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On the Smarandache-Pascal derived sequences of generalized Tribonacci numbers

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Abstract

For any sequence recurrence formula, the Smarandache-Pascal derived sequence $\{T_n\}$ of $\{b_n\}$ is defined by $T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot b_{k+1}$ for all $n \geq 2$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the combination number. The recurrence formula of $\{T_n\}$ is obtained by the properties of the third-order linear recurrence sequence.

Keywords: Smarandache-Pascal derived sequence; Tribonacci numbers; combination number; elementary method

1 Introduction

For any sequence $\{b_n\}$, a new sequence $\{T_n\}$ is defined by the following method: $T_1 = b_1$, $T_2 = b_1 + b_2$, $T_3 = b_1 + 2b_2 + b_3$, generally, $T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot b_{k+1}$ for all $n \geq 2$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the combination number. This sequence is called the Smarandache-Pascal derived sequence of $\{b_n\}$. It was introduced by professor Smarandache in [1] and studied by some authors. For example, Murthy and Ashbacher [2] proposed a series of conjectures related to Fibonacci numbers and the Smarandache-Pascal derived sequence, one of them is as follows.

Conjecture *Let $\{b_n\} = \{F_{8n+1}\} = \{F_1, F_9, F_{17}, F_{25}, \ldots\}$, $\{T_n\}$ be the Smarandache-Pascal derived sequence of $\{b_n\}$, then we have the recurrence formula*

$$T_{n+1} = 49 \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

Li and Han [3] studied these problems and proved a generalized conclusion as follows.

Proposition *Let $\{X_n\}$ be a second-order linear recurrence sequence with $X_0 = u$, $X_1 = v$, $X_{n+1} = aX_n + bX_{n-1}$ for all $n \geq 1$, where $a^2 + 4b > 0$. For any positive integer $d \geq 2$, we define the Smarandache-Pascal derived sequence of $\{X_{dn+1}\}$ as*

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1}.$$

Then we have the recurrence formula

$$T_{n+1} = (2 + A_d + b \cdot A_{d-2}) \cdot T_n - \left(1 + A_d + b \cdot A_{d-2} + (-b)^d\right) \cdot T_{n-1},$$

where the sequence $\{A_n\}$ is defined as $A_0 = 1$, $A_1 = a$, $A_{n+1} = a \cdot A_n + b \cdot A_{n-1}$ for all $n \geq 1$.

It is clear that if we take $b = 1$, then X_n is the Fibonacci polynomials, see [4–7].

The main purpose of this paper is, using the elementary method and the properties of the third-order linear recurrence sequence, to unify the above results by proving the following theorem.

Theorem *Let $\{X_n\}$ be a third-order linear recurrence sequence $X_{n+3} = a \cdot X_{n+2} + b \cdot X_{n+1} + c \cdot X_n$ with the initial values $X_0 = u$, $X_1 = v$ and $X_2 = w$ for all $n \geq 1$, where a, b and c are positive integers. For any positive integer $d \geq 2$, we define the Smarandache-Pascal derived sequence of $\{X_{dn+1}\}$ as*

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1}.$$

Then we have the recurrence formula

$$T_{n+1} = \frac{g_1g_3 - g_1g_6 + g_7}{g_3 - g_6} \cdot T_n + \frac{g_3g_4 - g_2g_6 - g_1g_7}{g_3 - g_6} \cdot T_{n-1} + \frac{g_3g_5 - g_2g_7}{g_3 - g_6} \cdot T_{n-2},$$

where

$$g_1 = f_1 + f_2 + c \cdot A_d f_5, \qquad g_2 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d f_6,$$
$$g_3 = c \cdot A_{d+1}^2 - c \cdot A_d f_2 + c \cdot A_d f_4, \qquad g_4 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d (f_5 + f_6),$$
$$g_5 = c \cdot A_d f_6, \qquad g_6 = c \cdot (A_{d+1} - A_d f_2 + A_d f_4 - A_d), \qquad g_7 = c \cdot A_d f_4$$

and

$$f_1 = b \cdot A_d + c \cdot A_{d-1} + 1, \qquad f_2 = 1 + A_{d+2},$$
$$f_3 = b \cdot A_{d-1} + c \cdot A_d, \qquad f_4 = 1 + c \cdot A_{d-1} - \frac{c \cdot A_d^2}{A_{d+1}},$$
$$f_5 = \frac{A_d}{A_{d+1}}, \qquad f_6 = b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1}A_d + A_d}{A_{d+1}},$$

the sequence $\{A_n\}$ is defined by $A_{n+3} = a \cdot A_{n+2} + b \cdot A_{n+1} + c \cdot A_n$ with the initial values $A_1 = 0$, $A_2 = 1$ and $A_3 = a$ for all $n \geq 1$.

From our theorem we know that if $\{b_n\}$ is a third-order linear recurrence sequence, then its Smarandache-Pascal derived sequence $\{T_n\}$ is also a third-order linear recurrence sequence.

2 Proof of the theorem

To complete the proof of our theorem, we need the following lemma.

Lemma *Let integers $m \geq 0$ and $n \geq 3$. If the sequence $\{X_n\}$ satisfies the recurrence relations $X_{n+3} = a \cdot X_{n+2} + b \cdot X_{n+1} + c \cdot X_n$, $n \geq 0$, then we have the identity*

$$X_{m+n} = A_n \cdot X_{m+2} + (b \cdot A_{n-1} + c \cdot A_{n-2}) \cdot X_{m+1} + c \cdot A_{n-1} \cdot X_m,$$

where A_n is defined by $A_{n+3} = a \cdot A_{n+2} + b \cdot A_{n+1} + c \cdot A_n$ with the initial values $A_1 = 0$, $A_2 = 1$ and $A_3 = a$ for all $n \geq 1$.

Proof Now we prove this lemma by mathematical induction. Note that the recurrence formula $X_{m+3} = a \cdot X_{m+2} + b \cdot X_{m+1} + c \cdot X_m = A_3 \cdot X_{m+2} + (b \cdot A_2 + c \cdot A_1) \cdot X_{m+1} + c \cdot A_2 \cdot X_m$ for all $n \geq 1$. That is, the lemma holds for $n = 3$ since

$$\begin{aligned} X_{m+4} &= a \cdot (a \cdot X_{m+2} + b \cdot X_{m+1} + c \cdot X_m) + b \cdot X_{m+2} + c \cdot X_{m+1} \\ &= \left(a^2 + b\right) \cdot X_{m+2} + (ab + c) \cdot X_{m+1} + ac \cdot X_m \\ &= A_4 \cdot X_{m+2} + (b \cdot A_3 + c \cdot A_2) \cdot X_{m+1} + c \cdot A_3 \cdot X_m. \end{aligned}$$

That is, the lemma holds for $n = 4$. Suppose that for all integers $2 \leq n \leq k$, we have $X_{m+n} = A_n \cdot X_{m+2} + (b \cdot A_{n-1} + c \cdot A_{n-2}) \cdot X_{m+1} + c \cdot A_{n-1} \cdot X_m$. Then, for $n = k + 1$, from the recurrence relations for X_m and the inductive hypothesis, we have

$$\begin{aligned} X_{m+k+1} &= a \cdot X_{m+k} + b \cdot X_{m+k-1} + c \cdot X_{m+k-2} \\ &= a \cdot \left(A_k \cdot X_{m+2} + (b \cdot A_{k-1} + c \cdot A_{k-2}) \cdot X_{m+1} + c \cdot A_{k-1} \cdot X_m\right) \\ &\quad + b \cdot \left(A_{k-1} \cdot X_{m+2} + (b \cdot A_{k-2} + c \cdot A_{k-3}) \cdot X_{m+1} + c \cdot A_{k-2} \cdot X_m\right) \\ &\quad + c \cdot \left(A_{k-2} \cdot X_{m+2} + (b \cdot A_{k-3} + c \cdot A_{k-4}) \cdot X_{m+1} + c \cdot A_{k-3} \cdot X_m\right) \\ &= (a \cdot A_k + b \cdot A_{k-1} + c \cdot A_{k-2}) \cdot X_{m+2} + \left(ab \cdot A_{k-1} + \left(ac + b^2\right) \cdot A_{k-2}\right. \\ &\quad \left. + 2bc \cdot A_{k-3} + c^2 \cdot A_{k-4}\right) \cdot X_{m+1} + c(a \cdot A_{k-1} + b \cdot A_{k-2} + c \cdot A_{k-3}) \cdot X_m \\ &= A_{k+1} \cdot X_{m+2} + \left(ab \cdot A_{k-1} + b^2 \cdot A_{k-2} + bc \cdot A_{k-3} + c \cdot A_{k-1}\right) \cdot X_{m+1} + c \cdot A_k \cdot X_m \\ &= A_{k+1} \cdot X_{m+2} + (b \cdot A_k + c \cdot A_{k-1}) \cdot X_{m+1} + c \cdot A_k \cdot X_m. \end{aligned}$$

That is, the lemma also holds for $n = k + 1$. This completes the proof of our lemma by mathematical induction. □

Now we use this lemma to complete the proof of our theorem. From the properties of the binomial coefficient $\binom{n}{k}$, we have

$$\begin{aligned} \binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} + \frac{1}{n-k}\right) = \binom{n}{k}. \end{aligned} \tag{1}$$

For any positive integer d, from the lemma we have $X_{dk+d+1} = A_{d+1} \cdot X_{dk+2} + (b \cdot A_d + c \cdot A_{d-1}) \cdot X_{dk+1} + c \cdot A_d \cdot X_{dk}$. By the definition of T_n, we may deduce that

$$\begin{aligned} T_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1} \\ &= X_1 + X_{dn+1} + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right) \cdot X_{dk+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+1} + \sum_{k=0}^{n-2} \binom{n-1}{k} \cdot X_{dk+d+1} + X_{dn+1} \\
&= T_n + \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+d+1} \\
&= T_n + \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot \bigl(A_{d+1} \cdot X_{dk+2} + (b \cdot A_d + c \cdot A_{d-1}) \cdot X_{dk+1} + c \cdot A_d \cdot X_{dk}\bigr) \\
&= (b \cdot A_d + c \cdot A_{d-1} + 1) \cdot T_n + A_{d+1} \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+2} \\
&\quad + c \cdot A_d \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk}.
\end{aligned}
$$

For convenience, we let $f_1(A_k) = b \cdot A_d + c \cdot A_{d-1} + 1$ (briefly f_1), then the above identity implies that

$$
T_{n+1} = f_1 \cdot T_n + A_{d+1} \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+2} + c \cdot A_d \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk}. \tag{2}
$$

From this identity, we can also deduce

$$
T_n = f_1 \cdot T_{n-1} + A_{d+1} \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+2} + c \cdot A_d \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk}
$$

and

$$
T_{n-1} = f_1 \cdot T_{n-2} + A_{d+1} \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk+2} + c \cdot A_d \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk}.
$$

They are equivalent to

$$
\sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+2} = \frac{1}{A_{d+1}} \left(T_n - f_1 \cdot T_{n-1} - c \cdot A_d \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} \right) \tag{3}
$$

and

$$
\sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk+2} = \frac{1}{A_{d+1}} \left(T_{n-1} - f_1 \cdot T_{n-2} - c \cdot A_d \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk} \right). \tag{4}
$$

On the other hand, from the lemma we also deduce $X_{dk+d+2} = A_{d+2} \cdot X_{dk+2} + (b \cdot A_{d+1} + c \cdot A_d) \cdot X_{dk+1} + c \cdot A_{d+1} \cdot X_{dk}$. Then we have

$$
\begin{aligned}
&\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+2} \\
&\quad = X_2 + X_{dn-d+2} + \sum_{k=1}^{n-2} \binom{n-1}{k} \cdot X_{dk+2}
\end{aligned}
$$

$$
\begin{aligned}
&= X_2 + X_{dn-d+2} + \sum_{k=1}^{n-2}\left(\binom{n-2}{k} + \binom{n-2}{k-1}\right) \cdot X_{dk+2} \\
&= \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk+d+2} + \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk+2} \\
&= \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot \bigl(A_{d+2} \cdot X_{dk+2} + (b \cdot A_{d+1} + c \cdot A_d) \cdot X_{dk+1} + c \cdot A_{d+1} \cdot X_{dk}\bigr) \\
&\quad + \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk+2} \\
&= (1 + A_{d+2}) \cdot \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk+2} + c \cdot A_{d+1}\sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk} \\
&\quad + (b \cdot A_{d+1} + c \cdot A_d) \cdot T_{n-1}.
\end{aligned} \tag{5}
$$

Similarly, applying formula (1) and identity (3), we have

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\binom{n-1}{k} \cdot X_{dk} \\
&\quad= X_0 + X_{dn-d} + \sum_{k=1}^{n-2}\binom{n-1}{k} \cdot X_{dk} \\
&\quad= X_0 + X_{dn-d} + \sum_{k=1}^{n-2}\left(\binom{n-2}{k} + \binom{n-2}{k-1}\right) \cdot X_{dk} \\
&\quad= \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot \bigl(A_d \cdot X_{dk+2} + (b \cdot A_{d-1} + c \cdot A_{d-2}) \cdot X_{dk+1} + c \cdot A_{d-1} \cdot X_{dk}\bigr) \\
&\qquad + \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk} \\
&\quad= A_d \cdot \sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk+2} + (1 + c \cdot A_{d-1})\sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk} \\
&\qquad + (b \cdot A_{d-1} + c \cdot A_{d-2}) \cdot T_{n-1} \\
&\quad= \frac{A_d}{A_{d+1}}\left(T_n - (b \cdot A_d + c \cdot A_{d-1} + 1) \cdot T_{n-1} - c \cdot A_d\sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk}\right) \\
&\qquad + (1 + c \cdot A_{d-1})\sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk} \\
&\qquad + (b \cdot A_{d-1} + c \cdot A_{d-2}) \cdot T_{n-1} \\
&\quad= \frac{A_d}{A_{d+1}} \cdot T_n + \left(b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1}A_d + A_d}{A_{d+1}}\right) \cdot T_{n-1} \\
&\qquad + \left(1 + c \cdot A_{d-1} - \frac{c \cdot A_d^2}{A_{d+1}}\right)\sum_{k=0}^{n-2}\binom{n-2}{k} \cdot X_{dk}.
\end{aligned} \tag{6}
$$

For convenience, we let

$$f_2 = 1 + A_{d+2}, \qquad f_3 = b \cdot A_{d+1} + c \cdot A_d, \qquad f_4 = 1 + c \cdot A_{d-1} - \frac{c \cdot A_d^2}{A_{d+1}},$$
$$f_5 = \frac{A_d}{A_{d+1}}, \qquad f_6 = b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1}A_d + A_d}{A_{d+1}},$$

then identities (5) and (6) imply that

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk+2} = f_2 \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+2} + c \cdot A_{d+1} \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} + f_3 \cdot T_{n-1}, \tag{7}$$

$$\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk} = f_4 \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} + f_5 T_n + f_6 T_{n-1}. \tag{8}$$

Combining (2), (3), (7) and (8), we deduce

$$\begin{aligned} T_{n+1} &= (f_1 + f_2 + c \cdot A_d f_5) \cdot T_n + (f_3 A_{d+1} - f_1 f_2 + c \cdot A_d f_6) \cdot T_{n-1} \\ &\quad + \left(c \cdot A_{d+1}^2 - c \cdot A_d f_2 + c \cdot A_d f_4\right) \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk}. \end{aligned} \tag{9}$$

Applying formula (1), we deduce

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{k-1}. \tag{10}$$

From this and identities (3) and (4), note that $X_{dk+d} = A_d \cdot X_{dk+2} + (b \cdot A_{d-1} + c \cdot A_{d-2}) \cdot X_{dk+1} + c \cdot A_{d-1} \cdot X_{dk}$, we have

$$\begin{aligned} &\sum_{k=0}^{n-1} \binom{n-1}{k} \cdot X_{dk} \\ &\quad = X_0 + X_{dn-d} + \binom{n-1}{n-2} \cdot X_{dn-2d} + \sum_{k=1}^{n-3} \binom{n-1}{k} \cdot X_{dk} \\ &\quad = X_0 + X_{dn-d} + (n-1) \cdot X_{dn-2d} + \sum_{k=1}^{n-3} \left(\binom{n-3}{k} + \binom{n-3}{k-1} + \binom{n-2}{k-1} \right) \cdot X_{dk} \\ &\quad = X_0 + X_{dn-d} + (n-1) \cdot X_{dn-2d} + \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk} - X_0 \\ &\qquad + \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk+d} - X_{dn-2d} + \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+d} - (n-2) \cdot X_{dn-2d} - X_{dn-d} \\ &\quad = \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk} + \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk+d} + \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+d} \end{aligned}$$

$$
\begin{aligned}
&= A_d \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk+2} + A_d \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk+2} + c \cdot A_{d-1} \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} \\
&\quad + (c \cdot A_{d-1} + 1) \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk} + (b \cdot A_{d-1} + c \cdot A_{d-2}) \cdot (T_{n-1} + T_{n-2}) \\
&= \frac{A_d}{A_{d+1}} T_n + \left(b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1} A_d}{A_{d+1}} \right) \cdot T_{n-1} \\
&\quad + \left(b \cdot A_{d-1} + c \cdot A_{d-2} - \frac{b \cdot A_d^2 + c \cdot A_{d-1} A_d + A_d}{A_{d+1}} \right) \cdot T_{n-2} \\
&\quad + \left(c \cdot A_{d-1} - \frac{c \cdot A_d^2}{A_{d+1}} \right) \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} \\
&\quad + \left(c \cdot A_{d-1} + 1 - \frac{c \cdot A_d^2}{A_{d+1}} \right) \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk} \\
&= f_5 T_n + (f_5 + f_6) T_{n-1} + f_6 T_{n-2} \\
&\quad + (f_4 - 1) \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} + f_4 \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk}.
\end{aligned}
\tag{11}
$$

Combining (2), (3), (7) and (11), we deduce

$$
\begin{aligned}
T_{n+1} &= (f_1 + f_2 + c \cdot A_d f_5) \cdot T_n + \bigl(f_3 A_{d+1} - f_1 f_2 + c \cdot A_d (f_5 + f_6) \bigr) \cdot T_{n-1} + c \cdot A_d f_6 \cdot T_{n-2} \\
&\quad + c(A_{d+1} - A_d f_2 + A_d f_4 - A_d) \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} \\
&\quad + c \cdot A_d f_4 \cdot \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk}.
\end{aligned}
\tag{12}
$$

From identity (9) we can also deduce

$$
\begin{aligned}
T_n &= (f_1 + f_2 + c \cdot A_d f_5) \cdot T_{n-1} + (f_3 A_{d+1} - f_1 f_2 + c \cdot A_d f_6) \cdot T_{n-2} \\
&\quad + \bigl(c \cdot A_{d+1}^2 - c \cdot A_d f_2 + c \cdot A_d f_4 \bigr) \cdot \sum_{k=0}^{n-3} \binom{n-3}{k} \cdot X_{dk}.
\end{aligned}
\tag{13}
$$

For convenience, we let

$$
\begin{aligned}
&g_1 = f_1 + f_2 + c \cdot A_d f_5, \qquad g_2 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d f_6, \\
&g_3 = c \cdot A_{d+1}^2 - c \cdot A_d f_2 + c \cdot A_d f_4, \qquad g_4 = f_3 A_{d+1} - f_1 f_2 + c \cdot A_d (f_5 + f_6), \\
&g_5 = c \cdot A_d f_6, \qquad g_6 = c \cdot (A_{d+1} - A_d f_2 + A_d f_4 - A_d), \qquad g_7 = c \cdot A_d f_4.
\end{aligned}
$$

Inserting (9) and (13) into (12), we deduce

$$
T_{n+1} = \frac{g_1 g_3 - g_1 g_6 + g_7}{g_3 - g_6} \cdot T_n + \frac{g_3 g_4 - g_2 g_6 - g_1 g_7}{g_3 - g_6} \cdot T_{n-1} + \frac{g_3 g_5 - g_2 g_7}{g_3 - g_6} \cdot T_{n-2}. \tag{14}
$$

This completes the proof of our theorem.

Remark In fact, using the above formulas, we can also obtain the recurrence formula of the Smarandache-Pascal derived sequence $\{T_n\}$ of $\{u_n\}$, where $\{u_n\}$ denotes the mth-order linear recursive sequences as follows:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_{m-1} u_{n-m+1} + a_m u_{n-m},$$

with initial values $u_i \in \mathbb{N}$ for $n > m$ and $0 \leq i < m$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZW obtained the theorems and completed the proof. JL and HZ corrected and improved the final version. All authors read and approved the final manuscript.

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References
1. Smarandache, F: Only Problems, Not Solutions. Xiquan Publishing House, Chicago (1993)
2. Murthy, A, Ashbacher, C: Generalized Partitions and New Ideas on Number Theory and Smarandache Sequences. Hexis, Phoenix (2005)
3. Li, X, Han, D: On the Smarandache-Pascal derived sequences and some of their conjectures. Adv. Differ. Equ. **2013**, 240 (2013)
4. Ma, R, Zhang, W: Several identities involving the Fibonacci numbers and Lucas numbers. Fibonacci Q. **45**, 164-170 (2007)
5. Yi, Y, Zhang, W: Some identities involving the Fibonacci polynomials. Fibonacci Q. **40**, 314-318 (2002)
6. Wang, T, Zhang, W: Some identities involving Fibonacci, Lucas polynomials and their applications. Bull. Math. Soc. Sci. Math. Roum. **55**, 95-103 (2012)
7. Riordan, J: Combinatorial Identities. Wiley, New York (1968)

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