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Qualitative properties of a cooperative degenerate Lotka-Volterra model

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Abstract
This paper considers a kind of degenerate parabolic systems. First, we consider the initial boundary value problem of the two-species degenerate parabolic cooperative system. By using the method of a parabolic regularization and energy estimate, we establish the existence of the weak solution of the problem. Then we establish the comparison principle and discuss the uniqueness and the uniform bound. At last, we consider the periodic boundary value problem of the system. By constructing a pair of ordered upper and lower solutions, we establish the existence of nontrivial nonnegative periodic solutions.

1 Introduction

In this paper, we consider the following degenerate parabolic cooperative system

$$u_t = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + u^\alpha(a - bu + cv), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

$$v_t = \operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) + v^\beta(d + eu - fv), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.2}$$

$$u(x,t) = 0, \qquad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $p,q > 2$, $0 < \alpha < p-1$, $0 < \beta < q-1$, $1 < (p-1-\alpha)(q-1-\beta)$, $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$, $d = d(x,t)$, $e = e(x,t)$, $f = f(x,t)$ are strictly positive smooth functions and periodic in time t with period $T > 0$, $u_0(x)$ and $v_0(x)$ are nonnegative smooth functions.

Our motivation for the present study comes from population dynamics, to be specific, such model can be used to describe the population dynamics behavior. We refer to [1, 2] for a survey on this model. The functions u and v represent the spatial densities of two species at time t, the diffusion terms $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\operatorname{div}(|\nabla v|^{q-2}\nabla v)$ represent the effect of dispersion in the habitat, which models a tendency to avoid crowding, and the speed of the diffusion is rather slow since $p,q > 2$, the boundary conditions (1.3) describe the living environment at the boundary, a, d are their respective net birth rate, b and f are intra-specific competitions, whereas c and e are those of inter-specific competitions.

Recently, degenerate cooperative systems have been the subject of extensive study, and most of the works are devoted to the existence, uniqueness, regularity properties and some other interesting properties of the weak solutions (one can see [3–8]). Since such models can describe nonlinear diffusion phenomenon, they are introduced into the discussion of population dynamics. For example, Vishnevskiĭ [9] studied the behavior at large time of solutions to mixed problems for weakly connected cooperative parabolic systems and obtained the monotonicity of the solutions. Pozio, Tesei [10] investigated the coexistence of prey-predator or competing species, subject to density dependent diffusion in an inhomogeneous habitat. They proved that coexistence arises in suitable domains, where favorable conditions are satisfied and also investigated the support properties and attractivity of the resulting stationary solutions. Later, Delgado and Suarez [11] studied the stability and uniqueness for a cooperative degenerate Lotka-Volterra model. For the semilinear case with $p = q = 2$, some results of this kind cooperative systems have already been obtained. The basic questions which have been considered are existence, uniqueness and boundary behavior of solutions, for details, one can see [12–16] and the references therein.

In this paper, we are particularly interested in the discussion of the existence of weak solutions of the initial boundary value problem and the nontrivial nonnegative periodic solutions to problem (1.1)-(1.3), as well as their attractivity character. When investigating this point, we shall make use of the results obtained in [17] for the case of a single equation and also the method of monotone iteration. First, by parabolic generalized method, we establish the existence of the global generalized solution of the initial boundary value problem (1.1)-(1.3). Then we establish the comparison principle and show that the weak solution of (1.1)-(1.3) is uniformly bounded under the condition that

$$b_l f_l > c_M e_M,$$

where $s_M = \sup\{s(x,t) | (x,t) \in \Omega \times \mathbb{R}\}$, $s_l = \inf\{s(x,t) | (x,t) \in \Omega \times \mathbb{R}\}$. At last, by constructing a pair of ordered upper and lower solutions, we establish the existence of the nontrivial nonnegative periodic solutions and the attractivity of the maximal periodic solution.

Since (1.1), (1.2) are degenerate at points where $\nabla u = 0$, $\nabla v = 0$, problem (1.1)-(1.4) might not have classical solutions in general. Therefore, we focus our main efforts on the discussion of weak solutions in the sense of the following.

Definition 1.1 A nonnegative vector function (u, v) is called a weak solution of the problem (1.1)-(1.4) if

$$\begin{aligned} &u \in L^p\big(0, T; W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T), \qquad u_t \in L^2(Q_T), \\ &v \in L^q\big(0, T; W_0^{1,q}(\Omega)\big) \cap L^\infty(Q_T), \qquad v_t \in L^2(Q_T), \end{aligned}$$

and for all $0 \le \tau < T$ and all test functions $\varphi_i \in C^1(\overline{Q_\tau})$ with $\varphi_i|_{\Omega\times[0,\tau)} = 0$ $(i = 1, 2)$, (u, v) satisfies

$$\begin{aligned} &\iint_{Q_\tau} u\frac{\partial \varphi_1}{\partial t} - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi_1 + u^\alpha(a - bu + cv)\varphi_1\,dx\,dt \\ &\quad = \int_\Omega u(x,\tau)\varphi_1(x,\tau)\,dx - \int_\Omega u_0(x)\varphi_1(x,0)\,dx, \qquad (1.5) \\ &\iint_{Q_\tau} v\frac{\partial \varphi_2}{\partial t} - |\nabla v|^{q-2}\nabla v \cdot \nabla\varphi_2 + v^\beta(d + eu - fv)\varphi_2\,dx\,dt \\ &\quad = \int_\Omega v(x,\tau)\varphi_2(x,\tau)\,dx - \int_\Omega v_0(x)\varphi_2(x,0)\,dx, \qquad (1.6) \end{aligned}$$

where $Q_\tau = \Omega \times (0, \tau)$.

Similarly, we can define a weak supersolution $(\overline{u}, \overline{v})$ (subsolution $(\underline{u}, \underline{v})$) if they satisfy the inequalities obtained by replacing '=' with '$\leq$' ('$\geq$') in (1.5), (1.6) with additional assumptions $\phi_i|_{\Omega\times[0,\tau)} \geq 0$ $(i = 1, 2)$.

Definition 1.2 A vector-valued function (u, v) is said to be a T-periodic solution of the problem (1.1)-(1.3) if it is a solution in $[0, T]$ such that $u(\cdot, 0) = u(\cdot, T)$, $v(\cdot, 0) = v(\cdot, T)$ in Ω. A vector-valued function $(\overline{u}, \overline{v})$ is said to be a T-periodic supersolution of problem (1.1)-(1.3), if it is a super-solution in $[0, T]$ such that $\overline{u}(\cdot, 0) \geq \overline{u}(\cdot, T)$, $\overline{v}(\cdot, 0) \geq \overline{v}(\cdot, T)$ in Ω. A vector-valued function $(\underline{u}, \underline{v})$ is said to be a T-periodic subsolution of problem (1.1)-(1.3), if it is a subsolution in $[0, T]$ such that $\underline{u}(\cdot, 0) \leq \underline{u}(\cdot, T)$, $\underline{v}(\cdot, 0) \leq \underline{v}(\cdot, T)$ in Ω.

This paper is organized as follows: In Section 2, we establish the existence and uniqueness of the weak solution of the problem (1.1)-(1.4). In Section 3, we establish the existence of the nontrivial nonnegative periodic solutions by constructing a pair of ordered upper and lower solutions and the method of monotone iteration technique.

2 Initial boundary value problem

To establish the existence of weak solution of the initial boundary value problem (1.1)-(1.4), we consider the following regularity problem

$$\frac{\partial u_\varepsilon}{\partial t} = \operatorname{div}\left(\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla u_\varepsilon\right) + u_\varepsilon^\alpha (a - bu_\varepsilon + cv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.1}$$

$$\frac{\partial v_\varepsilon}{\partial t} = \operatorname{div}\left(\left(|\nabla v_\varepsilon|^2 + \varepsilon\right)^{\frac{q-2}{2}} \nabla v_\varepsilon\right) + v_\varepsilon^\beta (d + eu_\varepsilon - fv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.2}$$

$$u_\varepsilon(x,t) = 0, \qquad v_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \tag{2.3}$$

$$u_\varepsilon(x,0) = u_{\varepsilon 0}(x), \qquad v_\varepsilon(x,0) = v_{\varepsilon 0}(x), \quad x \in \Omega, \tag{2.4}$$

where $u_{\varepsilon 0}(x)$ and $v_{\varepsilon 0}(x)$ are both nonnegative and bounded functions in $C_0^\infty(\Omega)$ and satisfy the following conditions:

$$0 \leq u_{\varepsilon 0} \leq \|u_0\|_{L^\infty(\Omega)}, \qquad 0 \leq v_{\varepsilon 0} \leq \|v_0\|_{L^\infty(\Omega)}, \tag{2.5}$$

$$u_{\varepsilon 0}^{p-1} \to u_0^{p-1}, \qquad v_{\varepsilon 0}^{q-1} \to v_0^{q-1}, \quad \text{in } W_0^{1,2}(\Omega) \text{ as } \varepsilon \to 0. \tag{2.6}$$

By the result of [18], the regularity problem (2.1)-(2.4) admits a classical solution $(u_\varepsilon, v_\varepsilon)$. So we just need to establish a necessary energy estimate for the classic solution $(u_\varepsilon, v_\varepsilon)$ and then establish the existence of weak solution of the initial boundary value problem by letting $\varepsilon \to 0$. For convenience, here and below, C denotes various positive constants independent of ε.

Lemma 2.1 *Assume that* $(u_\varepsilon, v_\varepsilon)$ *is a solution of the regularity problem*(2.1)-(2.4), *then there exist constants* $r, s > 0$ *which are sufficiently large such that*

$$\frac{1}{q - \beta - 1} < \frac{p + r - 2}{q + s - 2} < p - \alpha - 1,$$

and

$$\|u_\varepsilon\|_{L^r(Q_T)} \leq C, \qquad \|v_\varepsilon\|_{L^s(Q_T)} \leq C.$$

Proof Multiplying (2.1) by u_ε^{r-1} $(r > 1)$ and integrating over Ω, integrating by parts, we have

$$\int_\Omega \frac{\partial u_\varepsilon{}^r}{\partial t}\,dx \le -\frac{r(r-1)p^p}{(p+r-2)^p}\int_\Omega \bigl|\nabla u_\varepsilon^{\frac{p+r-2}{p}}\bigr|^p\,dx + r\int_\Omega u_\varepsilon^{\alpha+r-1}(a - bu_\varepsilon + cv_\varepsilon)\,dx. \tag{2.7}$$

By Poincaré's inequality, we have

$$K\int_\Omega u_\varepsilon^{p+r-2}\,dx \le \int_\Omega \bigl|\nabla u_\varepsilon^{\frac{p+r-2}{p}}\bigr|^p\,dx,$$

where K denotes a positive constant only dependent on $|\Omega|$, N. Substituting the formula above into (2.7), we have

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t}\,dx \le -\frac{Kr(r-1)p^p}{(p+r-2)^p}\int_\Omega u_\varepsilon^{p+r-2}\,dx + r\int_\Omega u_\varepsilon^{\alpha+r-1}(a - bu_\varepsilon + cv_\varepsilon)\,dx. \tag{2.8}$$

By Young's inequality, we have

$$rau_\varepsilon^{\alpha+r-1} \le \frac{Kr(r-1)p^p}{4(p+r-2)^p}u_\varepsilon^{p+r-2} + CK^{-\frac{\alpha+r-1}{p-\alpha-1}}, \tag{2.9}$$

$$rcu_\varepsilon^{\alpha+r-1}v_\varepsilon \le \frac{Kr(r-1)p^p}{4(p+r-2)^p}u_\varepsilon^{p+r-2} + CK^{-\frac{\alpha+r-1}{p-\alpha-1}}v_\varepsilon^{\frac{p+r-2}{p-\alpha-1}}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8), we have

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t}\,dx \le -\frac{Kr(r-1)p^p}{2(p+r-2)^p}\int_\Omega u_\varepsilon^{p+r-2}\,dx + CK^{-\frac{\alpha+r-1}{p-\alpha-1}}\int_\Omega v_\varepsilon^{\frac{p+r-2}{p-\alpha-1}}\,dx + CK^{-\frac{\alpha+r-1}{p-\alpha-1}}. \tag{2.11}$$

Similarly, multiplying (2.2) by v_ε^{s-1} $(s > 1)$ and integrating over Ω, we have

$$\int_\Omega \frac{\partial v_\varepsilon^s}{\partial t}\,dx \le -\frac{Ks(s-1)q^q}{2(q+s-2)^q}\int_\Omega v_\varepsilon^{q+s-2}\,dx + CK^{-\frac{\beta+s-1}{q-\beta-1}}\int_\Omega u_\varepsilon^{\frac{q+s-2}{q-\beta-1}}\,dx + CK^{-\frac{\beta+s-1}{q-\beta-1}}. \tag{2.12}$$

Combining (2.11) with (2.12), we have

$$\begin{aligned}\int_\Omega \left(\frac{\partial u_\varepsilon^r}{\partial t} + \frac{\partial v_\varepsilon^s}{\partial t}\right)dx &\le -\frac{Kr(r-1)p^p}{2(p+r-2)^p}\int_\Omega u_\varepsilon^{p+r-2}\,dx + CK^{-\frac{\alpha+r-1}{p-\alpha-1}}\int_\Omega v_\varepsilon^{\frac{p+r-2}{p-\alpha-1}}\,dx \\ &\quad - \frac{Ks(s-1)q^q}{2(q+s-2)^q}\int_\Omega v_\varepsilon^{q+s-2}\,dx + CK^{-\frac{\beta+s-1}{q-\beta-1}}\int_\Omega u_\varepsilon^{\frac{q+s-2}{q-\beta-1}}\,dx \\ &\quad + CK^{-\frac{\alpha+r-1}{p-\alpha-1}} + CK^{-\frac{\beta+s-1}{q-\beta-1}}.\end{aligned} \tag{2.13}$$

Since that

$$\frac{1}{q-\beta-1} < p - \alpha - 1,$$

we can choose r, s large enough such that

$$\frac{1}{q-\beta-1} < \frac{p+r-2}{q+s-2} < p - \alpha - 1.$$

Then by Young's inequality, we have

$$\int_\Omega u_\varepsilon^{\frac{q+s-2}{q-\beta-1}}\,dx \le \frac{r(r-1)(p-1)K^{\frac{q+s-2}{q-\beta-1}}}{2C(p+r-2)^p}\int_\Omega u_\varepsilon^{p+r-2}\,dx + CK^{-\gamma_1}, \tag{2.14}$$

$$\int_\Omega v_\varepsilon^{\frac{p+r-2}{p-\alpha-1}}\,dx \le \frac{s(s-1)(q-1)K^{\frac{p+r-2}{p-\alpha-1}}}{2C(q+s-2)^q}\int_\Omega v_\varepsilon^{q+s-2}\,dx + CK^{-\gamma_2}, \tag{2.15}$$

where

$$\gamma_1 = \frac{(q+s-2)^q}{(q-\beta-1)[(q-\beta-1)(p+r-2)-(q+s-2)]},$$
$$\gamma_2 = \frac{(p+r-2)^p}{(p-\alpha-1)[(p-\alpha-1)(q+s-2)-(p+r-2)]}.$$

Combine (2.13) with (2.14) and (2.15), when r, s are sufficiently large, we have

$$\begin{aligned}\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le{}& -\frac{K}{2}\int_\Omega\left(u_\varepsilon^{p+r-2}+v_\varepsilon^{q+s-2}\right)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right)\\ &+ CK^{-\frac{\alpha+r-1}{p-\alpha-1}} + CK^{-\frac{\beta+s-1}{q-\beta-1}},\end{aligned}$$

where

$$\theta_1 = \frac{(q+s-2)+(p+r-2)(\beta+s-1)}{(q-\beta-1)(p+r-2)-(q+s-2)}, \qquad \theta_2 = \frac{(p+r-2)+(q+s-2)(\alpha+r-1)}{(p-\alpha-1)(q+s-2)-(p+r-2)}.$$

Furthermore, by Young's inequality and Hölder's inequality, we can obtain

$$\begin{aligned}&\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx\\ &\quad\le -\frac{K}{2}\int_\Omega\left(u_\varepsilon^r+v_\varepsilon^s\right)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right) + K|\Omega| + CK^{-\frac{\alpha+r-1}{p-\alpha-1}} + CK^{-\frac{\beta+s-1}{q-\beta-1}}.\end{aligned}$$

Thus by Gronwall's inequality, we have

$$\int_\Omega\left(u_\varepsilon^r+v_\varepsilon^s\right)dx \le C.$$

The proof is complete. □

By Lemma 2.1 and choosing u_ε, v_ε, $\frac{\partial u_\varepsilon}{\partial t}$, $\frac{\partial v_\varepsilon}{\partial t}$ as the test functions, we can easily show the following estimates.

Lemma 2.2 *Assume that* $(u_\varepsilon, v_\varepsilon)$ *is a solution of problem* (2.1)-(2.4), *then*

$$\|\nabla u_\varepsilon\|_{L^p(Q_T)} \le C, \qquad \|\nabla v_\varepsilon\|_{L^q(Q_T)} \le C,$$
$$\left\|\nabla(u_\varepsilon)_t\right\|_{L^2(Q_T)} \le C, \qquad \left\|\nabla(v_\varepsilon)_t\right\|_{L^2(Q_T)} \le C.$$

In order to obtain the maximum norm estimate of the approximate solution, we introduce the following lemma.

Lemma 2.3 (*See* [19]) *Assume that $\varphi(t)$ is a nonnegative monotone increasing function defined in $[k_0, +\infty)$, satisfying*

$$\phi(h) \le \left(\frac{M}{h-k}\right)^{\alpha} \left[\phi(k)\right]^{\beta}, \quad \forall h > k \ge k_0,$$

where $\alpha > 0$, $\beta > 1$. Then we have

$$\varphi(k_0 + d) = 0,$$

with

$$d = M\left[\phi(k_0)\right]^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}.$$

Lemma 2.4 *Assume that $(u_\varepsilon, v_\varepsilon)$ is a solution of problem* (2.1)-(2.4), *then*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le C, \qquad \|v_\varepsilon\|_{L^\infty(Q_T)} \le C. \tag{2.16}$$

Proof Let $l = \|u_{\varepsilon 0}(x)\|_{L^\infty(\Omega)}$, multiplying (2.1) by $(u_\varepsilon - k)_+ \chi[t_1, t_2]$ and integrating over Q_T, where k denotes a various positive constant satisfying $k > l$, we have

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(u_\varepsilon - k)_+ \chi[t_1, t_2]\,dx\,dt \\
&\quad = \iint_{Q_T} \operatorname{div}\left(\left(|\nabla u_\varepsilon|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla u_\varepsilon\right)(u_\varepsilon - k)_+ \chi[t_1, t_2]\,dx\,dt \\
&\qquad + \iint_{Q_T} u_\varepsilon^\alpha (a - bu_\varepsilon + cv_\varepsilon)(u_\varepsilon - k)_+ \chi[t_1, t_2]\,dx\,dt.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
\frac{1}{2}\iint_{Q_T} \frac{\partial}{\partial t}(u_\varepsilon - k)_+^2 \chi[t_1, t_2]\,dx\,dt &\le -\iint_{Q_T} \left|\nabla(u_\varepsilon - k)_+\right|^p \chi[t_1, t_2]\,dx\,dt \\
&\quad + \iint_{Q_T} |u_\varepsilon|^{\alpha+1}(a + cv_\varepsilon)\chi[t_1, t_2]\,dx\,dt.
\end{aligned}$$

Let $I_k(t) = \int_\Omega (u_\varepsilon - k)_+^2\,dx$, we can see that $I_k(t)$ is absolutely continuous in $[0, T]$, and there exists a σ such that $I_k(\sigma) \overset{\Delta}{=} \sup I_k(t)$. Set $t_1 = \sigma - \varepsilon$, $t_2 = \sigma$, we have

$$\begin{aligned}
&\frac{1}{2\varepsilon}\int_{\sigma-\varepsilon}^{\varepsilon}\int_\Omega \frac{\partial}{\partial t}(u_\varepsilon - k)_+^2\,dx\,dt + \frac{1}{\varepsilon}\int_{\sigma-\varepsilon}^{\varepsilon}\int_\Omega \left|\nabla(u_\varepsilon - k)_+\right|^p\,dx\,dt \\
&\quad \le \frac{1}{\varepsilon}\int_{\sigma-\varepsilon}^{\varepsilon}\int_\Omega u_\varepsilon^{\alpha+1}(a + cv_\varepsilon)\,dx\,dt.
\end{aligned} \tag{2.17}$$

Since

$$\int_{\sigma-\varepsilon}^{\varepsilon}\int_\Omega \frac{\partial}{\partial t}(u_\varepsilon - k)_+^2\,dx\,dt = I_k(\sigma) - I_k(\sigma - \varepsilon) \ge 0,$$

we have

$$\frac{1}{\varepsilon}\int_{\sigma-\varepsilon}^{\varepsilon}\int_{\Omega}\left|\nabla(u_\varepsilon-k)_+\right|^p\,dx\,dt\le\frac{1}{\varepsilon}\int_{\sigma-\varepsilon}^{\varepsilon}\int_{\Omega}u_\varepsilon^{\alpha+1}(a+cv_\varepsilon)\,dx\,dt.$$

Letting $\varepsilon\to 0^+$, from (2.17), we have

$$\int_{\Omega}\left|\nabla\big(u_\varepsilon(x,\sigma)-k\big)_+\right|^p\,dx\le\int_{\Omega}u_\varepsilon^{\alpha+1}(x,\sigma)\big(a+cv_\varepsilon(x,\sigma)\big)\,dx.$$

Set

$$A_k(t)=\left\{x:u_\varepsilon(x,t)>k\right\},\qquad \mu_k=\sup_{t\in(0,T)}\left|A_k(t)\right|,$$

we have

$$\int_{A_k(\sigma)}\left|\nabla(u_\varepsilon-k)_+\right|^p\,dx\le\int_{A_k(\sigma)}u_\varepsilon^{\alpha+1}(a+cv_\varepsilon)\,dx.$$

By Sobolev's theorem

$$\left(\int_{A_k(\sigma)}(u_\varepsilon-k)_+^\omega\,dx\right)^{\frac{1}{\omega}}\le C(N,\omega,\Omega)\left(\int_{A_k(\sigma)}\left|\nabla(u_\varepsilon-k)_+\right|^p\,dx\right)^{\frac{1}{p}},$$

where

$$p<\omega<\begin{cases}+\infty, & p\ge N,\\ \frac{Np}{N-p}, & p<N.\end{cases}$$

Combining with Hölder's inequality, we have

$$\begin{aligned}\left(\int_{A_k(\sigma)}(u_\varepsilon-k)_+^\omega\,dx\right)^{\frac{p}{\omega}}&\le C\int_{A_k(\sigma)}\left|\nabla(u_\varepsilon-k)_+\right|^p\,dx\\&\le C\int_{A_k(\sigma)}u_\varepsilon^{\alpha+1}(a+v_\varepsilon)\,dx\\&\le C\left(\int_{A_k(\sigma)}u_\varepsilon^r\,dx\right)^{\frac{\alpha+1}{r}}\left(\int_{A_k(\sigma)}(a+v_\varepsilon)^{\frac{r}{r-\alpha-1}}\,dx\right)^{\frac{r-\alpha-1}{r}}\\&\le C\left(\int_{A_k(\sigma)}(a+v_\varepsilon)^{\frac{r}{r-\alpha-1}}\,dx\right)^{\frac{r-\alpha-1}{r}}\\&\le C\left(\int_{A_k(\sigma)}(a+v_\varepsilon)^s\,dx\right)^{\frac{1}{s}}\left|A_k(\sigma)\right|^{\frac{s(r-\alpha-1)-r}{sr}}\\&\le C\mu^{\frac{s(r-\alpha-1)-r}{sr}},\end{aligned}\tag{2.18}$$

where $r > \frac{\omega(\alpha+1)}{\omega-p}$, $s > \frac{\omega r}{\omega(r-\alpha-1)-2r}$, and C denotes a various positive constant which is independent of ε. Applying Hölder's inequality, we have

$$\begin{aligned} I_k(t) &\le I_k(\sigma) \\ &= \int_\Omega (u_\varepsilon - k)_+^2\,dx = \int_{A_k(\sigma)} (u_\varepsilon - k)_+^2\,dx \\ &\le \left(\int_{A_k(\sigma)} (u_\varepsilon - k)_+^\omega\,dx\right)^{\frac{2}{\omega}} \mu_k^{\frac{\omega-2}{\omega}} \\ &\le C\mu_k^{\frac{2s(r-\alpha-1)-2r}{psr}+\frac{\omega-2}{\omega}}. \end{aligned} \tag{2.19}$$

Furthermore, for any $h > k$, $t \in [0, T]$, we have

$$I_k(t) \ge \int_{A_k(t)} (u_\varepsilon - k)_+^2\,dx \ge (h-k)^2 \big|A_h(t)\big|.$$

Combining with (2.19), we have

$$\mu_h \le \frac{C}{(h-k)^2}\mu_k^{\frac{2s(r-\alpha-1)}{psr}+\frac{\omega-2}{\omega}}.$$

By Lemma 2.3, we have

$$\mu_{l+d} = \sup\big|A_{l+d}(t)\big| = 0,$$

where

$$\begin{aligned} d &= C^{\frac{1}{\omega}} \mu_l^{\frac{\beta-1}{2}} 2^{\frac{\beta}{\beta-1}}, \\ \beta &= \frac{2s(r-\alpha-1)}{psr} + \frac{\omega-2}{\omega} = 1 + \frac{2[s\omega(r-\alpha-1)-\omega r-psr]}{psr\omega} > 1. \end{aligned}$$

That is, $u_\varepsilon \le l + d$ a.e. in Q_T.

Similarly, we also have the same results for v_ε. The proof is complete. □

Theorem 2.1 *The initial boundary value problem* (1.1)-(1.4) *has a weak solution* (u, v).

Proof From Lemma 2.2, Lemma 2.4, we can see that there exists a subsequence $\{(u_{\varepsilon_k}, v_{\varepsilon_k})\}$ of $\{(u_\varepsilon, v_\varepsilon)\}$ and a vector valued function (u, v) satisfying

$$u \in L^p\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T), \qquad v \in L^q\big(0,T;W_0^{1,q}(\Omega)\big) \cap L^\infty(Q_T)$$

such that

$$\begin{aligned} &u_{\varepsilon_k} \to u, \quad \text{a.e. in } Q_T, \\ &\frac{\partial u_{\varepsilon_k}}{\partial t} \to \frac{\partial u}{\partial t}, \quad \text{weakly in } L^2(Q_T), \\ &\nabla u_{\varepsilon_k} \to \nabla u, \quad \text{weakly in } L^p(Q_T), \end{aligned}$$

$$|\nabla u_{\varepsilon_k}|^{p-2}\nabla u_{\varepsilon_k} \to W, \quad \text{weakly in } L^{\frac{p-1}{p}}(Q_T),$$
$$v_{\varepsilon_k} \to v, \quad \text{a.e. in } Q_T,$$
$$\frac{\partial v_{\varepsilon_k}}{\partial t} \to \frac{\partial v}{\partial t}, \quad \text{weakly in } L^2(Q_T),$$
$$\nabla v_{\varepsilon_k} \to \nabla v, \quad \text{weakly in } L^q(Q_T),$$
$$|\nabla v_{\varepsilon_k}|^{q-2}\nabla v_{\varepsilon_k} \to \overline{W}, \quad \text{weakly in } L^{\frac{q-1}{q}}(Q_T).$$

A rather standard argument as that in [20] shows that $W_{x_j} = |\nabla u|^{p-2}u_{x_j}$, $\overline{W}_{x_j} = |\nabla v|^{p-2}v_{x_j}$ a.e. in Q_T. Then we can prove that (u,v) meets Definition 1.1. Thus we complete the proof. □

In order to establish the uniqueness of the solution of (1.1)-(1.4), we need the following comparison principle.

Lemma 2.5 *Assume that* $(\underline{u},\underline{v})$ *is the subsolution of problem* (1.1)-(1.4), *and it has an initial condition* $(\underline{u}_0,\underline{v}_0)$, $(\bar{u},\bar{v})$ *is the supersolution, which has a positive lower bound of problem* (1.1)-(1.4) *and has an initial condition* $(\bar{u}_0,\bar{v}_0)$. *If* $\underline{u}_0 \le \bar{u}_0$, $\underline{v}_0 \le \bar{v}_0$, *then* $\underline{u}(x,t) \le \bar{u}(x,t)$, $\underline{v}(x,t) \le \bar{v}(x,t)$ *on* Q_T.

Proof Suppose that

$$\|\underline{u}(x,t)\|_{L^\infty(Q_T)}, \|\bar{u}(x,t)\|_{L^\infty(Q_T)}, \|\underline{v}(x,t)\|_{L^\infty(Q_T)}, \|\bar{v}(x,t)\|_{L^\infty(Q_T)} \le M,$$

and M is a positive constant, by Definition 1.1, we have

$$\begin{aligned}
&\int_0^t \int_\Omega -\underline{u}\frac{\partial\phi}{\partial t} + |\nabla \underline{u}|^{p-2}\nabla\underline{u}\nabla\phi\,dx\,d\tau + \int_\Omega \underline{u}(x,t)\phi(x,t)\,dx - \int_\Omega \underline{u}_0(x)\phi(x,0)\,dx \\
&\quad\le \int_0^t \int_\Omega \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\phi\,dx\,d\tau, \\
&\int_0^t \int_\Omega -\bar{u}\frac{\partial\phi}{\partial t} + |\nabla \bar{u}|^{p-2}\nabla\bar{u}\nabla\phi\,dx\,d\tau + \int_\Omega \bar{u}(x,t)\phi(x,t)\,dx - \int_\Omega \bar{u}_0(x)\phi(x,0)\,dx \\
&\quad\ge \int_0^t \int_\Omega \bar{u}^\alpha(a - b\bar{u} + c\bar{v})\phi\,dx\,d\tau.
\end{aligned}$$

Let

$$\phi(x,t) = H_\varepsilon\bigl(\underline{u}^{p-1}(x,t) - \bar{u}^{p-1}(x,t)\bigr),$$

and $H_\varepsilon(s)$ is the approximate monotonically increasing smooth function of function $H(s)$ and

$$H(s) = \begin{cases} 1, & s > 0, \\ 0, & \text{others.} \end{cases}$$

Obviously, $H'_\varepsilon(s) \to \delta(s)$ as $\varepsilon \to 0$. Then we have

$$\begin{aligned}
&\int_\Omega (\underline{u} - \bar{u}) H_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1})\,dx - \int_0^t \int_\Omega (\underline{u} - \bar{u}) \frac{\partial H_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1})}{\partial t}\,dx\,d\tau \\
&\qquad + \int_0^t \int_\Omega H'_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1}) \left|\nabla(\underline{u}^{p-1} - \bar{u}^{p-1})\right|^2 dx\,d\tau \\
&\quad \le \int_0^t \int_\Omega a(\underline{u}^\alpha - \bar{u}^\alpha) H_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1}) + c(\underline{u}^\alpha \underline{v} - \bar{u}^\alpha \bar{v}) H_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1})\,dx\,d\tau.
\end{aligned}$$

Let $\varepsilon \to 0$, and notice that

$$\int_0^t \int_\Omega H'_\varepsilon(\underline{u}^{p-1} - \bar{u}^{p-1}) \left|\nabla(\underline{u}^{p-1} - \bar{u}^{p-1})\right|^2 dx\,d\tau \ge 0,$$

we have

$$\int_\Omega \left[\underline{u}(x,t) - \bar{u}(x,t)\right]_+ dx \le C \int_0^t \int_\Omega (\underline{u}^\alpha - \bar{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha - \bar{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v} - \bar{v})_+\,dx\,d\tau,$$

where C is a positive number, which only depends on $\|a(x,t)\|_{C(Q_t)}$, $\|c(x,t)\|_{C(Q_t)}$. Let $(\overline{u}, \overline{v})$ be a supersolution, which has a lower bound σ, notice that for $x, y > 0$,

$$\begin{aligned}
&(x^\alpha - y^\alpha)_+ \le C(\alpha)(x - y)_+, \quad \alpha \ge 1, \\
&(x^\alpha - y^\alpha)_+ \le x^{\alpha-1}(x - y)_+ \le y^{\alpha-1}(x - y)_+, \quad \alpha < 1,
\end{aligned}$$

and $\|\underline{u}(x,t)\|_{L^\infty(Q_T)} \le M$, we have

$$\begin{aligned}
&\int_0^t \int_\Omega (\underline{u}^\alpha - \bar{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha - \bar{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v} - \bar{v})_+\,dx\,d\tau \\
&\quad \le C \int_0^t \int_\Omega (\underline{u} - \bar{u})_+ + (\underline{v} - \bar{v})_+\,dx\,d\tau.
\end{aligned}$$

So

$$\int_\Omega \left[\underline{u}(x,t) - \bar{u}(x,t)\right]_+ dx \le C \int_0^t \int_\Omega (\underline{u} - \bar{u})_+ + (\underline{v} - \bar{v})_+\,dx\,d\tau,$$

and C is a positive number, which depends on α, σ, M. Similarly, we have

$$\int_\Omega \left[\underline{v}(x,t) - \bar{v}(x,t)\right]_+ dx \le C \int_0^t \int_\Omega (\underline{u} - \bar{u})_+ + (\underline{v} - \bar{v})_+\,dx\,d\tau.$$

Then from Gronwall's lemma, we see that $\underline{u} \le \overline{u}$, $\underline{v} \le \bar{v}$. The proof is completed. □

Theorem 2.2 *Assume that* $b_l f_l > c_M e_M$, *then initial-boundary value problem* (1.1)-(1.4) *has a unique weak solution, which is uniformly bounded on* $\overline{\Omega} \times [0, \infty)$.

Proof It is easy to obtain the uniqueness of a weak solution of the initial-boundary value problem (1.1)-(1.4) by the comparison principle. In order to prove the uniform bound, we just need to construct a bounded positive supersolution. Let

$$\rho_1 = \frac{a_M f_l + d_M c_M}{b_l f_l - c_M e_M}, \qquad \rho_2 = \frac{a_M e_M + d_M b_l}{b_l f_l - c_M e_M},$$

for $b_l f_l > c_M e_M$, we have $\rho_1, \rho_2 > 0$ and

$$a_M - b_l \rho_1 + c_M \rho_2 = 0, \qquad d_M + e_M \rho_1 - f_l \rho_2 = 0.$$

Let

$$(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2),$$

and $\eta > 1$ is a constant such that $(u_0, v_0) \leq (\eta\rho_1, \eta\rho_2)$. Then we have

$$\overline{u}_t - \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0 \geq \overline{u}^{\alpha}(a - b\overline{u} + c\overline{v}),$$
$$\overline{v}_t - \operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = 0 \geq \overline{u}^{\beta}(d + e\overline{u} - f\overline{v}).$$

Namely, $(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2)$ is a positive supersolution of problem (1.1)-(1.4). So the weak solution (u, v) of (1.1)-(1.4) is uniformly bounded. □

3 Periodic solutions

In this section, we will establish the existence of the nontrivial nonnegative periodic solutions by constructing a pair of ordered upper and lower solutions and the method of monotone iteration technique.

Lemma 3.1 *Let $b_l f_l > c_M e_M$, then* (1.1)-(1.3) *has a pair of ordered T-periodic supersolutions and T-periodic subsolutions.*

Proof Firstly, we construct a T-periodic subsolution of problem (1.1)-(1.3). Let λ be the first characteristic value, and let ϕ be the uniqueness solution of the following problem:

$$-\operatorname{div}\big(|\nabla\phi|^{p-2}\nabla\phi\big) = \lambda|\phi|^{p-2}\phi, \quad x \in \Omega,$$
$$\phi = 0, \quad x \in \partial\Omega,$$

and let μ be the first characteristic value, and let ψ be the uniqueness solution of the following problem:

$$-\operatorname{div}\big(|\nabla\psi|^{q-2}\nabla\phi\big) = \lambda|\psi|^{q-2}\psi, \quad x \in \Omega,$$
$$\psi = 0, \quad x \in \partial\Omega.$$

According to the classic theory [17], we have

$$\lambda, \mu > 0, \qquad \phi(x), \psi(x) > 0 \quad \text{in } \Omega, \qquad |\nabla\phi| > 0, \qquad |\nabla\psi(x)| > 0 \quad \text{in } \partial\Omega,$$
$$M = \max\Big\{\max_{x\in\overline{\Omega}}\phi(x), \max_{x\in\overline{\Omega}}\psi(x)\Big\} < \infty.$$

Let

$$(\underline{u},\underline{v}) = \bigl(\varepsilon\phi^{\frac{p}{p-1}}(x), \varepsilon\psi^{\frac{q}{q-1}}(x)\bigr),$$

where $\varepsilon > 0$ is a small constant. We now show that $(\underline{u},\underline{v})$ is a subsolution of (1.1)-(1.3) and also is a T-periodic subsolution, since it is time independent. Choosing nonnegative function $\varphi_1(x,t) \in C^1(\bar{Q}_T)$ as the test function, we have

$$\begin{aligned}
&\iint_{Q_T} \underline{u}\frac{\partial\varphi_1}{\partial t} + \operatorname{div}\bigl(|\nabla\underline{u}|^{p-2}\nabla\underline{u}\bigr)\varphi_1 + \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt \\
&\quad + \int_\Omega \underline{u}(x,0)\varphi_1(x,0) - \underline{u}(x,T)\varphi_1(x,T)\,dx \\
&\quad = \iint_{Q_T} \bigl[\underline{u}^\alpha(a - b\underline{u} + c\underline{v}) + \operatorname{div}\bigl(|\nabla\underline{u}|^{p-2}\nabla\underline{u}\bigr)\bigr]\varphi_1\,dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt - \iint_{Q_T} |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi_1\,dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt - \left(\frac{p\varepsilon}{p-1}\right)^{p-1} \iint_{Q_T} |\nabla\phi|^{p-2}\phi\nabla\phi\nabla\varphi_1\,dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt \\
&\qquad - \left(\frac{p\varepsilon}{p-1}\right)^{p-1} \iint_{Q_T} |\nabla\phi|^{p-2}\nabla\phi\nabla(\phi\varphi_1) - |\nabla\phi|^p\varphi_1\,dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt \\
&\qquad - \left(\frac{p\varepsilon}{p-1}\right)^{p-1} \iint_{Q_T} -\operatorname{div}\bigl(|\nabla\phi|^{p-2}\nabla\phi\bigr)\phi\varphi_1 - |\nabla\phi|^p\varphi_1\,dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt \\
&\qquad - \left(\frac{p\varepsilon}{p-1}\right)^{p-1} \iint_{Q_T} \bigl(\lambda|\phi|^{p-2}\phi^2 - |\nabla\phi|^p\bigr)\varphi_1\,dx\,dt.
\end{aligned} \tag{3.1}$$

Similarly, for any nonnegative function $\varphi_2(x,t) \in C^1(\bar{Q}_T)$, we have

$$\begin{aligned}
&\iint_{Q_T} \underline{v}\frac{\partial\varphi_2}{\partial t} + \operatorname{div}\bigl(|\nabla\underline{v}|^{q-2}\nabla\underline{v}\bigr)\varphi_2 + \underline{v}^\beta(d + e\underline{u} - f\underline{v})\varphi_2\,dx\,dt \\
&\quad + \int_\Omega \underline{v}(x,0)\varphi_2(x,0) - \underline{v}(x,T)\varphi_2(x,T)\,dx \\
&\quad = \iint_{Q_T} \underline{v}^\beta(d + e\underline{u} - f\underline{v})\varphi_2\,dx\,dt \\
&\qquad - \left(\frac{q\varepsilon}{q-1}\right)^{q-1} \iint_{Q_T} \bigl(\mu|\psi|^{q-2}\psi^2 - |\nabla\psi|^q\bigr)\varphi_2\,dx\,dt.
\end{aligned} \tag{3.2}$$

Now we just need to show the nonnegativity of the right of (3.1) and (3.2). Since $\phi = \psi = 0$, $|\nabla\phi|, |\nabla\psi| > 0$ in $\partial\Omega$, there exists $\delta > 0$ such that

$$\begin{aligned}
&\lambda|\phi|^{p-2}\phi^2 - |\nabla\phi|^p \le 0,\\
&\mu|\psi|^{q-2}\psi^2 - |\nabla\psi|^q \le 0, \quad x \in \bar{\Omega}_\delta,
\end{aligned}$$

with

$$\bar{\Omega}_\delta = \bigl\{x \in \Omega | \operatorname{dist}(x, \partial\Omega) \le \delta\bigr\}.$$

Let

$$\varepsilon \le \min\left\{\frac{a_l}{b_M M^{p/p-1}}, \frac{d_l}{f_M M^{q/q-1}}\right\},$$

we have

$$\begin{aligned}
&\left(\frac{p\varepsilon}{p-1}\right)^{p-1}\int_0^T\int_{\Omega_\delta}\bigl(\lambda|\phi|^{p-2}\phi^2 - |\nabla\phi|^p\bigr)\varphi_1\,dx\,dt\\
&\quad \le 0 \le \int_0^T\int_{\Omega_\delta}\underline{u}^\alpha(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt,\\
&\left(\frac{q\varepsilon}{q-1}\right)^{q-1}\int_0^T\int_{\Omega_\delta}\bigl(\mu|\psi|^{q-2}\psi^2 - |\nabla\psi|^q\bigr)\varphi_2\,dx\,dt\\
&\quad \le 0 \le \int_0^T\int_{\Omega_\delta}\underline{v}^\beta(d + e\underline{u} - f\underline{v})\varphi_2\,dx\,dt.
\end{aligned}$$

Which show that $(\underline{u}, \underline{v})$ is a positive T-periodic subsolution of problem (1.1)-(1.3) in the domain $\bar{\Omega}_\delta \times (0, T)$. In addition, for some $\sigma > 0$, let

$$\phi(x), \psi(x) \ge \sigma > 0, \quad x \in \Omega \backslash \bar{\Omega}_\delta,$$

and take

$$\begin{aligned}
\varepsilon \le \min\biggl\{&\frac{a_l}{2b_M M^{p/p-1}}, \left(\frac{a_l(p-1)^{p-1}}{2\lambda p^{p-1}}\right)^{\frac{1}{p-1-\alpha}}\frac{1}{M^{p/p-1}},\\
&\frac{d_l}{2f_M M^{q/q-1}}, \left(\frac{d_l(q-1)^{q-1}}{2\mu q^{q-1}}\right)^{\frac{1}{q-1-\beta}}\frac{1}{M^{q/q-1}}\biggr\},
\end{aligned}$$

then we have

$$\begin{aligned}
&\varepsilon^\alpha\phi^{\frac{p\alpha}{p-1}}a - b\varepsilon^{\alpha+1}\phi^{\frac{p(\alpha+1)}{p-1}} + c\varepsilon^\alpha\phi^{\frac{p\alpha}{p-1}}\varepsilon\psi^{\frac{q}{q-1}} - \left(\frac{p\varepsilon}{p-1}\right)^{p-1}\lambda\phi^p \ge 0,\\
&\varepsilon^\beta\phi^{\frac{q\beta}{q-1}}d + e\varepsilon\phi^{\frac{p}{p-1}}\varepsilon^\beta\phi^{\frac{q\beta}{q-1}} - f\varepsilon^{\beta+1}\phi^{\frac{q(\beta+1)}{q-1}} - \left(\frac{q\varepsilon}{q-1}\right)^{q-1}\mu\psi^q \ge 0.
\end{aligned}$$

Namely

$$\iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1\,dx\,dt - \left(\frac{p\varepsilon}{p-1}\right)^{p-1} \iint_{Q_T} \bigl(\lambda|\phi|^{p-2}\phi^2 - |\nabla\phi|^p\bigr)\varphi_1\,dx\,dt \geq 0,$$
$$\iint_{Q_T} \underline{v}^{\beta}(d + e\underline{u} - f\underline{v})\varphi_2\,dx\,dt - \left(\frac{q\varepsilon}{q-1}\right)^{q-1} \iint_{Q_T} \bigl(\mu|\psi|^{q-2}\psi^2 - |\nabla\psi|^q\bigr)\varphi_2\,dx\,dt \geq 0.$$

By the related equalities above, we see that

$$(\underline{u}, \underline{v}) = \bigl(\varepsilon\phi^{\frac{p}{p-1}}(x), \varepsilon\psi^{\frac{q}{q-1}}(x)\bigr)$$

is a subsolution of (1.1)-(1.3) and also is a T-periodic subsolution.

Let

$$(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2),$$

and η, ρ_1, ρ_2 are chosen as those in Theorem 2.2. Obviously, $(\overline{u}, \overline{v})$ is a positive T-periodic supersolution of problem (1.1)-(1.3).

Obviously, by choosing a suitable positive constant η, ε, we have

$$\underline{u}(x,t) \leq \overline{u}(x,t), \qquad \underline{v}(x,t) \leq \overline{v}(x,t).$$

The proof is complete. □

Lemma 3.2 [21] *Let $u(x,t)$ be a weak solution of problem* (1.1)-(1.4). *Then there exist constants $\lambda \in (0,1)$ and $K > 0$ such that*

$$\bigl|u(x_1,t_1) - u(x_2,t_2)\bigr| \leq K\bigl(|x_1 - x_2|^{\lambda} + |t_1 - t_2|^{\lambda/p}\bigr)$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \overline{Q}_T$.

For the solution $(u(x,t), v(x,t))$ of problem (1.1)-(1.4) with the initial condition $(u_0(x), v_0(x))$, we can define the Poincaré mapping as follows:

$$P_t : L^{\infty}(\Omega) \times L^{\infty}(\Omega) \to L^{\infty}(\Omega) \times L^{\infty}(\Omega).$$

According to Lemma 2.5, Lemma 3.2 and Theorem 2.1, we can see that the mapping P_t is well defined in $t > 0$ and also an ordered preserving and compact map.

Theorem 3.1 *Let $b_l f_l > c_M e_M$, and there exists a pair of nontrivial nonnegative T-periodic subsolutions $(\underline{u}(x,t), \underline{v}(x,t))$ and T-periodic supersolutions $(\overline{u}(x,t), \overline{v}(x,t))$ of problem* (1.1)-(1.3) *with $\underline{u}(x,0) \leq \overline{u}(x,0)$, $\underline{v}(x,0) \leq \overline{v}(x,0)$. Then problem* (1.1)-(1.3) *has a pair of nontrivial nonnegative periodic solutions*

$$\bigl(u_*(x,t), v_*(x,t)\bigr), \qquad \bigl(u^*(x,t), v^*(x,t)\bigr),$$

which satisfy

$$\underline{u}(x,t) \le u_*(x,t) \le u^*(x,t) \le \bar{u}(x,t), \qquad \underline{v}(x,t) \le v_*(x,t) \le v^*(x,t) \le \bar{v}(x,t). \tag{3.3}$$

Proof Take $\overline{u}(x,t)$, $\underline{u}(x,t)$ as those in Lemma 3.1. By choosing suitable $B(x_0,\delta)$, $B(x_0,\delta')$, Ω', k_1, k_2, K, we can obtain $\underline{u}(x,0) \le \overline{u}(x,0)$. According to Lemma 2.5, we have

$$P_T\big(\underline{u}(\cdot,0)\big) \ge \underline{u}(\cdot,T).$$

By Definition 1.2, we have $P_T(\underline{u}(\cdot,0)) \ge \underline{u}(\cdot,0)$. Then we have $P_{(k+1)T}(\underline{u}(\cdot,0)) \ge P_{kT}(\underline{u}(\cdot,0))$, $\forall k \in \mathbb{N}$. Similarly, we can also obtain $P_{(k+1)T}(\bar{u}(\cdot,0)) \ge P_{kT}(\bar{u}(\cdot,0))$, $\forall k \in \mathbb{N}$. Then by using Lemma 2.5, we have

$$P_{kT}\big(\underline{u}(\cdot,0)\big) \ge P_{kT}\big(\bar{u}(\cdot,0)\big), \quad \forall k \in \mathbb{N}.$$

Hence, we can see that

$$u_*(x,0) = \lim_{k\to\infty} P_{kT}\big(\underline{u}(x,0)\big), \qquad u^*(x,0) = \lim_{k\to\infty} P_{kT}\big(\bar{u}(x,0)\big)$$

for almost every $x \in \Omega$. Since P_T is a compact operator, the limit above also exists in $L^\infty(\Omega)$. In addition, $(u_*(x,0), u^*(x,0))$ are also the fixed points of the Poincare mapping P_T. Using the method similar to that in [22], we can prove that the even extension of function $u_*(x,t)$, which is the solution of problem (1.1)-(1.4) with initial value $u_*(x,0)$, is just the nontrivial nonnegative periodic solution of problem (1.1)-(1.3). The existence of $u^*(x,t)$ can be obtained similarly. In addition, by Lemma 2.5, we can conclude (3.3). The proof is complete. □

Now we consider the asymptotic behavior of the corresponding initial boundary value. Using the similar method as document [22], we have the following results.

Theorem 3.2 *If* $b_l f_l > c_M e_M$, *then there exists a maximal periodic solution* $(\overline{u}(x,t), \overline{v}(x,t))$ *of problem* (1.1)-(1.3). *In addition, let* $(u(x,t), v(x,t))$ *be the solution of the initial boundary problem with the nonnegative initial value* $(u_0(x), v_0(x))$, *then for any* $\varepsilon > 0$, *there exists a time* T *which is large enough such that*

$$0 \le u(x,t) \le \bar{u}(x,t) + \varepsilon, \qquad 0 \le v(x,t) \le \bar{v}(x,t) + \varepsilon, \quad x \in \Omega, t \ge T.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JS and BW carried out the proof of the main part of this article, DZ corrected the manuscript, and participated in its design and coordination. All authors have read and approved the final manuscript.

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