried out the exact computational formula for $S(L_n(x), L_{n+1}(x))$, $S(L_{2n-1}(x), L_{2n}(x))$. HZ participated in the research and summary of the study. All authors read and approved the final manuscript.

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