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Double almost lacunary statistical convergence of order α

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Abstract

In this paper, we define and study lacunary double almost statistical convergence of order α. Further, some inclusion relations have been examined. We also introduce a new sequence space by combining lacunary double almost statistical convergence and Orlicz function.

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1 Introduction

The notion of convergence of a real sequence was extended to a statistical convergence by Fast [1] (see also Schoenberg [2]) as follows. If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then $K(m,n)$ denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural density of the subset K is defined by

$$\overline{d}(K) = \lim_{n\to\infty} \sup \frac{K(1,n)}{n} \quad \text{and} \quad \underline{d}(K) = \lim_{n\to\infty} \inf \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$, then we say that the natural density of K exists, and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $x = (x_k)$ of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has a natural density zero.

Statistical convergence turned out to be one of the most active areas of research in summability theory after the works of Fridy [3] and Šalát [4]. For some very interesting investigations concerning statistical convergence, one may consult the papers of Cakalli [5], Miller [6], Maddox [7] and many others, where more references on this important summability method can be found.

On the other hand, in [8, 9], a different direction was given to the study of statistical convergence, where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence. It was observed in [8] that the behaviour of this new convergence was not exactly parallel to that of statistical convergence, and some basic properties were obtained. One can also see [10] for related works.

In this paper, we define and study lacunary double almost statistical convergence of order α. Also some inclusion relations have been examined.

Let w_2 be the set of all real or complex double sequences. By the convergence of a double sequence, we mean the convergence on the Pringsheim sense, that is, double sequence $x = (x_{ij})$ has a Pringsheim limit L, denoted by P-$\lim x = L$, provided that given $\epsilon > 0$, and there exists $N \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i,j \geq N$. We shall describe such an x more briefly as 'P-convergent' (see, [11]). We denote by c_2 the space of P-convergent sequences. A double sequence $x = (x_{ij})$ is bounded if $\|x\| = \sup_{i,j\geq 0} |x_{ij}| < \infty$. Let l_2^∞ and c_2^∞ be the set of all real or complex bounded double sequences and the set bounded and convergent double sequences, respectively. Moricz and Rhoades [12] defined the almost convergence of double sequence as follows: $x = (x_{ij})$ is said to be almost convergent to a number L if

$$P\text{-}\lim_{p,q\to\infty} \sup_{m,n} \left| \frac{1}{(p+1)(q+1)} \sum_{i=m}^{m+p} \sum_{j=n}^{n+q} x_{ij} - L \right| = 0,$$

that is, the average value of (x_{ij}) taken over any rectangle

$$D = \bigl\{(i,j) : m \leq i \leq m+p, n \leq j \leq n+q\bigr\},$$

tends to L as both p and q tend to ∞, and this convergence is uniform in m and n. We denote the space of almost convergent double sequence by $\hat{c}_2$, as

$$\hat{c}_2 = \Bigl\{x = (x_{ij}) : \lim_{k,l\to\infty} \bigl|t_{klpq}(x) - L\bigr| = 0, \text{ uniformly in } p,q\Bigr\},$$

where

$$t_{klpq}(x) = \frac{1}{(k+1)(l+1)} \sum_{i=p}^{k+p} \sum_{j=q}^{l+q} x_{ij}.$$

The notion of almost convergence for single sequences was introduced by Lorentz [13] and some others.

A double sequence x is called *strongly double almost convergent* to a number L if

$$P\text{-}\lim_{k,l\to\infty} \frac{1}{(k+1)(l+1)} \sum_{i=p}^{k+p} \sum_{j=q}^{l+q} |x_{ij} - L| = 0, \quad \text{uniformly in } p,q.$$

By $[\hat{c}_2]$, we denote the space of strongly almost convergent double sequences.

The notion of strong almost convergence for single sequences has been introduced by Maddox [7].

The idea of statistical convergence was extended to double sequences by Mursaleen and Edely [14]. More recent developments on double sequences can be found in [8, 15–18]. For the single sequences; statistical convergence of order α and strongly p-Cesàro summability of order α introduced by Çolak [9]. Quite recently, in [10], Çolak and Bektaş generalized this notion by using de la Valée-Poussin mean.

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers, and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \leq n$ and $j \leq m$.

Then the lower asymptotic density of K is defined as

$$P\text{-}\liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $(\frac{K_{m,n}}{mn})_{m,n=1,1}^{\infty,\infty}$ has a limit, we say that K has a natural density and is defined as

$$P\text{-}\lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$, where $\mathbb{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P\text{-}\lim_{m,n} \frac{K_{m,n}}{mn} \leq P\text{-}\lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(*i.e.*, the set K has a double natural density zero).

Mursaleen and Edely [14] presented the notion of a statistical convergence for the double sequence $x = (x_{ij})$ as follows: A real double sequence $x = (x_{ij})$ is said to be statistically convergent to L, provided that for each $\epsilon > 0$

$$P\text{-}\lim_{m,n} \frac{1}{mn} \left|\left\{(i,j) : i \leq m \text{ and } j \leq n, |x_{ij} - L| \geq \epsilon\right\}\right| = 0.$$

We now write the following definition.

The double statistical convergence of order α is defined as follows. Let $0 < \alpha \leq 1$ be given. The sequence (x_{ij}) is said to be statistically convergent of order α if there is a real number L such that

$$P\text{-}\lim_{mn\to\infty} \frac{1}{(mn)^\alpha} \left|\left\{i \leq m \text{ and } j \leq n : |x_{ij} - L| \geq \epsilon\right\}\right| = 0$$

for every $\epsilon > 0$, in this, case we say that x is double statistically convergent of order α to L. In this case, we write $S_2^\alpha\text{-}\lim x_{ij} = L$. The set of all double statistically convergent sequences of order α will be denoted by S_2^α. If we take $\alpha = 1$ in this definition , we can have the previous definition.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Fridy and Orhan [19] introduced the idea of lacunary statistical convergence for single sequence as follows.

The number sequence $x = (x_i)$ is said to be lacunary statistically convergent to the number ℓ if for each $\epsilon > 0$,

$$\lim_n \frac{1}{h_r} \left|\left\{k \in I_r : |x_i - L| \geq \epsilon\right\}\right| = 0.$$

In this case, we write $S_\theta\text{-}\lim_i x_i = \ell$, and we denote the set of all lacunary statistically convergent sequences by S_θ.

Definition 1.1 By a double lacunary $\theta_{rs} = \{(k_r l_s)\}$, $r, s = 0, 1, 2, \ldots$, where $k_0 = 0$ and $l_0 = 0$, we shall mean two increasing sequences of nonnegative integers with

$$h_r = k_r - k_{k-1} \to \infty \quad \text{as } r \to \infty$$

and

$$\bar{h}_s = l_s - l_{s-1} \to \infty \quad \text{as } s \to \infty.$$

Let us denote $k_{rs} = k_r l_s$, $h_{rs} = h_r \bar{h}_s$ and the intervals determined by θ_{rs} will be denoted by $I_{rs} = \{(k,l) : k_{r-1} < k \leq k_r \text{ and } l_{s-1} < l \leq l_s\}$, $q_r = \frac{k_r}{k_{r-1}}$, $\bar{q}_s = \frac{l_s}{l_{s-1}}$, and $q_{rs} = q_r \bar{q}_s$. We will denote the set of all double lacunary sequences by $\mathbf{N}_{\theta_{rs}}$.

Let $K \subseteq N \times N$ have double lacunary density $\delta_2^\theta(K)$ if

$$P\text{-}\lim_{rs} \frac{1}{h_{rs}} \bigl|\bigl\{(k,l) \in I_{rs} : (k,l) \in K\bigr\}\bigr|$$

exists.

Example 1 Let $\theta = \{(2^r - 1, 3^s - 1)\}$ and $K = \{(k, 2l) : k, l \in N \times N\}$. Then $\delta_2^\theta(K) = 0$. But it is obvious that $\delta_2(K) = 1/2$.

In 2005, Patterson and Savaş [17] studied double lacunary statistical convergence by giving the definition for complex sequences as follows.

Definition 1.2 Let θ_{rs} be a double lacunary sequence; the double number sequence x is S_θ^2-convergent to L, provided that for every $\epsilon > 0$,

$$P\text{-}\lim_{rs} \frac{1}{h_{rs}} \bigl|\bigl\{(k,l) \in I_{rs} : |x_{kl} - L| \geq \epsilon\bigr\}\bigr| = 0.$$

In this case, write $S_\theta^2\text{-}\lim x = L$ or $x_{kl} \xrightarrow{P} L(S_\theta^2)$.

More investigation in this direction and more applications of double lacunary and double sequences can be found in [20–22] and [23].

2 Main results

In this section, we define lacunary double almost statistically convergent sequences of order α. Also we shall prove some inclusion theorems.

We now have the following.

Definition 2.1 Let $0 < \alpha \leq 1$ be given. The sequence $x = (x_{ij}) \in w_2$ is said to be $\hat{S}_{\theta_{rs}}^\alpha$-statistical convergence of order α if there is a real number L such that

$$P\text{-}\lim_{rs} \frac{1}{h_{rs}^\alpha} \bigl|\bigl\{(k,l) \in I_{rs} : \bigl|t_{klpq}(x) - L\bigr| \geq \epsilon\bigr\}\bigr| = 0, \quad \text{uniformly in } p, q,$$

where h_{rs}^α denote the αth power $(h_{rs})^\alpha$ of h_{rs}. In case $x = (x_{ij})$ is $\hat{S}_{\theta_{rs}}^\alpha$-statistically convergent of order α to L, we write $\hat{S}_{\theta_{rs}}^\alpha\text{-}\lim x_{ij} = L$. We denote the set of all $\hat{S}_{\theta_{rs}}^\alpha$-statistically convergent sequences of order α by $\hat{S}_{\theta_{rs}}^\alpha$.

We know that the $\hat{S}^{\alpha}_{\theta_{rs}}$-statistical convergence of order α is well defined for $0 < \alpha \le 1$, but it is not well defined for $\alpha > 1$ in general. It is easy to see by taking $x = (x_{ij})$ as fixed.

Definition 2.2 Let $0 < \alpha \le 1$ be any real number, and let t be a positive real number. A sequence x is said to be strongly $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summable of order α, if there is a real number L such that

$$P\text{-}\lim_{rs} \frac{1}{h^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} \left|t_{klpq}(x) - L\right|^t = 0, \quad \text{uniformly in } p, q.$$

If we take $\alpha = 1$, the strong $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summability of order α reduces to the strong $\hat{w}_{\theta_{rs}}(t)$-summability.

We denote the set of all strongly $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summable sequence of order α by $\hat{w}^{\alpha}_{\theta_{rs}}(t)$.

We now state the following theorem.

Theorem 2.1 *If* $0 < \alpha \le \beta \le 1$, *then* $\hat{S}^{\alpha}_{\theta_{rs}} \subset \hat{S}^{\beta}_{\theta_{rs}}$.

Proof Let $0 < \alpha \le \beta \le 1$. Then

$$\frac{1}{h^{\beta}_{rs}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \ge \epsilon\right\}\right| \le \frac{1}{h^{\alpha}_{rs}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \ge \epsilon\right\}\right|$$

for every $\epsilon > 0$, and finally, we have that $\hat{S}^{\alpha}_{\theta_{rs}} \subset \hat{S}^{\beta}_{\theta_{rs}}$. This proves the result. □

Theorem 2.2 *For any lacunary sequences* θ, $\hat{S}^{\alpha}_{2} \subseteq \hat{S}^{\alpha}_{\theta_{rs}}$, *if* $\liminf q_r > 1$ *and* $\liminf \bar{q}_s > 1$.

Proof Suppose that $\liminf q^{\alpha}_r > 1$ and $\liminf q^{\alpha}_s > 1$, $\liminf q^{\alpha}_r = \alpha_1$ and $\liminf q^{\alpha}_s = \alpha_2$, say. Write $\beta_1 = (\alpha_1 - 1)/2$ and $\beta_2 = (\alpha_2 - 1)/2$. Then there exist a positive integer r_0 and s_0 such that $q^{\alpha}_r \ge 1 + \beta_1$ for $r \ge r_0$ and $q_s \ge 1 + \beta_2$ for $s \ge s_0$. Hence for $r \ge r_0$, and $s \ge s_0$,

$$\begin{aligned} h^{\alpha}_{rs} \frac{1}{(k_r l_s)^{\alpha}} &= 1 - \left(\frac{k^{\alpha}_{r-1}}{k^{\alpha}_r}\right) \times 1 - \left(\frac{l^{\alpha}_{s-1}}{l^{\alpha}_s}\right) \\ &= \left(1 - \frac{1}{q^{\alpha}_r}\right) \times \left(1 - \frac{1}{q^{\alpha}_s}\right) \\ &\ge 1 - \frac{1}{(1+\beta_1)} \times 1 - \frac{1}{(1+\beta_2)} \\ &= \frac{\beta_1}{1+\beta_1} \times \frac{\beta_2}{1+\beta_2}. \end{aligned}$$

Take any $(x_{kl}) \in \hat{S}^{\alpha}_2$, and $\hat{S}^{\alpha}_2\text{-}\lim_{(k,l)\to\infty} x_{kl} = L$, say. We prove that $\hat{S}^{\alpha}_{\theta_{rs}}\text{-}\lim_{(k,l)\to\infty} x_{kl} = L$. Then for $r \ge r_0$ and $s \ge s_0$, we have

$$\begin{aligned} &\frac{1}{(k_r l_s)^{\alpha}} \left|\left\{k \le k_r, l \le l_s : \left|t_{klpq}(x) - L\right| \ge \epsilon\right\}\right| \\ &\quad \ge \frac{1}{(k_r l_s)^{\alpha}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \ge \epsilon\right\}\right| \end{aligned}$$

$$= h_{rs}^{\alpha} \frac{1}{(k_r l_s)^{\alpha}} \frac{1}{h_{rs}^{\alpha}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right|$$
$$\geq \frac{\beta_1}{1+\beta_1} \times \frac{\beta_2}{1+\beta_2} \frac{1}{h_{rs}^{\alpha}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right|.$$

Therefore, $\hat{S}_{\theta_{rs}}^{\alpha}\text{-}\lim_{(k,l)\to\infty} x(k,l) = L$. □

Remark 2.1 The converse of this result is true for $\alpha = 1$. However, for $\alpha < 1$ it is not clear, and we leave it as an open problem.

Theorem 2.3 *For any double lacunary sequence* θ_{rs}, $\hat{S}_{\theta_{rs}}^{\alpha} \subseteq \hat{S}_2^{\alpha}$ *if* $\limsup_r q_r^{\alpha} < \infty$ *and* $\limsup_s q_s^{\alpha} < \infty$.

Proof Suppose that $\limsup_r q_r^{\alpha} < \infty$ and $\limsup_s q_s^{\alpha} < \infty$. Then there exists $H > 0$ such that $q_r^{\alpha} < H$ and $q_s^{\alpha} < H$ for all r and s. Suppose that $x_{kl} \to L(S_{\theta_{rs}^{\alpha}})$ and

$$N_{rs} = \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right|$$

by the definition of $x_{kl} \to L(S_{\theta_{rs}})$ given $\epsilon > 0$, there exists $r_0, s_0 \in N$ such that $\frac{N_{rs}}{h_{rs}^{\alpha}} < \epsilon$ for all $r > r_0$ and $s > s_0$. Let

$$M := \max\{N_{rs} : 1 \leq r \leq r_0 \text{ and } 1 \leq s \leq s_0\}.$$

Let n and m be such that $k_{r-1} < m \leq k_r$ and $l_{s-1} < n \leq l_s$. Therefore, we obtain the following:

$$\frac{1}{(mn)^{\alpha}} \left|\left\{k \leq m \text{ and } l \leq n : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right|$$
$$\leq \frac{1}{(k_{r-1} l_{s-1})^{\alpha}} \left|\left\{k \leq k_r \text{ and } l \leq l_s : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right|$$
$$= \frac{1}{(k_{r-1} l_{s-1})^{\alpha}} \left\{\sum_{i,j=1,1}^{r,s} N_{i,j}\right\}$$
$$\leq \frac{M r_0 s_0}{(k_{r-1} l_{s-1})^{\alpha}} + \frac{1}{(k_{r-1} l_{s-1})^{\alpha}} \left\{\sum_{i,j=r_0+1,r_0+1}^{r,s} N_{i,j}\right\}$$
$$\leq \frac{M r_0 s_0}{(k_{r-1} l_{s-1})^{\alpha}} + \frac{1}{(k_{r-1} l_{s-1})^{\alpha}} \left\{\sum_{i,j=r_0+1,r_0+1}^{r,s} \frac{N_{i,j} h_{i,j}^{\alpha}}{h_{i,j}^{\alpha}}\right\}$$
$$\leq \frac{M r_0 s_0}{k_{r-1} l_{s-1}} + \frac{1}{(k_{r-1} l_{s-1})^{\alpha}} \left(\sup_{i,j \geq r_0, r_0} \frac{N_{i,j}}{h_{i,j}^{\alpha}}\right) \left\{\sum_{i,j=r_0+1,r_0+1}^{r,s} h_{i,j}^{\alpha}\right\}$$
$$\leq \frac{M r_0 s_0}{(k_{r-1} l_{s-1})^{\alpha}} + \epsilon \left\{\sum_{i,j=r_0+1,r_0+1}^{r,s} h_{i,j}^{\alpha}\right\}$$
$$\leq \frac{M r_0 s_0}{(k_{r-1} l_{s-1})^{\alpha}} + \epsilon H^2.$$

This completes the proof of the theorem. □

Theorem 2.4 *Let* $0 < \alpha \leq \beta \leq 1$ *and* t *be a positive real number, then* $\hat{w}_{\theta_{rs}}^{\alpha}(t) \subseteq \hat{w}_{\theta_{rs}}^{\beta}(t)$.

Proof Let $x = (x_{ij}) \in \hat{w}^{\alpha}_{\theta_{rs}}(t)$. Then given α and β such that $0 < \alpha \leq \beta \leq 1$ and a positive real number t we write

$$\frac{1}{h^{\beta}_{rs}} \sum_{(k,l)\in I_{rs}} \left|t_{klpq}(x) - L\right|^t \leq \frac{1}{h^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} \left|t_{klpq}(x) - L\right|^t,$$

and we get that $\hat{w}^{\alpha}_{\theta_{rs}}(t) \subseteq \hat{w}^{\beta}_{\theta_{rs}}(t)$. □

As a consequence of Theorem 2.4, we have the following.

Corollary 2.1 *Let $0 < \alpha \leq \beta \leq 1$ and t be a positive real number. Then*:

(i) *If $\alpha = \beta$, then $\hat{w}^{\alpha}_{\theta_{rs}}(t) = \hat{w}^{\beta}_{\theta_{rs}}(t)$.*

(ii) $\hat{w}^{\alpha}_{\theta_{rs}}(t) \subseteq \hat{w}_{\theta_{rs}}(t)$ *for each $\alpha \in (0,1]$ and* $0 < t < \infty$.

Theorem 2.5 *Let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$ and $0 < t < \infty$. If a sequence is a strongly $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summable sequence of order α, to L, then it is $\hat{S}^{\beta}_{\theta_{rs}}$-statistically convergent of order β, to L, i.e., $\hat{w}^{\alpha}_{\theta_{rs}}(t) \subset \hat{S}^{\beta}_{\theta_{rs}}$.*

Proof For any sequence $x = (x_{ij})$ and $\epsilon > 0$, we write

$$\begin{aligned}\sum_{(k,l)\in I_{rs}} \left|t_{klpq}(x) - L\right|^t &= \sum_{\substack{(k,l)\in I_{rs} \\ |t_{klpq}(x)-L|\geq\epsilon}} \left|t_{klpq}(x) - L\right|^t + \sum_{\substack{(k,l)\in I_{rs} \\ |t_{klpq}(x)-L|<\epsilon}} \left|t_{klpq}(x) - L\right|^t \\ &\geq \sum_{\substack{(k,l)\in I_{rs} \\ |t_{klpq}(x)-L|\geq\epsilon}} \left|t_{klpq}(x) - L\right|^t \geq \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right| \cdot \epsilon^t\end{aligned}$$

and so that

$$\begin{aligned}\frac{1}{h^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} \left|t_{klpq}(x) - L\right|^t &\geq \frac{1}{h^{\alpha}_{rs}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right| \cdot \epsilon^t \\ &\geq \frac{1}{h^{\beta}_{rs}} \left|\left\{(k,l) \in I_{rs} : \left|t_{klpq}(x) - L\right| \geq \epsilon\right\}\right| \cdot \epsilon^t,\end{aligned}$$

this shows that if $x = (x_{ij})$ is strongly $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summable sequence of order α to L, then it is $\hat{S}^{\beta}_{\theta_{rs}}$-statistically convergent of order β to L. This completes the proof. □

We have the following.

Corollary 2.2 *Let α be fixed real numbers such that $0 < \alpha \leq 1$ and $0 < t < \infty$.*

(i) *If a sequence is strongly $\hat{w}^{\alpha}_{\theta_{rs}}(t)$-summable sequence of order α to L, then it is $\hat{S}^{\alpha}_{\theta_{rs}}$-statistically convergent of order α to L, i.e., $\hat{w}^{\alpha}_{\theta_{rs}}(t) \subset \hat{S}^{\alpha}_{\theta_{rs}}$.*

(ii) $\hat{w}^{\alpha}_{\theta_{rs}}(t) \subset \hat{S}_{\theta_{rs}}$, *for* $0 < \alpha \leq 1$.

3 New sequence space

In this section, we study the inclusion relations between the set of $\hat{S}^{\alpha}_{\theta_{rs}}$-statistical convergent sequences of order α and strongly $\hat{w}^{\alpha}_{\theta_{rs}}[M,t]$-summable sequences of order α with respect to an Orlicz function M.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Lindenstrauss and Tzafriri [24] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p ($1 \leq p < \infty$). The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [25]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. Whereas the Orlicz sequence spaces are the generalization of l_p spaces, the l_p-spaces find themselves enveloped in Orlicz spaces [26].

Recall in [25] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is continuous, convex, nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [27], Savaş [28–33] and many others.

Definition 3.1 Let M be an Orlicz function, $t = (t_{kl})$ be a sequence of strictly positive real numbers, and let $\alpha \in (0,1]$ be any real number. Now, we write

$$\hat{w}^{\alpha}_{\theta_{rs}}[M,t] = \biggl\{ x = (x_{kl}) : P\text{-}\lim_{rs} \frac{1}{h^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}} = 0, \text{ uniformly in } p,q, \text{ for some } L \text{ and } \rho > 0 \biggr\}.$$

If $x \in \hat{w}^{\alpha}_{\theta_{rs}}[M,t]$, then we say that x is strongly double almost lacunary summable of order α with respect to the Orlicz function M.

If we consider various assignments of M, θ_{rs} and t in the sequence spaces above, we are granted the following:

(1) If $M(x) = x$, $\theta = 2^{rs}$, and $t_{k,l} = 1$ for all (k,l) then $\hat{w}^{\alpha}_{\theta_{rs}}[M,t] = [\hat{w}^{\alpha}]$.
(2) If $t_{k,l} = 1$ for all (k,l), then $\hat{w}^{\alpha}_{\theta_{rs}}[M,t] = \hat{w}^{\alpha}_{\theta_{rs}}[M]$.
(3) If $t_{k,l} = 1$ for all (k,l) and $\theta = 2^{rs}$, then $\hat{w}^{\alpha}_{\theta_{rs}}[M,t] = \hat{w}^{\alpha}[M]$.
(4) If $\theta = 2^{rs}$, then $\hat{w}^{\alpha}_{\theta}[M,t] = \hat{w}^{\alpha}[M,t]$.

In the followings theorems, we shall assume that $t = (t_{kl})$ is bounded and $0 < h = \inf_{kl} t_{kl} \leq t_{kl} \leq \sup_{kl} t_{kl} = H < \infty$.

Theorem 3.1 *Let $\alpha, \beta \in (0,1]$ be real numbers such that $\alpha \leq \beta$, and let M be an Orlicz function, then $\hat{w}^{\alpha}_{\theta_{rs}}[M,t] \subset \hat{S}^{\beta}_{\theta_{rs}}$.*

Proof Let $x \in \hat{w}^{\alpha}_{\theta}[M,t]$, $\epsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $(k,l) \in I_{rs}$, $|t_{klpq}(x) - L| \geq \epsilon$ and $(k,l) \in I_{rs}$, $|t_{klpq}(x) - L| < \epsilon$, respectively. Since $h^{\alpha}_{rs} \leq h^{\beta}_{rs}$ for each r, s we write

$$\begin{aligned} &\frac{1}{h^{\alpha}_{rs}} \sum_{(k,l)\in I_{rs}} \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}} \\ &\quad= \frac{1}{h^{\alpha}_{rs}} \biggl[\sum_1 \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}} + \sum_2 \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}}\biggr] \\ &\quad\geq \frac{1}{h^{\beta}_{rs}} \biggl[\sum_1 \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}} + \sum_2 \biggl[\frac{M(|t_{klpq}(x) - L|)}{\rho}\biggr]^{t_{kl}}\biggr] \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{h_{rs}^{\beta}}\left[\sum_{1}\left[M(\epsilon/\rho)\right]\right]^{t_{kl}} \\
&\geq \frac{1}{h_{rs}^{\beta}}\sum_{1}\min\left(\left[M(\epsilon_1)\right]^{h},\left[M(\epsilon_1)\right]^{H}\right), \quad \epsilon_1=\frac{\epsilon}{\rho} \\
&\geq \frac{1}{h_{rs}^{\beta}}\left|\left\{(k,l)\in I_{rs}:\left|t_{klpq}(x)-L\right|\geq\epsilon\right\}\right|\min\left(\left[M(\epsilon_1)\right]^{h},\left[M(\epsilon_1)\right]^{H}\right).
\end{aligned}
$$

Since $x\in\hat{w}_{\theta_{rs}}^{\alpha}[M,t]$, the left hand side of the inequality above tends to zero as $r,s\to\infty$ uniformly in p, q. Hence the right hand side tends to zero as $r,s\to\infty$ uniformly in p, q, and, therefore, $x\in\hat{S}_{\theta_{rs}}^{\beta}$. This proves the result. □

Corollary 3.1 *Let $\alpha\in(0,1]$ and M be an Orlicz function, then $\hat{w}_{\theta_{rs}}^{\alpha}[M,t]\subset\hat{S}_{\theta_{rs}}^{\alpha}$.*

We finally prove the following theorem.

Theorem 3.2 *Let M be an Orlicz function, and let $x=(x_{ij})$ be a bounded sequence, then $\hat{S}_{\theta_{rs}}^{\alpha}\subset\hat{w}_{\theta_{rs}}^{\alpha}[M,t]$.*

Proof Suppose that $x\in\ell_2^{\infty}$ and $\hat{S}_{\theta_{rs}}^{\alpha}$-$\lim x_{ij}=L$. Since $x\in\ell_2^{\infty}$, then there is a constant $K>0$ such that $|t_{klpq}(x)|\leq K$. Given $\epsilon>0$, we write for all p, q

$$
\begin{aligned}
\frac{1}{h_{rs}^{\alpha}}\sum_{(k,l)\in I_{rs}}\left[M\left(\frac{|t_{klpq}(x)-L|}{\rho}\right)\right]^{r_{kl}} &= \frac{1}{h_{rs}^{\alpha}}\sum_{1}\left[M\left(\frac{|t_{klpq}(x)-L|}{\rho}\right)\right]^{r_{kl}} \\
&\quad+\frac{1}{h_{rs}^{\alpha}}\sum_{2}\left[M\left(\frac{|t_{klpq}(x)-L|}{\rho}\right)\right]^{r_{kl}} \\
&\leq \frac{1}{h_{rs}^{\alpha}}\sum_{1}\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^{h},\left[M\left(\frac{K}{\rho}\right)\right]^{H}\right\} \\
&\quad+\frac{1}{h_{rs}^{\alpha}}\sum_{2}\left[M\left(\frac{\epsilon}{\rho}\right)\right]^{t_{kl}} \\
&\leq \max\left\{\left[M(T)\right]^{h},\left[M(T)\right]^{H}\right\} \\
&\quad\times\frac{1}{h_{rs}^{\alpha}}\left|\left\{(k,l)\in I_{rs}:\left|t_{klpq}(x)-L\right|\geq\epsilon\right\}\right| \\
&\quad+\max\left\{\left[M(\epsilon_1)\right]^{h},\left[M(\epsilon_1)\right]^{H}\right\}, \quad \frac{K}{\rho}=T,\ \frac{\epsilon}{\rho}=\epsilon_1.
\end{aligned}
$$

Therefore, $x\in x\in\hat{w}_{\theta_{rs}}^{\alpha}[M,t]$. This proves the result. □

Competing interests
The author declares that they have no competing interests.

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