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Generalized difference strongly summable sequence spaces of fuzzy real numbers defined by ideal convergence and Orlicz function

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Abstract

We study some new generalized difference strongly summable sequence spaces of fuzzy real numbers using ideal convergence and an Orlicz function in connection with de la Vallèe Poussin mean. We give some relations related to these sequence spaces also.

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Keywords: Orlicz function; difference operator; ideal convergence; de la Vallèe Poussin mean

1 Introduction

Let ℓ_∞, c and c_0 be the Banach space of bounded, convergent and null sequences $x = (x_k)$, respectively, with the usual norm $\|x\| = \sup_n |x_n|$.

A sequence $x \in \ell_\infty$ is said to be almost convergent if all of its Banach limits coincide.

Let $\hat{c}$ denote the space of all almost convergent sequences.

Lorentz [1] proved that,

$$\hat{c} = \Big\{x \in \ell_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\Big\},$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + \cdots + x_{m+n}}{m+1}.$$

The following space of strongly almost convergent sequence was introduced by Maddox [2]:

$$[\hat{c}] = \Big\{x \in \ell_\infty : \lim_m t_{m,n}(|x - Le|) \text{ exists uniformly in } n \text{ for some } L\Big\},$$

where, $e = (1, 1, \ldots)$.

Let σ be a one-to-one mapping from the set of positive integers into itself such that $\sigma^m(n) = \sigma^{m-1}(\sigma(n))$, $m = 1, 2, 3, \ldots$, where $\sigma^m(n)$ denotes the mth iterative of the mapping σ in n, see [3].

Schaefer [3] proved that

$$V_\sigma = \left\{x \in \ell_\infty : \lim_k t_{km}(x) = L \text{ uniformly in } m \text{ for some } L = \sigma - \lim x\right\},$$

where,

$$t_{km}(x) = \frac{x + x_{\sigma(m)} + \cdots + x_{\sigma^k(m)}}{k+1}, \quad t_{-1,m} = 0.$$

Thus, we say that a bounded sequence $x = (x_k)$ is σ-convergent if and only if $x \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$.

A sequence $x = (x_k)$ is said to be strongly σ-convergent (Mursaleen [4]) if there exists a number ℓ such that

$$\frac{1}{k}\sum_{i=1}^{k} |x_{\sigma^i(m)} - \ell| \to 0, \quad \text{as } k \to \infty \text{ uniformly in } m. \tag{1}$$

We write $[V_\sigma]$ to denote the set of all strong σ-convergent sequences, and when (1) holds, we write $[V_\sigma] - \lim x = \ell$.

Taking $\sigma(m) = m + 1$, we obtain $[V_\sigma] = [\hat{c}]$. Then the strong σ-convergence generalizes the concept of strong almost convergence.

We also note that

$$[V_\sigma] \subset V_\sigma \subset \ell_\infty.$$

The notion of ideal convergence was first introduced by Kostyrko *et al.* [5] as a generalization of statistical convergence, which was later studied by many other authors.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space,

$$\ell_M = \left\{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space, which is called an Orlicz sequence space.

Kizmaz [7] studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ of crisp sets. The notion is defined as follows:

$$Z(\Delta) = \left\{x = (x_k) : (\Delta x_k) \in Z\right\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in N$.

The spaces above are Banach spaces, normed by

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k|.$$

The generalized difference is defined as follows:
For $m \geq 1$ and $n \geq 1$,

$$Z(\Delta_m^n) = \{x = (x_k) : (\Delta_m^n x_k) \in Z\}$$

for $Z = \ell_\infty, c$ and c_0.

This generalized difference has the following binomial representation:

$$\Delta_m^n x_k = \sum_{r=0}^{n} (-1)^r \binom{n}{r} x_{k+rm}.$$

The concept of fuzzy set theory was introduced by Zadeh in the year 1965. It has been applied for the studies in almost all the branches of science, where mathematics is used. Workers on sequence spaces have also applied the notion and introduced sequences of fuzzy real numbers and studied their different properties.

2 Definitions and preliminaries

A fuzzy real number X is a fuzzy set on R, *i.e.*, a mapping $X : R \to I$ $(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semicontinuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$, it is open in the usual topology of R.

The class of all *upper semicontinuous, normal, convex* fuzzy real numbers is denoted by $R(I)$.

Define $\overline{d} : R(I) \times R(I) \to R$ by $\overline{d}(X, Y) = \sup_{0<\alpha\leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$. Then it is well known that $(R(I), \overline{d})$ is a complete metric space.

A sequence $X = (X_n)$ of fuzzy real numbers is said to converge to the fuzzy number X_0, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\overline{d}(X_n, X_0) < \varepsilon$ for all $n \geq n_0$.

Let X be a nonempty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive, *i.e.*, $A, B \in I \Rightarrow A \cup B \in I$ and hereditary, *i.e.*, $A \in I$, $B \subseteq A \Rightarrow B \in I$.

A sequence (X_k) of fuzzy real numbers is said to be *I-convergent* to a fuzzy real number $X_0 \in X$ if for each $\varepsilon > 0$, the set

$$E(\varepsilon) = \{k \in N : \overline{d}(X_k, X_0) \geq \varepsilon\} \text{ belongs to } I.$$

The fuzzy number X_0 is called the I-limit of the sequence (X_k) of fuzzy numbers, and we write $I - \lim X_k = X_0$.

The generalized de la Vallé-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$ for $n = 1, 2, \ldots$.

Then a sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L [8] if $t_n(x) \to L$ as $n \to \infty$, and we write

$$[V,\lambda]_0 = \left\{x : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0\right\},$$

$$[V,\lambda] = \left\{x : x - \ell e \in [V,\lambda]_0 \text{ for some } \ell \in C\right\},$$

$$[V,\lambda]_\infty = \left\{x : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| < \infty\right\}$$

for the sets of sequences that are, respectively, strongly summable to zero, strongly summable, and strongly bounded by de la Vallé-Poussin method.

We also note that Nuray and Savas [9] defined the sets of sequence spaces such as strongly σ-summable to zero, strongly σ-summable and strongly σ-bounded with respect to the modulus function, see [10].

In this article, we define some new sequence spaces of fuzzy real numbers by using Orlicz function with the notion of generalized de la Vallèe Poussin mean, generalized difference sequences and ideals. We will also introduce and examine certain new sequence spaces using the tools above.

3 Main results

Let I be an admissible ideal of N, let M be an Orlicz function. Let $r = (r_k)$ be a sequence of real numbers such that $r_k > 0$ for all k, and $\sup_k r_k < \infty$. This assumption is made throughout the paper.

In this article, we have introduced the following sequence spaces,

$$\begin{aligned}
&\left[V_\sigma, \lambda, \Delta_p^q, M, r\right]_0^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, \overline{0})}{\rho}\right)\right\}^{r_k} \geq \varepsilon,\right.\right. \\
&\qquad \left.\left.\text{uniformly in } m\right\} \in I\right\} \quad \text{for some } \rho > 0, \\
&\left[V_\sigma, \lambda, \Delta_p^q, M, r\right]^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)\right\}^{r_k} \geq \varepsilon\right\} \in I\right\} \\
&\quad \text{for some } \rho > 0, X_0 \in R(I), \\
&\left[V_\sigma, \lambda, \Delta_p^q, M, r\right]_\infty^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \exists K > 0, \text{ s.t. } \left\{\sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, \overline{0})}{\rho}\right)\right\}^{r_k} \geq K\right\} \in I\right\} \\
&\quad \text{for some } \rho > 0.
\end{aligned}$$

In particular, if we take $r_k = 1$ for all k, we have

$$\begin{aligned}
&\left[V_\sigma, \lambda, \Delta_p^q, M\right]_0^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)} \overline{0})}{\rho}\right)\right\} \geq \varepsilon,\right.\right. \\
&\qquad \left.\left. \text{uniformly in } m\right\} \in I\right\} \quad \text{for some } \rho > 0, \\
&\left[V_\sigma, \lambda, \Delta_p^q, M\right]^{I(F)} \\
&\quad = \left\{(X_K) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)\right\} \geq \varepsilon\right\} \in I\right\} \\
&\qquad \text{for some } \rho > 0, X_0 \in R(I), \\
&\left[V_\sigma, \lambda, \Delta_p^q, M\right]_\infty^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \exists K > 0 \left\{\sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, \overline{0})}{\rho}\right)\right\} \geq K\right\} \in I\right\}.
\end{aligned}$$

Similarly, when $\sigma(m) = m+1$, then $[V, \lambda, \Delta_p^q, M, r]_0^{I(F)}$, $[V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$ and $[V_\sigma, \lambda, \Delta_p^q, M, r]_\infty^{I(F)}$ are reduced to

$$\begin{aligned}
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]_0^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{k+m}, \overline{0})}{\rho}\right)\right\}^{r_k} \geq \varepsilon\right\} \in I\right\} \\
&\qquad \text{uniformly in } m \text{ for some } \rho > 0. \\
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]^{I(F)} \\
&\quad = \left\{(X_k) \in w^F : \forall \varepsilon > 0 \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{k+m}, X_0)}{\rho}\right)\right\}^{r_k} \geq \varepsilon\right\} \in I\right\} \\
&\qquad \text{for some } \rho > 0, X_0 \in R(I). \\
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]_\infty^{I(F)} = \left\{x : \exists K > 0, \text{ s.t. } \left\{\sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \left\{M\left(\frac{\overline{d}(\Delta_p^q X_{k+m}, \overline{0})}{\rho}\right)\right\}^{r_k} \geq k\right\} \in I\right\} \\
&\qquad \text{for some } \rho > 0.
\end{aligned}$$

In particular, if we put $r_k = r$, for all k, then we have the spaces

$$\begin{aligned}
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]_0^{I(F)} = \left[\hat{V}, \lambda, \Delta_p^q, M\right]_0^{I(F)}, \\
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]^{I(F)} = \left[\hat{V}, \lambda, \Delta_p^q, M\right]^{I(F)}, \\
&\left[\hat{V}, \lambda, \Delta_p^q, M, r\right]_\infty^{I(F)} = \left[\hat{V}, \lambda, \Delta_p^q, M\right]_\infty^{I(F)}.
\end{aligned}$$

Further, when $\lambda_n = n$, for $n = 1, 2, \ldots$, the sets $[\hat{V}, \lambda, \Delta_p^q, M]_0^{I(F)}$ and $[\hat{V}, \lambda, \Delta_p^q, M]^{I(F)}$ are reduced to $[\hat{c}_0(M, \Delta_p^q)]^{I(F)}$ and $[\hat{c}(M, \Delta_p^q)]^{I(F)}$, respectively.

Now, if we consider $M(x) = x$, then we can easily obtain

$$\begin{aligned}&\big[V_\sigma, \lambda, \Delta_p^q, r\big]_0^{I(F)}\\ &\quad= \biggl\{(X_k) \in w^F : \forall \varepsilon > 0 \biggl\{ n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \bigl(\overline{d}\bigl(\Delta_p^q X_{\sigma^k(m)}, \overline{0}\bigr)\bigr)^{r_k} \geq \varepsilon, \\ &\qquad \text{uniformly in } m \biggr\} \in I \biggr\},\\ &\big[V_\sigma, \lambda, \Delta_p^q, r\big]^{I(F)}\\ &\quad= \biggl\{(X_k) \in w^F : \forall \varepsilon > 0 \biggl\{ n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \bigl(\overline{d}\bigl(\Delta_p^q X_{\sigma^k(m)}, X_0\bigr)\bigr)^{r_k} \geq \varepsilon \biggr\} \in I \biggr\}\\ &\quad \text{for } X_0 \in R(I).\\ &\big[V_\sigma, \lambda, \Delta_p^q, r\big]_\infty^{I(F)} = \biggl\{(X_k) \in w^F : \exists K > 0 \text{ s.t. } \biggl\{ \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \bigl(\overline{d}\bigl(\Delta_p^q X_{\sigma^k(m)}, \overline{0}\bigr)\bigr)^{r_k} \geq K \biggr\} \in I \biggr\}.\end{aligned}$$

If $X \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$ with $\{\frac{1}{\lambda_n} \sum_{k \in I_n} \{M(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho})\}^{r_k} \geq \varepsilon\} \in I$ as $n \to \infty$ uniformly in m, then we write $X_k \to X_0 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$.

The following well-known inequality will be used later.

If $0 \leq r_k \leq \sup r_k = H$ and $C = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{r_k} \leq C\bigl\{|a_k|^{r_k} + |b_k|^{r_k}\bigr\} \tag{2}$$

for all k and $a_k, b_k \in C$.

Lemma 3.1 (See [9]) *Let $r_k > 0$, $s_k > 0$. Then $c_0(s) \subset c_0(r)$ if and only if $\lim_{k\to\infty} \inf \frac{r_k}{s_k} > 0$, where $c_0(r) = \{x : |x_k|^{r_k} \to 0 \text{ as } k \to \infty\}$.*

Note that no other relation between (r_k) and (s_k) is needed in Lemma 3.1.

Theorem 3.2 *Let $\lim_{k\to\infty} \inf r_k > 0$. Then $X_k \to X_0$ implies that $X_k \to X_0 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$. Let $\lim_{k\to\infty} r_k = r > 0$. If $X_k \to X_0 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$, then X_0 is unique.*

Proof Let $X_k \to X_0$.

By the definition of Orlicz function, we have for all $\varepsilon > 0$,

$$\biggl\{ n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} M\biggl(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\biggr) \geq \varepsilon \biggr\} \in I.$$

Since $\lim_{k\to\infty} \inf r_k > 0$, it follows that

$$\biggl\{ n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \biggl\{ M\biggl(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\biggr) \biggr\}^{r_k} \geq \varepsilon \biggr\} \in I,$$

and, consequently, $X_k \to X_0 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$.

Let $\lim_{k\to\infty} r_k = r > 0$. Suppose that $X_k \to Y_1 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$, $X_k \to Y_2 \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$ and $(\overline{d}(Y_1, Y_2))^{r_k} = a > 0$.

Now, from (2) and the definition of Orlicz function, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(Y_1, Y_2)}{\rho}\right)^{r_k} \le \frac{C}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, Y_1)}{\rho}\right)^{r_k} + \frac{C}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, Y_2)}{\rho}\right)^{r_k}.$$

Since

$$\left\{n \in N : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, Y_1)}{\rho}\right)^{r_k} \ge \varepsilon\right\} \in I,$$

$$\left\{n \in N : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, Y_2)}{\rho}\right)^{r_k} \ge \varepsilon\right\} \in I.$$

Hence,

$$\left\{n \in N : \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(Y_1, Y_2)}{\rho}\right)^{r_k} \ge \varepsilon\right\} \in I. \tag{3}$$

Further, $M(\frac{\overline{d}(Y_1,Y_2)}{\rho})^{r_k} \to M(\frac{a}{\rho})^r$ as $k \to \infty$, and therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(Y_1, Y_2)}{\rho}\right)^{r_k} = M\left(\frac{a}{\rho}\right)^r. \tag{4}$$

From (3) and (4), it follows that $M(\frac{a}{\rho}) = 0$, and by the definition of an Orlicz function, we have $a = 0$.

Hence, $Y_1 = Y_2$, and this completes the proof. □

Theorem 3.3 (i) *Let* $0 < \inf_k r_k \le r_k \le 1$. *Then*

$$[V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)} \subset [V_\sigma, \lambda, \Delta_p^q, M]^{I(F)}.$$

(ii) *Let* $0 < r_k \le \sup_k r_k < \infty$. *Then*

$$[V_\sigma, \lambda, \Delta_p^q, M]^{I(F)} \subset [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}.$$

Proof (i) Let $X \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$. Since $0 < \inf_k r_k \le 1$, we get

$$\frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right) \le \frac{1}{\lambda_n}\sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)^{r_k}.$$

So,

$$\left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)\right\} \geq \varepsilon, \text{ uniformly in } m\right\}$$

$$\subseteq \left\{n \in N : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0}{\rho}\right)\right\}^{r_k} \geq \varepsilon, \text{ uniformly in } m\right\} \in I,$$

and hence, $X \in [V_\sigma, \lambda, \Delta_p^q, M]^{I(F)}$.

(ii) Let $r \geq 1$ and $\sup_k r_k < \infty$. Let $X \in [V_\sigma, \lambda, \Delta_p^q, M]^{I(F)}$. Then for each k, $0 < \varepsilon < 1$, there exists a positive integer N such that

$$\frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right) \leq \varepsilon < 1$$

for all $m \geq N$. This implies that

$$\frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)^{r_k} \leq \frac{1}{\lambda_n} \sum_{k\in I_n} M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right).$$

So,

$$\left\{n \in N : \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)\right\}^{r_k} \geq \varepsilon, \text{ uniformly in } m\right\}$$

$$\subseteq \left\{n \in N : \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, X_0)}{\rho}\right)\right\} \geq \varepsilon, \text{ uniformly in } m\right\} \in I.$$

Therefore, $X \in [V_\sigma, \lambda, \Delta_p^q, M, r]^{I(F)}$.

This completes the proof. □

Theorem 3.4 *Let $X^F(V_\sigma, \lambda, \Delta_p^{q-1})$ stand for $[V_\sigma, \lambda, \Delta_p^{q-1}, M, r]_0^{I(F)}$, $[V_\sigma, \lambda, \Delta_p^{q-1}, M, r]^{I(F)}$ or $[V_\sigma, \lambda, \Delta_p^{q-1}, M, r]_\infty^{I(F)}$ and $m \geq 1$. Then the inclusion $X^F(V_\sigma, \lambda, \Delta_p^{q-1}) \subset X^F(V_\sigma, \lambda, \Delta_p^q)$ is strict. In general, $X^F(V_\sigma, \lambda, \Delta_p^i) \subset X(V_\sigma, \lambda, \Delta_p^q)$ for all $i = 1, 2, 3, \ldots, p-1$ and the inclusion is strict.*

Proof Let us take $[V_\sigma, \lambda, \Delta_p^{q-1}, M, r]_0^{I(F)}$.

Let $X = (X_k) \in [V_\sigma, \lambda, \Delta_p^{q-1}, M, r]_0^{I(F)}$. Then for given $\varepsilon > 0$, we have

$$\left\{n \in N : \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^{q-1} X_{\sigma^k(m)}, \overline{0})}{\rho}\right)\right\}^{r_k} \geq \varepsilon\right\} \in I \quad \text{for some } \rho > 0.$$

Since M is non-decreasing and convex, it follows that

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^q X_{\sigma^k(m)}, \overline{0})}{\rho}\right)\right\}^{r_k}$$

$$= \frac{1}{\lambda_n} \sum_{k\in I_n} \left\{ M\left(\frac{\overline{d}(\Delta_p^{q-1} X_{\sigma^{k+1}(m)}, \Delta_p^{q-1} X_{\sigma^k(m)})}{\rho}\right)\right\}^{r_k}$$

$$\leq D\frac{1}{\lambda_n}\sum_{k\in I_n}\left(\left[\frac{1}{2}M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k+1}(m)},\overline{0})}{\rho}\right)\right]^{r_k}+\left[\frac{1}{2}M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k}(m)},\overline{0})}{\rho}\right)\right]^{r_k}\right)$$
$$\leq D\frac{1}{\lambda_n}\sum_{k\in I_n}\left(\left[M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k+1}(m)},\overline{0})}{\rho}\right)\right]^{r_k}+\left[M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k}(m)},\overline{0})}{\rho}\right)\right]^{r_k}\right).$$

Hence, we have

$$\left\{n\in N:\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{M\left(\frac{\overline{d}(\Delta_p^{q}X_{\sigma^{k}(m)},\overline{0})}{\rho}\right)\right\}^{r_k}\geq\varepsilon\right\}$$
$$\subseteq\left\{n\in N:D\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k+1}(m)},\overline{0})}{\rho}\right)\right\}^{r_k}\geq\frac{\varepsilon}{2}\right\}$$
$$\cup\left\{n\in N:D\frac{1}{\lambda_n}\sum_{k\in I_n}\left\{M\left(\frac{\overline{d}(\Delta_p^{q-1}X_{\sigma^{k}(m)},\overline{0})}{\rho}\right)\right\}^{r_k}\geq\frac{\varepsilon}{2}\right\}.$$

Since the set on the right-hand side belongs to I, so does the left-hand side. The inclusion is strict as the sequence $X=(k^r)$, for example, belongs to $[V_\sigma,\lambda,\Delta_p^q,M]_0^{I(F)}$ but does not belong to $[V_\sigma,\lambda,\Delta_p^{q-1},M]_0^{I(F)}$ for $M(x)=x$ and $r_k=1$ for all k. □

Theorem 3.5 *$[V_\sigma,\lambda,\Delta_p^q,M,r]_0^{I(F)}$ and $[V_\sigma,\lambda,\Delta_p^q,M,r]^{I(F)}$ are complete metric spaces, with the metric defined by*

$$\overline{d_\sigma}(X,Y)=\sum_{m=1}^{pq}\overline{d}(X_{\sigma^k(m)},Y_{\sigma^k(m)})$$
$$+\inf\left\{\rho^{\frac{r_k}{H}}:\sup_{m,n}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left(M\left(\frac{\overline{d}(\Delta_p^qX_{\sigma^k(m)},\Delta_p^qY_{\sigma^k(m)})}{\rho}\right)\right)^H\right)\leq 1\right.$$
$$\left.\textit{for some } \rho>0\right\},$$

where $H=\max(1,(\sup_k r_k))$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both of the authors contributed equally. The authors also read the galley proof and approved the final copy of the manuscript.

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