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Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials

Dae San Kim[1] and Taekyun Kim[2*]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we consider higher-order Frobenius-Euler polynomials, associated with poly-Bernoulli polynomials, which are derived from polylogarithmic function. These polynomials are called higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials. The purpose of this paper is to give various identities of those polynomials arising from umbral calculus.

1 Introduction

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order α ($\alpha \in \mathbb{R}$) are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [1–5])}. \tag{1.1}$$

When $x = 0$, $H_n^{(\alpha)}(\lambda) = H_n^{(\alpha)}(0|\lambda)$ are called the Frobenius-Euler numbers of order α. As is well known, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{(see [6–8])}. \tag{1.2}$$

When $x = 0$, $\mathbb{B}_n^{(\alpha)} = \mathbb{B}_n^{(\alpha)}(x)$ is called the nth Bernoulli number of order α. In the special case, $\alpha = 1$, $\mathbb{B}_n^{(1)}(x) = B_n(x)$ is called the nth Bernoulli polynomial. When $x = 0$, $B_n = B_n(0)$ is called the nth ordinary Bernoulli number. Finally, we recall that the Euler polynomials of order α are given by

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \text{(see [9–13])}. \tag{1.3}$$

When $x = 0$, $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ is called the nth Euler number of order α. In the special case, $\alpha = 1$, $E_n^{(1)}(x) = E_n(x)$ is called the nth ordinary Euler polynomial. The classical polylogarithmic function $Li_k(x)$ is defined by

$$Li_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (k \in \mathbb{Z}) \text{ (see [7])}. \tag{1.4}$$

As is known, poly-Bernoulli polynomials are defined by the generating function to be

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} \quad (cf.\ [7]). \tag{1.5}$$

Let $\mathbb{C}$ be the complex number field, and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.6}$$

Now, we use the notation $\mathbb{P} = \mathbb{C}[x]$. In this paper, $\mathbb{P}^*$ will be denoted by the vector space of all linear functionals on $\mathbb{P}$. Let us assume that $\langle L|p(x)\rangle$ be the action of the linear functional L on the polynomial $p(x)$, and we remind that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \in \mathcal{F} \tag{1.7}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad \text{for all } n \geq 0 \text{ (see [14, 15])}. \tag{1.8}$$

From (1.7) and (1.8), we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad \text{(see [14, 15])}, \tag{1.9}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let us consider $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^n\rangle}{k!}t^k$. Then we see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$, and so $L = f_L(t)$ as linear functionals. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional (see [14]). We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $o(f(t))$ of a nonzero power series $f(t)$ is the smallest integer k, for which the coefficient of t^k does not vanish. A series $f(t)$ is called a delta series if $o(f(t)) = 1$, and an invertible series if $o(f(t)) = 0$. Let $f(t), g(t) \in \mathcal{F}$. Then we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \quad \text{(see [14])}. \tag{1.10}$$

For $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$, $o(g(t)) = 0$, there exists a unique sequence $S_n(x)$ ($\deg S_n(x) = n$) such that $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$ for $n, k \geq 0$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [14, 15]). Let $f(t) \in \mathcal{F}$ and $p(t) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!}. \tag{1.11}$$

From (1.11), we note that

$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle. \tag{1.12}$$

By (1.12), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [14, 15])}. \tag{1.13}$$

From (1.13), we easily derive the following equation

$$e^{yt} p(x) = p(x+y), \quad \langle e^{yt} | p(x) \rangle = p(y). \tag{1.14}$$

For $p(x) \in \mathbb{P}, f(t) \in \mathcal{F}$, it is known that

$$\langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle = \langle f'(t) | p(x) \rangle \quad \text{(see [14])}. \tag{1.15}$$

Let $S_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(x))} e^{y\bar{f}(t)} = \sum_{n=0}^{\infty} S_n(y) \frac{t^n}{n!} \quad \text{for all } y \in \mathbb{C}, \tag{1.16}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$, and

$$f(t) S_n(x) = n S_{n-1}(x) \quad \text{(see [14, 15])}. \tag{1.17}$$

The Stirling number of the second kind is defined by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^m}{m!} \quad (m \in \mathbb{Z}_{\geq 0}). \tag{1.18}$$

For $S_n(x) \sim (g(t), t)$, it is well known that

$$S_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) S_n(x) \quad (n \geq 0) \text{ (see [14, 15])}. \tag{1.19}$$

Let $S_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then we have

$$S_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \tag{1.20}$$

where

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l\left(\bar{f}(t)\right)^m \middle| x^n \right\rangle \quad \text{(see [14, 15])}. \tag{1.21}$$

In this paper, we study higher-order Frobeniuns-Euler polynomials associated with poly-Bernoulli polynomials, which are called higher-order Frobenius-Euler and poly-Beroulli mixed-type polynomials. The purpose of this paper is to give various identities of those polynomials arising from umbral calculus.

2 Higher-order Frobenius-Euler polynomials, associated poly-Bernoulli polynomials

Let us consider the polynomials $T_n^{(r,k)}(x|\lambda)$, called higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials, as follows:

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} T_n^{(r,k)}(x|\lambda)\frac{t^n}{n!}, \tag{2.1}$$

where $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, $r, k \in \mathbb{Z}$.

When $x = 0$, $T_n^{(r,k)}(\lambda) = T_n^{(r,k)}(0|\lambda)$ is called the nth higher-order Frobenius-Euler and poly-Bernoulli mixed type number.

From (1.16) and (2.1), we note that

$$T_n^{(r,k)}(x|\lambda) \sim \left(g_{r,k}(t) = \left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right). \tag{2.2}$$

By (1.17) and (2.2), we get

$$tT_n^{(r,k)}(x|\lambda) = nT_{n-1}^{(r,k)}(x|\lambda). \tag{2.3}$$

From (2.1), we can easily derive the following equation

$$\begin{aligned} T_n^{(r,k)}(x|\lambda) &= \sum_{l=0}^{n}\binom{n}{l} H_{n-l}^{(r)}(\lambda)B_l^{(k)}(x) \\ &= \sum_{l=0}^{n}\binom{n}{l} H_{n-l}^{(r)}(x|\lambda)B_l^{(k)}. \end{aligned} \tag{2.4}$$

By (1.16) and (2.2), we get

$$T_n^{(r,k)}(x|\lambda) = \frac{1}{g_{r,k}(t)}x^n = \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}}x^n. \tag{2.5}$$

In [7], it is known that

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}x^n = \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(x-j)^n. \tag{2.6}$$

Thus, by (2.5) and (2.6), we get

$$\begin{aligned} T_n^{(r,k)}(x|\lambda) &= \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}}x^n \\ &= \sum_{m=0}^{\infty}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\left(\frac{1-\lambda}{e^t-\lambda}\right)^r(x-j)^n \\ &= \sum_{m=0}^{n}\frac{1}{(m+1)^k}\sum_{j=0}^{m}(-1)^j\binom{m}{j}H_n^{(r)}(x-j|\lambda). \end{aligned} \tag{2.7}$$

By (1.1), we easily see that

$$H_n^{(r)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) x^l. \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.1 *For $r, k \in \mathbb{Z}$, $n \geq 0$, we have*

$$\begin{aligned} T_n^{(r,k)}(x|\lambda) &= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda)(x-j)^l \\ &= \sum_{l=0}^{n} \left\{ \binom{n}{l} H_{n-l}^{(r)}(\lambda) \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \right\} (x-j)^l. \end{aligned}$$

In [7], it is known that

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m! S_2(n-j,m) \right\} x^j. \tag{2.9}$$

By (2.5) and (2.9), we get

$$\begin{aligned} T_n^{(r,k)}(x|\lambda) &= \left(\frac{1-\lambda}{e^t-\lambda} \right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^n \\ &= \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m! S_2(n-j,m) \right\} \left(\frac{1-\lambda}{e^t-\lambda} \right)^r x^j \\ &= \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^{n-m-j}}{(m+1)^k} \binom{n}{j} m! S_2(n-j,m) \right\} H_j^{(r)}(x|\lambda). \end{aligned} \tag{2.10}$$

Therefore, by (2.8) and (2.10), we obtain the following theorem.

Theorem 2.2 *For $r, k \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$, we have*

$$T_n^{(r,k)}(x|\lambda) = \sum_{l=0}^{n} \left\{ \sum_{j=l}^{n} \sum_{m=0}^{n-j} (-1)^{n-m-j} \binom{n}{j} \binom{j}{l} \frac{m!}{(m+1)^k} H_{j-l}^{(r)}(\lambda) S_2(n-j,m) \right\} x^l.$$

From (1.19) and (2.2), we have

$$T_{n+1}^{(r,k)}(x|\lambda) = \left(x - \frac{g'_{r,k}(t)}{g_{r,k}(t)} \right) T_n^{(r,k)}(x|\lambda). \tag{2.11}$$

Now, we note that

$$\begin{aligned}\frac{g'_{r,k}(t)}{g_{r,k}(t)} &= \left(\log g_{r,k}(t)\right)' \\ &= \left(r\log\left(e^t-\lambda\right) - r\log(1-\lambda) + \log\left(1-e^{-t}\right) - \log Li_k\left(1-e^t\right)\right)' \\ &= r + \frac{r\lambda}{e^t\lambda} + \left(\frac{t}{e^t-1}\right)\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{tLi_k(1-e^{-t})}.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we get

$$\begin{aligned}T_{n+1}^{(r,k)}(x|\lambda) &= xT_n^{(r,k)}(x|\lambda) - rT_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda}\left(\frac{1-\lambda}{e^t-\lambda}\right)^{r+1}\frac{Li_k(1-e^{-t})}{1-e^{-t}}x^n \\ &\quad - \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{t(1-e^{-t})}\left(\frac{t}{e^t-1}\right)x^n \\ &= (x-r)T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda}T_n^{(r+1,k)}(x|\lambda) \\ &\quad - \sum_{l=0}^{n}\binom{n}{l}B_{n-l}\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{t(1-e^{-t})}x^l.\end{aligned} \tag{2.13}$$

It is easy to show that

$$\begin{aligned}\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{1-e^{-t}} &= \frac{1}{1-e^{-t}}\sum_{n=1}^{\infty}\left\{\frac{(1-e^{-t})^n}{n^k} - \frac{(1-e^{-t})^n}{n^{k-1}}\right\} \\ &= \left(\frac{1-e^{-t}}{2^k} - \frac{1-e^{-t}}{2^{k-1}}\right) + \cdots \\ &= \left(\frac{1}{2^k} - \frac{1}{2^{k-1}}\right)t + \cdots.\end{aligned} \tag{2.14}$$

For any delta series $f(t)$, we have

$$\frac{f(t)}{t}x^n = f(t)\frac{1}{n+1}x^{n+1}. \tag{2.15}$$

Thus, by (2.13), (2.14) and (2.15), we get

$$\begin{aligned}T_{n+1}^{(r,k)}(x|\lambda) &= (x-r)T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda}T_n^{(r+1,k)}(x|\lambda) \\ &\quad - \sum_{l=0}^{n}\binom{n}{l}B_{n-l}\frac{1}{l+1}\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t}) - Li_{k-1}(1-e^{-t})}{1-e^{-t}}x^{l+1} \\ &= (x-r)T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda}T_n^{(r+1,k)}(x|\lambda) \\ &\quad - \sum_{l=0}^{n}\frac{\binom{n}{l}}{l+1}B_{n-l}\left\{T_{l+1}^{(r,k)}(x|\lambda) - T_{l+1}^{(r,k-1)}(x|\lambda)\right\} \\ &= (x-r)T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda}T_n^{(r+1,k)}(x|\lambda)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{n+1} \sum_{l=1}^{n+1} \binom{n+1}{l} B_{n+1-l} \left\{ T_l^{(r,k)}(x|\lambda) - T_l^{(r,k-1)}(x|\lambda) \right\} \\
&= (x-r) T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda} T_n^{(r+1,k)}(x|\lambda) \\
&\quad - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_{n+1-l} \left\{ T_l^{(r,k)}(x|\lambda) - T_l^{(r,k-1)}(x|\lambda) \right\} \\
&= (x-r) T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda} T_n^{(r+1,k)}(x|\lambda) \\
&\quad - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left\{ T_{n+1-l}^{(r,k)}(x|\lambda) - T_{n+1-l}^{(r,k-1)}(x|\lambda) \right\}.
\end{aligned}
\tag{2.16}
$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.3 *For $r, k \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$, we have*

$$
\begin{aligned}
T_{n+1}^{(r,k)}(x|\lambda) &= (x-r) T_n^{(r,k)}(x|\lambda) - \frac{r\lambda}{1-\lambda} T_n^{(r+1,k)}(x|\lambda) \\
&\quad - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left\{ T_{n+1-l}^{(r,k)}(x|\lambda) - T_{n+1-l}^{(r,k-1)}(x|\lambda) \right\}.
\end{aligned}
$$

Remark 1 If $r = 0$, then we have

$$
\sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{(1-e^{-t})} e^{xt} = \sum_{n=0}^{\infty} T_n^{(0,k)}(x|\lambda) \frac{t^n}{n!}.
\tag{2.17}
$$

Thus, by (2.17), we get $B_n^{(k)}(x) = T_n^{(0,k)}(x|\lambda)$.

From (2.4), we have

$$
\begin{aligned}
txT_n^{(r,k)}(x|\lambda) &= t\left(x \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) B_l^{(k)}(x) \right) \\
&= \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) \left\{ lxB_{l-1}^{(k)}(x) + B_l^{(k)}(x) \right\} \\
&= nx \sum_{l=0}^{n-1} \binom{n-1}{l} H_{n-1-l}^{(r)}(\lambda) B_l^{(k)}(x) + \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) B_l^{(k)}(x) \\
&= nxT_{n-1}^{(r,k)}(x|\lambda) + T_n^{(r,k)}(x|\lambda).
\end{aligned}
\tag{2.18}
$$

Applying t on both sides of Theorem 2.3, we get

$$
\begin{aligned}
&(n+1) T_n^{(r,k)}(x|\lambda) \\
&\quad = nxT_{n-1}^{(r,k)}(x|\lambda) + T_n^{(r,k)}(x|\lambda) - rnT_{n-1}^{(r,k)}(x|\lambda) - \frac{rn\lambda}{1-\lambda} T_{n-1}^{(r+1,k)}(x|\lambda) \\
&\qquad - \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l \left\{ (n+1-l) T_{n-l}^{(r,k)}(x|\lambda) - (n+1-l) T_{n-l}^{(r,k-1)}(x|\lambda) \right\}.
\end{aligned}
\tag{2.19}
$$

Thus, by (2.19), we have

$$\begin{aligned}(n+1)T_n^{(r,k)}(x|\lambda)+n\left(r-\frac{1}{2}-x\right)T_{n-1}^{(r,k)}(x|\lambda)+\sum_{l=0}^{n-2}\binom{n}{l}B_{n-l}T_l^{(r,k)}(x|\lambda)\\=-\frac{r\lambda n}{1-\lambda}T_{n-1}^{(r+1,k)}(x|\lambda)+\sum_{l=0}^{n}\binom{n}{l}B_{n-l}T_l^{(r,k-1)}(x|\lambda).\end{aligned}\tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 2.4 *For $r,k\in\mathbb{Z}$, $n\in\mathbb{Z}$ with $n\geq 2$, we have*

$$\begin{aligned}(n+1)T_n^{(r,k)}(x|\lambda)+n\left(r-\frac{1}{2}-x\right)T_{n-1}^{(r,k)}(x|\lambda)+\sum_{l=0}^{n-2}\binom{n}{l}B_{n-l}T_l^{(r,k)}(x|\lambda)\\=-\frac{r\lambda n}{1-\lambda}T_{n-1}^{(r+1,k)}(x|\lambda)+\sum_{l=0}^{n}\binom{n}{l}B_{n-l}T_l^{(r,k-1)}(x|\lambda).\end{aligned}$$

From (1.14) and (2.5), we note that

$$\begin{aligned}T_n^{(r,k)}(y|\lambda)&=\left\langle\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^n\right\rangle\\&=\left\langle\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|xx^{n-1}\right\rangle.\end{aligned}\tag{2.21}$$

By (1.15) and (2.21), we get

$$\begin{aligned}T_n^{(r,k)}(y|\lambda)&=\left\langle\partial_t\left(\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{yt}\right)\middle|x^{n-1}\right\rangle\\&=\left\langle\left(\partial_t\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\right)\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{yt}\middle|x^{n-1}\right\rangle\\&\quad+\left\langle\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\left(\partial_t\frac{Li_k(1-e^{-t})}{1-e^{-t}}\right)e^{yt}\middle|x^{n-1}\right\rangle\\&\quad+\left\langle\left(\frac{1-\lambda}{e^t-\lambda}\right)^r\frac{Li_k(1-e^{-t})}{1-e^{-t}}\partial_te^{yt}\middle|x^{n-1}\right\rangle.\end{aligned}\tag{2.22}$$

Therefore, by (2.22), we obtain the following theorem.

Theorem 2.5 *For $r,k\in\mathbb{Z}$, $n\geq 1$, we have*

$$\begin{aligned}T_n^{(r,k)}(x|\lambda)&=(x-r)T_{n-1}^{(r,k)}(x|\lambda)-\frac{r\lambda}{1-\lambda}T_{n-1}^{(r+1,k)}(x|\lambda)\\&\quad+\sum_{l=0}^{n-1}\left\{(-1)^{n-1-l}\binom{n-1}{l}\sum_{m=0}^{n-1-l}(-1)^m\frac{(m+1)!}{(m+2)^k}S_2(n-1-l,m)\right\}H_l^{(r)}(x-1|\lambda).\end{aligned}$$

Now, we compute $\langle(\frac{1-\lambda}{e^t-\lambda})^rLi_k(1-e^{-t})|x^{n+1}\rangle$ in two different ways.

On the one hand,

$$\begin{aligned}
&\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r Li_k\bigl(1-e^{-t}\bigr)\middle| x^{n+1}\right\rangle \\
&\quad = \left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}}\middle| \bigl(1-e^{-t}\bigr)x^{n+1}\right\rangle \\
&\quad = \left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}}\middle| x^{n+1}-(x-1)^{n+1}\right\rangle \\
&\quad = \sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}}\middle| x^m\right\rangle \\
&\quad = \sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}\bigl\langle 1\big| T_m^{(r,k)}(x|\lambda)\bigr\rangle \\
&\quad = \sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}T_m^{(r,k)}(\lambda).
\end{aligned} \tag{2.23}$$

On the other hand, we get

$$\begin{aligned}
&\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r Li_k\bigl(1-e^{-t}\bigr)\middle| x^{n+1}\right\rangle \\
&\quad = \left\langle Li_k\bigl(1-e^{-t}\bigr)\middle| \left(\frac{1-\lambda}{e^t-\lambda}\right)^r x^{n+1}\right\rangle \\
&\quad = \left\langle \int_0^t \bigl(Li_k\bigl(1-e^{-s}\bigr)\bigr)'\,ds\middle| H_{n+1}^{(r)}(x|\lambda)\right\rangle \\
&\quad = \left\langle \int_0^t e^{-s}\frac{Li_k(1-e^{-s})}{(1-e^{-s})}\,ds\middle| H_{n+1}^{(r)}(x|\lambda)\right\rangle \\
&\quad = \sum_{l=0}^{n}\left(\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}B_m^{(k-1)}\right)\frac{1}{l!}\left\langle \int_0^t s^l\,ds\middle| H_{n+1}^{(r)}(x|\lambda)\right\rangle \\
&\quad = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}\frac{B_m^{(k-1)}}{(l+1)!}\bigl\langle t^{l+1}\big| H_{n+1}^{(r)}(x|\lambda)\bigr\rangle \\
&\quad = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{l}{m}\binom{n+1}{l+1}(-1)^{l-m}B_m^{(k-1)}H_{n-l}^{(r)}(\lambda).
\end{aligned} \tag{2.24}$$

Therefore, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.6 *For $r,k\in\mathbb{Z}$, $n\in\mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n+1}{m}(-1)^{n-m}T_m^{(r,k)}(\lambda) \\
&\quad = \sum_{l=0}^{n}\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\binom{n+1}{l+1}B_m^{(k-1)}H_{n-l}^{(r)}(\lambda).
\end{aligned}$$

Now, we consider the following two Sheffer sequences:

$$\begin{aligned} &T_n^{(r,k)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right), \\ &\mathbb{B}^{(s)} \sim \left(\left(\frac{e^t-1}{t}\right)^s, t\right), \end{aligned} \tag{2.25}$$

where $s \in \mathbb{Z}_{\geq 0}$, $r, k \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. Let us assume that

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n\cdot m}\mathbb{B}_m^{(s)}(x). \tag{2.26}$$

By (1.21) and (2.26), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^t-1}{t}\right)^s \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} t^m \Big| x^n \right\rangle \\ &= \frac{1}{m!}\left\langle \left(\frac{e^t-1}{t}\right)^s \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \Big| t^m x^n \right\rangle \\ &= \binom{n}{m}\left\langle \left(\frac{e^t-1}{t}\right)^s \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \Big| x^{n-m} \right\rangle \\ &= \binom{n}{m}\sum_{l=0}^{n-m} \frac{s!}{(l+s)!} S_2(l+s,s) \left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \Big| t^l x^{n-m} \right\rangle \\ &= \binom{n}{m}\sum_{l=0}^{n-m} \frac{s!l!}{(l+s)!} \frac{(n-m)_l}{l!} S_2(l+s,s) \langle 1 | T_{n-m-l}^{(r,k)}(x|\lambda)\rangle \\ &= \binom{n}{m}\sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{s+l}{l}} S_2(l+s,s) T_{n-m-l}^{(r,k)}(\lambda). \end{aligned} \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.7 *For $r, k \in \mathbb{Z}$, $s \in \mathbb{Z}_{\geq 0}$, we have*

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{s+l}{l}} S_2(l+s,s) T_{n-m-l}^{(r,k)}(\lambda) \right\} \mathbb{B}_m^{(s)}(x).$$

From (1.3) and (2.1), we note that

$$\begin{aligned} &T_n^{(r,k)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right), \\ &E_n^{(r,s)}(x) \sim \left(\left(\frac{e^t+1}{2}\right)^s, t\right), \end{aligned} \tag{2.28}$$

where $r, k \in \mathbb{Z}$, $s \in \mathbb{Z}_{\geq 0}$.

By the same method, we get

$$T_n^{(r,k)}(x|\lambda) = \frac{1}{2^s}\sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{j=0}^{s} \binom{s}{j} T_{n-m}^{(r,k)}(j) \right\} E_m^{(s)}(x). \tag{2.29}$$

From (1.1) and (2.1), we note that

$$\begin{aligned} &T_n^{(r,k)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right), \\ &H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right), \end{aligned} \tag{2.30}$$

where $r, k \in \mathbb{Z}$, and $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 1$, $\mu \neq 1$, $s \in \mathbb{Z}_{\geq 0}$.

Let us assume that

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\mu). \tag{2.31}$$

By (1.21) and (2.31), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{e^t-\mu}{1-\mu}\right)^s \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} t^m \middle| x^n \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\mu)^s}\left\langle \left(e^t-\mu\right)^s \middle| \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \left\langle e^{jt} \middle| T_{n-m}^{(r,k)}(x|\lambda)\right\rangle \\ &= \frac{\binom{n}{m}}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} T_{n-m}^{(r,k)}(j|\lambda). \end{aligned} \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.8 *For $r, k \in \mathbb{Z}$, $s \in \mathbb{Z}_{\geq 0}$, we have*

$$T_n^{(r,k)}(x|\lambda) = \frac{1}{(1-\mu)^s} \sum_{m=0}^{n} \left\{ \binom{n}{m} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} T_{n-m}^{(r,k)}(j|\lambda) \right\} H_m^{(s)}(x|\mu).$$

It is known that

$$\begin{aligned} &T_n^{(r,k)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t\right), \\ &(x)_n \sim \left(1, e^t-1\right). \end{aligned} \tag{2.33}$$

Let

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n,m} (x)_m. \tag{2.34}$$

Then, by (1.21) and (2.34), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \left(e^t-1\right)^m \middle| x^n \right\rangle \\ &= \sum_{l=0}^{\infty} \frac{S_2(l+m,m)}{(l+m)!} \left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \middle| t^{m+l} x^n \right\rangle \end{aligned}$$

$$= \sum_{l=0}^{n-m} \frac{S_2(l+m,m)}{(l+m)!}(n)_{m+l}\left\langle 1 \middle| \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^{n-m-l} \right\rangle$$
$$= \sum_{l=0}^{n-m} \binom{n}{l+m} S_2(l+m,m) T_{n-m-l}^{(r,k)}(\lambda). \tag{2.35}$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.9 *For $r,k \in \mathbb{Z}$, we have*

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \binom{n}{l+m} S_2(l+m,m) T_{n-m-l}^{(r,k)}(\lambda) \right\} (x)_m.$$

Finally, we consider the following two Sheffer sequences:

$$\begin{aligned} &T_n^{(r,k)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^r \frac{1-e^{-t}}{Li_k(1-e^{-t})}, t \right), \\ &x^{[n]} \sim \left(1, 1-e^{-t}\right), \end{aligned} \tag{2.36}$$

where $x^{[n]} = x(x+1)\cdots(x+n-1)$.

Let us assume that

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} C_{n,m} x^{[m]}. \tag{2.37}$$

Then, by (1.21) and (2.37), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \left(1-e^{-t}\right)^m \middle| x^n \right\rangle \\ &= \sum_{l=0}^{\infty} \frac{(-1)^l S_2(l+m,m)}{(l+m)!} \left\langle \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} \middle| t^{m+l} x^n \right\rangle \\ &= \sum_{l=0}^{n-m} \frac{(-1)^l S_2(l+m,m)}{(l+m)!} (n)_{m+l} \left\langle 1 \middle| \left(\frac{1-\lambda}{e^t-\lambda}\right)^r \frac{Li_k(1-e^{-t})}{1-e^{-t}} x^{n-m-l} \right\rangle \\ &= \sum_{l=0}^{n-m} (-1)^l \binom{n}{l+m} S_2(l+m,m) T_{n-m-l}^{(r,k)}(\lambda). \end{aligned} \tag{2.38}$$

Therefore, by (2.37) and (2.38), we obtain the following theorem.

Theorem 2.10 *For $r,k \in \mathbb{Z}$, $n \geq 0$, we have*

$$T_n^{(r,k)}(x|\lambda) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} (-1)^l \binom{n}{l+m} S_2(l+m,m) T_{n-m-l}^{(r,k)}(\lambda) \right\} x^{[m]}.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

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References
1. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. **22**(3), 399-406 (2012)
2. Kim, T: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$. Rocky Mt. J. Math. **41**(1), 239-247 (2011)
3. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory **132**(1), 2854-2865 (2012)
4. Ryoo, C: A note on the Frobenius-Euler polynomials. Proc. Jangjeon Math. Soc. **14**(4), 495-501 (2011)
5. Ryoo, CS, Agarwal, RP: Exploring the multiple Changhee q-Bernoulli polynomials. Int. J. Comput. Math. **82**(4), 483-493 (2005)
6. Kim, DS, Kim, T, Kim, YH, Lee, SH: Some arithmetic properties of Bernoulli and Euler numbers. Adv. Stud. Contemp. Math. **22**(4), 467-480 (2012)
7. Kim, DS, Kim, T: Poly-Bernoulli polynomials arising from umbral calculus (communicated)
8. Kim, T: Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function. Russ. J. Math. Phys. **12**(2), 186-196 (2005)
9. Can, M, Cenkci, M, Kurt, V, Simsek, Y: Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions. Adv. Stud. Contemp. Math. **18**(2), 135-160 (2009)
10. Ding, D, Yang, J: Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials. Adv. Stud. Contemp. Math. **20**(1), 7-21 (2010)
11. Kim, T, Choi, J: A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on $\mathbb{Z}_p$. Adv. Stud. Contemp. Math. **22**(2), 215-223 (2012)
12. Kurt, B, Simsek, Y: On the generalized Apostol-type Frobenius-Euler polynomials. Adv. Differ. Equ. **2013**, 1 (2013)
13. Simsek, Y, Yurekli, O, Kurt, V: On interpolation functions of the twisted generalized Frobenius-Euler numbers. Adv. Stud. Contemp. Math. **15**(2), 187-194 (2007)
14. Roman, S: The Umbral Calculus. Pure and Applied Mathematics, vol. 111. Academic Press, New York (1984)
15. Roman, S, Rota, G-C: The umbral calculus. Adv. Math. **27**(2), 95-188 (1978)

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