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Generic diffeomorphisms with weak limit shadowing

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Abstract

In this paper, we show that if a C^1-generic diffeomorphism has the weak limit shadowing property on the chain recurrent set, then the diffemorphism satisfies Axiom A and the no-cycle condition.
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1 Introduction

Let M be a closed C^∞ manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \text{Diff}(M)$ and Λ be a closed f-invariant set. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ ($-\infty \le a < b \le \infty$) in M is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$.

We say that f has the *shadowing property* on Λ if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f ($-\infty \le a < b \le \infty$), there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. In the dynamical systems, the shadowing theory is a very useful notion. In fact, it deals with the stability theorem (see [1]). For instance, Robinson [2] proved that if a diffeomorphism f is structurally stable, then it has the shadowing property. In [3] Sakai showed that f belongs to the C^1-interior of the shadowing property if and only if f is structurally stable. In this paper, we deal with another shadowing property, that is, the weak limit shadowing property which was studied by [4].

We say that f has the *weak limit shadowing property* on Λ (or Λ is *weak limit shadowable* for f) if there exists a $\delta > 0$ with the following property: if a sequence $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ is a δ-pseudo orbit of f, for which relations $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$ and $d(f^{-1}(x_{i+1}), x_i) \to 0$ as $i \to -\infty$ hold, then there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Note that if f has the limit shadowing property, then f has the weak limit shadowing property. But the converse is not true (see [4, Example 4]). Denote by $P(f)$ the set of periodic points of f. Then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$, where $\Omega(f)$ is the set of non-wandering points of f, and $\mathcal{R}(f)$ is the set of chain recurrent points of f. Note that if f satisfies Axiom A and the no-cycle condition, then $\Omega(f) = \mathcal{R}(f)$. We say that f has the *s-limit shadowing property* on Λ if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any δ-limit pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$, and $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Clearly, the weak limit shadowing property is a weak notion of the *s*-limit shadowing property. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$

and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\| D_x f^n |_{E_x^s} \right\| \leq C \lambda^n \quad \text{and} \quad \left\| D_x f^{-n} |_{E_x^u} \right\| \leq C \lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then f is Anosov. Very recently, Sakai [5] showed that if a C^1-generic diffeomorphism f has the s-limit shadowing property on $\mathcal{R}(f)$, then f satisfies Axiom A and the no-cycle condition. The result is motivation for this study. The main theorem of the paper is as follows.

Theorem 1.1 *For C^1-generic f, if f has the weak limit shadowing property on $\mathcal{R}(f)$, then f satisfies Axiom* A *and the no-cycle condition.*

2 Proof of Theorem 1.1

Let M be as before and $f \in \text{Diff}(M)$. Let $p \in P(f)$ be a hyperbolic saddle with period $\pi(p) > 0$. The stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ are defined as follows. It is well known that if p is a hyperbolic periodic point of f with a period k, then the sets

$$W^s(p) = \left\{ x \in M : f^{kn}(x) \to p \text{ as } n \to \infty \right\} \quad \text{and}$$
$$W^u(p) = \left\{ x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty \right\}$$

are C^1-injectively immersed submanifolds of M. Let $p, q \in P(f)$ be saddles. Let $P(f)$ be the set of periodic points of f. Denote by $\mathcal{O}_f(p)$ the periodic f-orbit of $p \in P(f)$. We denote $p \sim q$ if the intersections $W^s(\mathcal{O}_f(p)) \pitchfork W^u(\mathcal{O}_f(q)) \neq \emptyset$ and $W^u(\mathcal{O}_f(p)) \pitchfork W^s(\mathcal{O}_f(q)) \neq \emptyset$. Then we know that if $p \sim q$, then $\text{index}(p) = \text{index}(q)$. Here $\text{index}(p)$ is the dimension of the stable manifold of p, that is, $\dim W^s(p)$.

Proposition 2.1 *There is a residual set $\mathcal{G}_1 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}$, if $f|_{\mathcal{R}(f)}$ has the weak limit shadowing property, then for any saddles $p, q \in P(f)$, $\text{index}(p) = \text{index}(q)$.*

To prove Proposition 2.1, we need the following lemma.

Lemma 2.2 *Let $p, q \in P(f)$ be saddles. If f has the weak limit shadowing property on $\mathcal{R}(f)$, then $W^s(\mathcal{O}_f(p)) \cap W^u(\mathcal{O}_f(q)) \neq \emptyset$.*

Proof Suppose that f has the weak limit shadowing property on $\mathcal{R}(f)$. For any saddles $p, q \in P(f)$, we show that $W^u(\mathcal{O}_f(p)) \cap W^s(\mathcal{O}_f(q)) \neq \emptyset$. For the sake of simplicity, we may assume that $f(p) = p$ and $f(q) = q$. Let $\delta > 0$ be the number of the weak limit shadowing property of f such that $d(p, q) < \delta$. We construct δ-limit pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}} \subset \mathcal{R}(f)$ as follows. (i) $x_0 = p$, (ii) $x_{-i} = f^{-i}(p)$ for all $i > 0$, (iii) $x_i = f^i(q)$ for all $i \geq 1$. Then the δ-limit pseudo orbit

$$\xi = \left\{ \ldots, x_{-1}, p = (x_0), q(= x_1), x_2, \ldots \right\} = \{\ldots, p, p, q, q, \ldots\},$$

and it is clear that $\xi \subset \mathcal{R}(f)$. Since f has the weak shadowing property on $\mathcal{R}(f)$, there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Then $f^i(y) \to p$ as $i \to -\infty$ and $f^{1+i}(y) \to q$ as $i \to \infty$. Hence, $y \in W^s(p)$ and $f(y) \in W^s(q)$. Thus, $W^u(p) \cap W^s(q) \neq \emptyset$. □

The following is called the Kupka-Smale theorem.

Lemma 2.3 *There is a residual set $\mathcal{G}_1 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}$, every periodic point is hyperbolic, and the stable manifolds and the unstable manifolds of periodic points are all transverse.*

Proof of Proposition 2.1 Let $f \in \mathcal{G}_1$, and let $p, q \in P(f)$ be saddles. Suppose that f has the weak limit shadowing property on $\mathcal{R}(f)$. Let $\delta > 0$ be the number of the weak limit shadowing property of f such that $d(p, q) < \delta$. Then we will drive a contradiction, we may assume that $\mathrm{index}(p) \neq \mathrm{index}(q)$. Then we know that $\dim W^s(p) + \dim W^u(q) < \dim M$ or $\dim W^u(p) + \dim W^s(q) < \dim M$. In this proof, we consider that $\dim W^s(p) + \dim W^u(q) < \dim M$ (the other case is similar). Since $f \in \mathcal{G}_1$, $W^s(p) \cap W^u(q) = \emptyset$. Since f has the weak limit shadowing property on $\mathcal{R}(f)$, by Lemma 2.2, $W^s(p) \cap W^u(q) \neq \emptyset$. This is a contradiction. □

Let $p \in P(f)$ be a hyperbolic saddle with a period $\pi(p) > 0$. Then there are the local stable manifold $W^s_\epsilon(p)$ and the unstable manifold $W^u_\epsilon(p)$ of p for some $\epsilon = \epsilon(p) > 0$. It is easily seen that if $d(f^n(x), f^n(p)) \leq \epsilon$ for all $n \geq 0$, then $x \in W^s_\epsilon(p)$, and if $d(f^n(x), f^n(p)) \leq \epsilon$ for all $n \leq 0$, then $x \in W^u_\epsilon(p)$. The following lemma shows that if f has the s-limit shadowing property on $\mathcal{R}(f)$, then the numbers of sinks and sources are finite (see [5, Lemma 2]). From the above facts, we show that if f has the weak limit shadowing property on $\mathcal{R}(f)$, then the numbers of sinks and sources are finite.

Lemma 2.4 *Let f have the weak limit shadowing property on $\mathcal{R}(f)$, and let $\delta > 0$ be the number of the weak limit shadowing property of f. For any saddle $q \in P(f)$, if $p \in P(f)$ is a sink or a source, then $d(p, q) \geq \delta$.*

Proof We will derive a contradiction. Suppose that $q \in P(f)$ is a saddle and $p \in P(f)$ is a sink with $d(p, q) < \delta$. For the sake of simplicity, we may assume that $f(p) = p, f(q) = q$. Since q is a saddle, there is $\epsilon(q) > 0$ such that if for any $x \in M, d(f^i(x), f^i(q)) \leq \epsilon(q)$ as $i \to \infty$, then $x \in W^s_{\epsilon(q)}(q)$, and if $x \in M, d(f^i(x), f^i(q)) \leq \epsilon(q)$ as $i \to -\infty$, then $x \in W^u_{\epsilon(q)}(q)$. Then we may assume that $d(p, q) > \epsilon(q)$. Then we construct a δ-limit pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}} \subset \mathcal{R}(f)$ as follows. Put $x_{-i} = f^{-i}(p)$ for $i \geq 0$ and $x_i = f^i(q)$ for $i \geq 1$. Then $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is clearly a δ-limit pseudo orbit of f, and $\xi = \{x_i\}_{i \in \mathbb{Z}} \subset \mathcal{R}(f)$. Since f has the weak limit shadowing property on $\mathcal{R}(f)$, there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. Since p is a sink, $d(f^{-i}(y), x_{-i}) = d(f^{-i}(y), p) \to 0$ as $i \to \infty$. Then $y = p$. Since $d(f^i(y), x_i) = d(f^i(y), q) \to 0$ as $i \to \infty$, there is $k > 0$ such that $d(f^{k+i}(y), f^{k+i}(q)) = d(f^{k+i}(y), q) \leq \epsilon(q)$ for $i \geq 0$. Then $f^k(y) \in W^s_{\epsilon(q)}(q)$. Since $y = p$, we know that $d(p, q) \leq \epsilon(q)$. This is a contradiction. □

Let p be a periodic point of f, and let $0 < \delta < 1$. We say p has a δ-*weak eigenvalue* provided $D_p f^{\pi(p)}$ has an eigenvalue λ such that $(1 - \delta)^{\pi(p)} < |\lambda| < (1 + \delta)^{\pi(p)}$. We say that the periodic point has a real spectrum if all of its eigenvalues are real and a simple spectrum if all of its eigenvalues have multiplicity one. The following lemma will play a crucial role in our proof.

Lemma 2.5 [6, Lemma 5.1] *There is a residual set $\mathcal{G}_2 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_2$,*

(a) *for any $\eta > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f, there exist $g \in \mathcal{U}(f)$ and $p_g, q_g \in P(g)$ with the same period such that $d(p_g, q_g) < \eta$, then there exist $p, q \in P(f)$ with the same period such that $d(p, q) < \eta$;*

(b) *for any $\eta > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f, there exist $g \in \mathcal{U}(f)$ and $p_g \in P(g)$ with an η-weak eigenvalue, then there exist $p \in P(f)$ with a 2η-weak eigenvalue;*

(c) *for any $\eta > 0$, if $q \in P(f)$ with an η-weak eigenvalue and a real spectrum, then there exists $p \in P(f)$ with an η-weak eigenvalue with a simple real spectrum.*

Lemma 2.6 [7, Lemma 5.1] *There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$, for any $\eta > 0$, if for any C^1-neighborhood $\mathcal{U}(f)$ of f, there exist $g \in \mathcal{U}(f)$ and $p_g, q_g \in P(g)$ with the same period such that $d(p_g, q_g) < \eta$ with different indices, then there exist $p, q \in P(f)$ with the same period such that $d(p, q) < \eta$ with different indices.*

The following so-called Franks lemma will play an essential role in our proof.

Lemma 2.7 *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exists $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $g' \in \mathcal{U}(f)$ such that $g'(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$ and $D_{x_i}g' = L_i$ for all $1 \le i \le N$.*

If $p \in P(f)$ is hyperbolic, then for any $g \in \mathcal{U}(f)$, there is a unique hyperbolic periodic point $p_g \in P(g)$ nearby p such that $\pi(p_g) = \pi(p)$ and $\mathrm{index}(p_g) = \mathrm{index}(p)$, where $\mathrm{index} = \dim W^s(p)$. Such a p_g is called the *continuation* of p.

Lemma 2.8 *There is a residual set $\mathcal{G}_4 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_4$, if f has the weak limit shadowing property on $\mathcal{R}(f)$, then there is $\eta > 0$ such that f has no η-weak eigenvalue.*

Proof Let $f \in \mathcal{G}_4 = \mathcal{G}_2 \cap \mathcal{G}_3$. To derive a contradiction, we may assume that for any $\eta > 0$, there is a hyperbolic periodic point q_g of g (C^1-nearby f) such that q_g has an η-weak eigenvalue and a simple real spectrum. Let $\delta > 0$ be the number of the weak limit shadowing property of f such that $0 < \eta < \delta/2$. For the sake of simplicity, we assume that q_g is a fixed point. By Lemma 2.7, there is h C^1-close to g and h C^1-nearby f such that q_h has 1 as an eigenvalue. By Lemma 2.7 and as in the proof of [8, Lemma 2.4], we can construct an h^l ($l > 0$)-invariant small arc $\mathcal{I}_{q_h}$ of h^l containing q_h such that $d(p_h, r_h) < \eta$, where p_h, r_h are the end points of $\mathcal{I}_{q_h}$, $h^l(p_h) = p_h$, $h^l(r_h) = r_h$ and p_h, r_h are hyperbolic saddles and different indices. Since $f \in \mathcal{G}_4$, there exist $p, r \in P(f)$ with the same period such that $d(p, r) < \eta$ with different indices. Since f has the weak limit shadowing property on $\mathcal{R}(f)$, and by Lemma 2.4, $p, r \in P(f)$ are saddles. Since $d(p, r) < \delta$, by Proposition 2.1, we know that $\mathrm{index}(p) = \mathrm{index}(r)$. But, since $\mathrm{index}(p) \ne \mathrm{index}(r)$, this is a contradiction. □

Denote by $\mathcal{F}(M)$ the C^1-interior of the set of diffeomorphisms of M whose periodic points are all hyperbolic. In [9], Hayashi showed that if $f \in \mathcal{F}(M)$, then f satisfies Axiom A and the no-cycle condition. To prove Theorem 1.1, it is enough to show $f \in \mathcal{F}(M)$.

End of proof of Theorem 1.1 Let $f \in \mathcal{G}_4$, and let f have the weak limit shadowing property on $\mathcal{R}(f)$. If not, then $f \notin \mathcal{F}(M)$. There is g C^1-closed to f and a non-hyperbolic periodic

point p_g such that the point p_g has an $\eta/2$ weak eigenvalue. Since $f \in \mathcal{G}_4$, there is $p \in P(f)$ such that p has an η-weak eigenvalue. By Lemma 2.8, this is a contradiction. □

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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