Wang *Advances in Difference Equations* 2013, **2013**:247
http://www.advancesindifferenceequations.com/content/2013/1/247

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

New recurrence formulae for the Apostol-Bernoulli and Apostol-Euler polynomials

Jingzhe Wang*

*Correspondence: wangjingzhe729@126.com
Department of Mathematics, Northwest University, Xi'an, Shaanxi 710127, People's Republic of China

Abstract
The main purpose of this paper is by using the generating function methods and some combinatorial techniques to establish some new recurrence formulae for the Apostol-Bernoulli and Apostol-Euler polynomials. It turns out that some known results in (He and Wang in Adv. Differ. Equ. 2012:209, 2012) are obtained as special cases.
MSC: 11B68; 05A19

Keywords: Bernoulli numbers and polynomials; Apostol-Bernoulli numbers and polynomials; combinatorial identities

1 Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad \bigl(|t| < 2\pi\bigr) \tag{1.1}$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \bigl(|t| < \pi\bigr). \tag{1.2}$$

In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called the classical Bernoulli numbers and Euler numbers, respectively. These polynomials and numbers play important roles in various branches of mathematics including number theory, combinatorics, special functions and analysis, and there exist numerous interesting properties for them, see, for example, [1–3].

In 2007, Agoh and Dilcher [4] made use of some connections between the classical Bernoulli numbers and the Stirling numbers of the second kind to establish a quadratic recurrence formula on the classical Bernoulli numbers, which was generalized to the classical Bernoulli polynomials by He and Zhang [5]. More recently, He and Wang [6] extended the Agoh and Dilcher's quadratic recurrence formula on the classical Bernoulli numbers to the Apostol-Bernoulli and Apostol-Euler polynomials. As further applications, they derived some corresponding results related to some formulae of products of the classical Bernoulli and Euler polynomials and numbers stated in Nielsen's book [1].

We begin by recalling now the Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of (real or complex) higher order α, which were introduced by Luo and Srivastava [7, 8]

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}\quad\bigl(|t+\log\lambda|<2\pi;1^{\alpha}:=1\bigr) \tag{1.3}$$

and

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}\quad\bigl(|t+\log\lambda|<\pi;1^{\alpha}:=1\bigr). \tag{1.4}$$

Especially, the case $\alpha = 1$ in (1.3) and (1.4) is called the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$, respectively. Moreover, we call $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0;\lambda)$ the Apostol-Bernoulli numbers and $\mathcal{E}_n(\lambda) = 2^n\mathcal{E}_n(1/2;\lambda)$ the Apostol-Euler numbers. It is worth of mentioning that the Apostol-Bernoulli polynomials were introduced by Apostol [9] (see also Srivastava [10] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. For more results on these polynomials and numbers, one is referred to [11–16].

In this paper, we only consider the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$. By applying the generating function methods and some combinatorial techniques, developed in [6, 17], we establish some new recurrence formulae for the Apostol-Bernoulli and Apostol-Euler polynomials, by virtue of which, some known results including the ones presented in [6], are obtained as special cases.

2 Recurrence formulae for Apostol-Bernoulli polynomials

In what follows, we shall always denote by $\delta_{1,\lambda}$ the Kronecker symbol given by $\delta_{1,\lambda} = 0$ or 1, according to $\lambda \neq 1$ or $\lambda = 1$, and we also denote by $\beta_n(x;\lambda) = \mathcal{B}_{n+1}(x;\lambda)/(n+1)$ for any non-negative integer n. We first state the following.

Theorem 2.1 *Let k, m, n be any non-negative integers. Then*

$$\begin{aligned}&\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}\beta_{i+j}(x;\mu)\beta_{k+m+n-i-j}(y;\lambda\mu)\\&\quad=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\beta_{m-i}\left(x;\frac{1}{\lambda}\right)\beta_{k+n+i}(x+y;\lambda\mu)+\beta_m(y;\lambda)\beta_{k+n}(x+y;\mu)\\&\qquad-\frac{\delta_{1,\lambda}\beta_{k+m+n+1}(x+y;\lambda\mu)}{m+1}-\frac{\delta_{1,\mu}\beta_{k+m+n+1}(y;\lambda\mu)}{k+n+1}\\&\qquad-\frac{\delta_{1,\lambda\mu}(-1)^m m!(k+n)!}{(k+m+n+1)!}\beta_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).\end{aligned} \tag{2.1}$$

Proof Multiplying both sides of the identity

$$\frac{1}{\lambda e^u-1}\frac{1}{\mu e^v-1}=\left(\frac{\lambda e^u}{\lambda e^u-1}+\frac{1}{\mu e^v-1}\right)\frac{1}{\lambda\mu e^{u+v}-1} \tag{2.2}$$

by $e^{xv+y(u+v)}$ yields

$$\frac{e^{xv}}{\mu e^{v}-1}\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}=\frac{e^{yu}}{\lambda e^{u}-1}\frac{e^{(x+y)v}}{\mu e^{v}-1}-\frac{\lambda e^{(1-x)u}}{\lambda e^{u}-1}\frac{e^{(x+y)(u+v)}}{\lambda\mu e^{u+v}-1}. \tag{2.3}$$

Since $\mathcal{B}_0(x;\lambda)=1$ when $\lambda=1$ and $\mathcal{B}_0(x;\lambda)=0$ when $\lambda\neq 1$ (see, *e.g.*, [7]), then by setting $\mathcal{B}_0(x;\lambda)=\delta_{1,\lambda}$, we get

$$\frac{e^{xu}}{\lambda e^{u}-1}-\frac{\delta_{1;\lambda}}{u}=\sum_{m=0}^{\infty}\beta_m(x;\lambda)\frac{u^m}{m!}. \tag{2.4}$$

More generally, the Taylor theorem gives

$$\begin{aligned}\frac{e^{x(u+v)}}{\lambda e^{u+v}-1}-\frac{\delta_{1;\lambda}}{u+v}&=\sum_{n=0}^{\infty}\frac{\partial^n}{\partial u^n}\left(\frac{e^{xu}}{\lambda e^{u}-1}-\frac{\delta_{1;\lambda}}{u}\right)\frac{v^n}{n!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(x;\lambda)\frac{u^m}{m!}\frac{v^n}{n!}.\end{aligned} \tag{2.5}$$

Hence, applying (2.4) and (2.5) to (2.3), we get

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\beta_n(x;\mu)\frac{v^n}{n!}+\frac{\delta_{1,\mu}}{v}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(y;\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}+\frac{\delta_{1,\lambda\mu}}{u+v}\right)\\&\quad=\left(\sum_{m=0}^{\infty}\beta_m(y;\lambda)\frac{u^m}{m!}+\frac{\delta_{1,\mu}}{u}\right)\left(\sum_{n=0}^{\infty}\beta_n(x+y;\mu)\frac{v^n}{n!}+\frac{\delta_{1,\mu}}{v}\right)\\&\qquad-\lambda\left(\sum_{m=0}^{\infty}\beta_m(1-x;\lambda)\frac{u^m}{m!}+\frac{\delta_{1,\lambda}}{u}\right)\\&\qquad\times\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(x+y;\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}+\frac{\delta_{1,\lambda\mu}}{u+v}\right).\end{aligned} \tag{2.6}$$

Obverse that

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(x;\lambda)\frac{u^{m-1}}{m!}\frac{v^n}{n!}\\&\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta_{m+n+1}(x;\lambda)}{m+1}\frac{u^m}{m!}\frac{v^n}{n!}+\frac{1}{u}\sum_{n=0}^{\infty}\beta_n(x;\lambda)\frac{v^n}{n!}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(x;\lambda)\frac{u^{m}}{m!}\frac{v^{n-1}}{n!}\\&\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\beta_{m+n+1}(x;\lambda)}{n+1}\frac{u^m}{m!}\frac{v^n}{n!}+\frac{1}{v}\sum_{m=0}^{\infty}\beta_m(x;\lambda)\frac{u^m}{m!}.\end{aligned} \tag{2.8}$$

It follows from (2.6)-(2.8) that

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}\beta_n(x;\mu)\frac{v^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(y,\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\
&\quad=\left(\sum_{m=0}^{\infty}(-1)^m\beta_m\left(x;\frac{1}{\lambda}\right)\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n}(x+y;\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\
&\qquad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\beta_m(y,\lambda)\beta_n(x+y;\mu)-\frac{\delta_{1,\mu}\beta_{m+n+1}(y,\lambda\mu)}{n+1}\right.\\
&\qquad\left.-\frac{\delta_{1,\lambda}\beta_{m+n+1}(x+y,\lambda\mu)}{m+1}\right\}\frac{u^m}{m!}\frac{v^n}{n!}+\mathrm{M},
\end{aligned}\tag{2.9}$$

where M is denoted by

$$\begin{aligned}
\mathrm{M}&=\frac{\delta_{1,\mu}}{v}\sum_{m=0}^{\infty}\beta_m(y;\lambda)\frac{u^m}{m!}+\frac{\delta_{1,\lambda}}{u}\sum_{n=0}^{\infty}\beta_n(x+y;\mu)\frac{v^n}{n!}+\frac{\delta_{1,\lambda}\delta_{1,\mu}}{uv}\\
&\quad-\frac{\delta_{1,\mu}}{v}\sum_{m=0}^{\infty}\beta_m(y;\lambda\mu)\frac{u^m}{m!}-\frac{\delta_{1,\lambda\mu}}{u+v}\sum_{n=0}^{\infty}\beta_n(x;\mu)\frac{v^n}{n!}-\frac{\delta_{1,\mu}\delta_{1,\lambda\mu}}{v(u+v)}\\
&\quad-\frac{\lambda\delta_{1,\lambda\mu}}{u+v}\sum_{m=0}^{\infty}\beta_m(1-x;\lambda)\frac{u^m}{m!}-\frac{\delta_{1,\lambda}}{u}\sum_{n=0}^{\infty}\beta_n(x+y;\lambda\mu)\frac{v^n}{n!}-\frac{\delta_{1,\lambda}\delta_{1,\lambda\mu}}{u(u+v)}.
\end{aligned}\tag{2.10}$$

In view of (2.4), we have

$$\begin{aligned}
\mathrm{M}&=\frac{\delta_{1,\mu}}{v}\left(\frac{e^{yu}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u}\right)+\frac{\delta_{1,\lambda}}{u}\left(\frac{e^{(x+y)v}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)+\frac{\delta_{1,\lambda}\delta_{1,\mu}}{uv}\\
&\quad-\frac{\delta_{1,\mu}}{v}\left(\frac{e^{yu}}{\lambda\mu e^u-1}-\frac{\delta_{1,\lambda\mu}}{u}\right)-\frac{\delta_{1,\lambda\mu}}{u+v}\left(\frac{e^{xv}}{\mu e^v-1}-\frac{\delta_{1,\mu}}{v}\right)-\frac{\delta_{1,\mu}\delta_{1,\lambda\mu}}{v(u+v)}\\
&\quad-\frac{\lambda\delta_{1,\lambda\mu}}{u+v}\left(\frac{e^{(1-x)u}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u}\right)-\frac{\delta_{1,\lambda}}{u}\left(\frac{e^{(x+y)v}}{\lambda\mu e^v-1}-\frac{\delta_{1,\lambda\mu}}{v}\right)-\frac{\delta_{1,\lambda}\delta_{1,\lambda\mu}}{u(u+v)}\\
&=-\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{e^{(1-x)u}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u}+\frac{e^{xv}}{\frac{1}{\lambda}e^v-1}-\frac{\delta_{1,\frac{1}{\lambda}}}{v}\right).
\end{aligned}\tag{2.11}$$

If applying $u^n=\sum_{k=0}^{n}\binom{n}{k}(u+v)^k(-v)^{n-k}$ to (2.4), in view of changing the order of the summation, we obtain

$$\begin{aligned}
\frac{e^{(1-x)u}}{\lambda e^u-1}-\frac{\delta_{1,\lambda}}{u}&=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\beta_n(1-x;\lambda)}{n!}\binom{n}{k}(u+v)^k(-v)^{n-k}\\
&=\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\beta_n(1-x;\lambda)}{n!}\binom{n}{k+1}(u+v)^{k+1}(-v)^{n-(k+1)}\\
&\quad+\sum_{n=0}^{\infty}\beta_n(1-x;\lambda)\frac{(-v)^n}{n!}.
\end{aligned}\tag{2.12}$$

So from (2.4), (2.11), (2.12) and the symmetric distributions for the Apostol-Bernoulli polynomials $\lambda\mathcal{B}_n(1-x;\lambda) = (-1)^n\mathcal{B}_n(x;\frac{1}{\lambda})$, $n \geq 0$ (see, *e.g.*, [7]), we get

$$\mathrm{M} = -\delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^k\frac{\beta_n(x;\frac{1}{\lambda})}{n!}\binom{n}{k+1}(u+v)^k v^{n-(k+1)}. \tag{2.13}$$

Since $(u+v)^k = \sum_{m=0}^{k}\binom{k}{m}u^m v^{k-m}$ for non-negative integer k, then the identity above can be rewritten as

$$\mathrm{M} = -\delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\sum_{n=k+1}^{\infty}(-1)^k\frac{\beta_n(x;\frac{1}{\lambda})}{n!}\binom{n}{k+1}\binom{k}{m}u^m v^{n-m-1}. \tag{2.14}$$

It follows from (2.14) that

$$\begin{aligned}\mathrm{M} &= -\delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}(-1)^m\frac{\beta_n(x;\frac{1}{\lambda})}{n!}u^m v^{n-m-1}\\ &= -\delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\frac{m!n!\beta_{m+n+1}(x;\frac{1}{\lambda})}{(m+n+1)!}\frac{u^m}{m!}\frac{v^n}{n!}.\end{aligned} \tag{2.15}$$

Thus, combining (2.9) and (2.15), and then making k-times derivative with respect to v, we get

$$\begin{aligned}&\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{n=0}^{\infty}\beta_{n+j}(x;\mu)\frac{v^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n+k-j}(y,\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\ &= \left(\sum_{m=0}^{\infty}(-1)^m\beta_m\left(x;\frac{1}{\lambda}\right)\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{k+m+n}(x+y;\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\ &\quad+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\beta_m(y,\lambda)\beta_{k+n}(x+y;\mu)-\frac{\delta_{1,\mu}\beta_{k+m+n+1}(y,\lambda\mu)}{k+n+1}\right.\\ &\quad\left.-\frac{\delta_{1,\lambda}\beta_{k+m+n+1}(x+y,\lambda\mu)}{m+1}-\delta_{1,\lambda\mu}(-1)^m\frac{m!(k+n)!\beta_{k+m+n+1}(x;\frac{1}{\lambda})}{(k+m+n+1)!}\right\}\frac{u^m}{m!}\frac{v^n}{n!},\end{aligned} \tag{2.16}$$

which together with the Cauchy product arises the desired result after comparing the coefficients of $u^m v^n/m!n!$. □

It follows that we show a special case of Theorem 2.1. We have the following formula of products of the Apostol-Bernoulli polynomial due to He and Wang [17].

Corollary 2.2 *Let m, n be any positive integers. Then*

$$\begin{aligned}\mathcal{B}_m(x;\lambda)\mathcal{B}_n(y;\mu) &= n\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{B}_{m-i}\left(y-x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+i}(y;\lambda\mu)}{n+i}\\ &\quad+m\sum_{i=0}^{n}\binom{n}{i}\mathcal{B}_{n-i}(y-x;\mu)\frac{\mathcal{B}_{m+i}(x;\lambda\mu)}{m+i}\\ &\quad+(-1)^{m+1}\delta_{1,\lambda\mu}\frac{m!n!}{(m+n)!}\mathcal{B}_{m+n}\left(y-x;\frac{1}{\lambda}\right).\end{aligned} \tag{2.17}$$

Proof Setting $k = 0$ in Theorem 2.1, we get

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\frac{\mathcal{B}_{n+1-i}(x;\mu)}{n+1-i}\frac{\mathcal{B}_{m+1+i}(y;\lambda\mu)}{m+1+i} \\
&\quad=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\frac{\mathcal{B}_{m+1-i}(x;\frac{1}{\lambda})}{m+1-i}\frac{\mathcal{B}_{n+1+i}(x+y;\lambda\mu)}{n+1+i} \\
&\qquad+\frac{\mathcal{B}_{m+1}(y;\lambda)}{m+1}\frac{\mathcal{B}_{n+1}(x+y;\mu)}{n+1}-\frac{\delta_{1,\lambda}\mathcal{B}_{m+n+2}(x+y;\lambda\mu)}{(m+1)(m+n+2)} \\
&\qquad-\frac{\delta_{1,\mu}\mathcal{B}_{m+n+2}(y;\lambda\mu)}{(n+1)(m+n+2)}-\frac{\delta_{1,\lambda\mu}(-1)^{m}m!n!}{(m+n+2)!}\mathcal{B}_{m+n+2}\left(x;\frac{1}{\lambda}\right). \qquad (2.18)
\end{aligned}$$

Note that for any negative integers i, n,

$$\frac{1}{n+1-i}\binom{n}{i}=\frac{1}{n+1}\binom{n+1}{i}. \qquad (2.19)$$

It follows from (2.18) and (2.19) that

$$\begin{aligned}
&\frac{1}{n+1}\sum_{i=0}^{n+1}\binom{n+1}{i}\mathcal{B}_{n+1-i}(x;\mu)\frac{\mathcal{B}_{m+1+i}(y;\lambda\mu)}{m+1+i} \\
&\quad=-\frac{1}{m+1}\sum_{i=0}^{m+1}\binom{m+1}{i}(-1)^{m+1-i}\mathcal{B}_{m+1-i}\left(x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+1+i}(x+y;\lambda\mu)}{n+1+i} \\
&\qquad+\frac{\mathcal{B}_{m+1}(y;\lambda)}{m+1}\frac{\mathcal{B}_{n+1}(x+y;\mu)}{n+1}-\frac{\delta_{1,\lambda\mu}(-1)^{m}m!n!}{(m+n+2)!}\mathcal{B}_{m+n+2}\left(x;\frac{1}{\lambda}\right). \qquad (2.20)
\end{aligned}$$

Thus, replacing x by $y-x$ and y by x in (2.20) gives the desired result. □

We now use Theorem 2.1 to give another new recurrence formula for the Apostol-Bernoulli polynomials.

Theorem 2.3 *Let k, m, n be any non-negative integers. Then for $x+y+z=1$,*

$$\begin{aligned}
&(-1)^{k}\sum_{i=0}^{k}\binom{k}{i}\beta_{m+i}(x;\mu)\beta_{k+n-i}(y;\lambda\mu)+\frac{(-1)^{m}}{\mu}\sum_{i=0}^{m}\binom{m}{i}\beta_{n+i}(y;\lambda)\beta_{k+m-i}\left(z;\frac{1}{\mu}\right) \\
&\qquad+\frac{(-1)^{n}}{\lambda\mu}\sum_{i=0}^{n}\binom{n}{i}\beta_{k+i}\left(z;\frac{1}{\lambda\mu}\right)\beta_{m+n-i}\left(x;\frac{1}{\lambda}\right) \\
&\quad=-\left\{\delta_{1,\lambda\mu}\frac{(-1)^{k+n}k!n!}{(k+n+1)!}\beta_{k+m+n+1}\left(x;\frac{1}{\lambda}\right)\right. \\
&\qquad+\delta_{1,\mu}\frac{(-1)^{k+m}k!m!}{(k+m+1)!}\beta_{k+m+n+1}(y;\lambda\mu) \\
&\qquad\left.+\frac{\delta_{1,\lambda}}{\lambda\mu}\frac{(-1)^{m+n}m!n!}{(m+n+1)!}\beta_{k+m+n+1}\left(z;\frac{1}{\lambda\mu}\right)\right\}. \qquad (2.21)
\end{aligned}$$

Proof We firstly prove that for any non-negative integers k, m, n,

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\beta_{k+i}(x;\mu)\beta_{m+n-i}(y;\lambda\mu)\\
&\quad=\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\beta_{n+i}(x+y;\mu)\beta_{k+m-i}(y;\lambda)\\
&\qquad+\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\beta_{n+i}(x+y;\lambda\mu)\beta_{k+m-i}\left(x;\frac{1}{\lambda}\right)\\
&\qquad-\delta_{1,\lambda}\frac{k!m!}{(k+m+1)!}\beta_{k+m+n+1}(x+y;\lambda\mu)-\delta_{1,\mu}\frac{(-1)^k k!n!}{(k+n+1)!}\beta_{k+m+n+1}(y;\lambda\mu)\\
&\qquad-\delta_{1,\lambda\mu}\frac{(-1)^m m!n!}{(m+n+1)!}\beta_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).
\end{aligned}\tag{2.22}$$

We shall use induction on k. Clearly, (2.22) holds trivially when $k = 0$ in Theorem 2.1. Now assume (2.22) for any smaller value of k. In light of (2.1), we have

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\beta_{k+i}(x;\mu)\beta_{m+n-i}(y;\lambda\mu)\\
&\quad=\sum_{i=0}^{k+m}\binom{m}{i-k}(-1)^{k+m-i}\beta_{n+i}(x+y)\beta_{k+m-i}\left(x;\frac{1}{\lambda}\right)+\beta_m(y;\lambda)\beta_{n+k}(x+y;\mu)\\
&\qquad-\sum_{j=0}^{k-1}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}\beta_{k+m+n-i-j}(y;\lambda\mu)\beta_{i+j}(x;\mu)-\delta_{1,\lambda}\frac{\beta_{k+m+n+1}(x+y;\lambda\mu)}{m+1}\\
&\qquad-\delta_{1,\mu}\frac{\beta_{k+m+n+1}(y;\lambda\mu)}{k+n+1}-\delta_{1,\lambda\mu}\frac{(-1)^m m!(k+n)!}{(k+m+n+1)!}\beta_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).
\end{aligned}\tag{2.23}$$

Since (2.22) holds for any smaller value of k then

$$\begin{aligned}
&\sum_{j=0}^{k-1}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}\beta_{i+j}(x;\mu)\beta_{k+m+n-i-j}(y;\lambda\mu)\\
&\quad=\sum_{i=0}^{k+m}(-1)^i\beta_{n+i}(x+y;\mu)\beta_{k+m-i}(y;\lambda)\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{j}{i}\\
&\qquad+\sum_{i=0}^{k+m}(-1)^{k+m-i}\beta_{n+i}(x+y;\lambda\mu)\beta_{k+m-i}\left(x;\frac{1}{\lambda}\right)\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{k+m-j}{i}\\
&\qquad-\delta_{1,\lambda}\frac{\beta_{k+m+n+1}(x+y;\lambda\mu)}{k+m+1}\sum_{j=0}^{k-1}\frac{\binom{k}{j}}{\binom{k+m}{j}}\\
&\qquad-\delta_{1,\mu}\frac{\beta_{k+m+n+1}(y;\lambda\mu)}{n+1}\sum_{j=0}^{k-1}(-1)^j\frac{\binom{k}{j}}{\binom{n+1+j}{n+1}}\\
&\qquad-\delta_{1,\lambda\mu}\frac{(-1)^{k+m}\beta_{k+m+n+1}(x;\frac{1}{\lambda})}{n+1}\sum_{j=0}^{k-1}(-1)^j\frac{\binom{k}{j}}{\binom{k+m+n+1-j}{n+1}}.
\end{aligned}\tag{2.24}$$

Note that for non-negative integers i, k, m, n,

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{m+j}{i}=(-1)^k\binom{m}{i-k}, \tag{2.25}$$

$$\sum_{j=0}^{k}\frac{\binom{k}{j}}{\binom{m+k}{j}}=\frac{k+m+1}{m+1}, \tag{2.26}$$

$$\sum_{j=0}^{k}(-1)^j\frac{\binom{k}{j}}{\binom{m+n+j+1}{n+1}}=\frac{m!(n+k)!(n+1)}{(k+m+n+1)!} \tag{2.27}$$

by using induction on k. It follows from (2.25)-(2.27) that

$$\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{j}{i}=(-1)^k\binom{0}{i-k}-(-1)^k\binom{k}{i}, \tag{2.28}$$

$$\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{m+k-j}{i}=\binom{m}{i-k}-(-1)^k\binom{m}{i}, \tag{2.29}$$

$$\frac{1}{k+m+1}\sum_{j=0}^{k-1}\frac{\binom{k}{j}}{\binom{m+k}{j}}=\frac{1}{m+1}-\frac{k!m!}{(k+m+1)!}, \tag{2.30}$$

$$\frac{1}{n+1}\sum_{j=0}^{k-1}(-1)^j\frac{\binom{k}{j}}{\binom{n+j+1}{n+1}}=\frac{1}{k+n+1}-\frac{(-1)^k k!n!}{(k+n+1)!}, \tag{2.31}$$

$$\frac{1}{n+1}\sum_{j=0}^{k-1}(-1)^j\frac{\binom{k}{j}}{\binom{m+n+k+1-j}{n+1}}=\frac{(-1)^k m!(n+k)!}{(k+m+n+1)!}-\frac{(-1)^k m!n!}{(m+n+1)!}. \tag{2.32}$$

Hence, applying the five identities (2.28)-(2.32) above to (2.24), and then combining (2.23), gives (2.22). Thus, by setting $x+y+z=1$ in (2.22), and using the symmetric distributions of the Apostol-Bernoulli polynomials, we complete the proof of Theorem 2.3 by replacing n by k, k by m and m by n. □

We now give a special case of Theorem 2.3. We have the following quadratic recurrence formula for the Apostol-Bernoulli polynomials, presented in [6].

Corollary 2.4 *Let k, m, n be any non-negative integers. Then*

$$\begin{aligned}\sum_{i=0}^{n}&\binom{n}{i}\mathcal{B}_{k+i}(x;\mu)\mathcal{B}_{m+n-i}(y;\lambda\mu)\\&=\sum_{i=0}^{k+m}(-1)^{k+i}\left\{n\binom{k}{i}-m\binom{k}{i-1}\right\}\mathcal{B}_{n-1+i}(x+y;\mu)\frac{\mathcal{B}_{k+m+1-i}(y;\lambda)}{k+m+1-i}\\&\quad+\sum_{i=0}^{k+m}(-1)^{m+i}\left\{n\binom{m}{i}-k\binom{m}{i-1}\right\}\mathcal{B}_{n-1+i}(x+y;\lambda\mu)\frac{\mathcal{B}_{k+m+1-i}(x;\frac{1}{\lambda})}{k+m+1-i}\\&\quad-\delta_{1,\lambda}\frac{k!m!(n+\delta(k,m)(k+m+1))}{(k+m+1)!}\mathcal{B}_{k+m+n}(x+y;\lambda\mu),\end{aligned} \tag{2.33}$$

where $\delta(k,m) = -1$ *when* $k = m = 0$, $\delta(k,m) = 0$ *when* $k = 0$, $m \geq 1$ *or* $m = 0$, $k \geq 1$, *and* $\delta(k,m) = 1$, *otherwise*.

Proof Since the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ satisfy the difference equation $\partial/\partial x(\mathcal{B}_n(x;\lambda)) = n\mathcal{B}_{n-1}(x;\lambda)$ for any positive integer n (see, *e.g.*, [7]), so by substituting n for k, k for m and m for n and making the derivative operation $\partial/\partial x \cdot \partial/\partial y$ in both sides of Theorem 2.3, and then using the symmetric distributions of the Apostol-Bernoulli polynomials, the desired result follows immediately. □

Remark 2.5 In fact, we can also use Theorem 2.3 to obtain the above formula of products of the Apostol-Bernoulli polynomials. For example, setting $m = 0$ and substituting $y - x$ for x, x for y, n for k and m for n in Theorem 2.3, with the help of the symmetric distributions of the Apostol-Bernoulli polynomials, Corollary 2.2 follows immediately. For some applications of Corollaries 2.2 and 2.4, the interested readers may consult [6, 17, 18].

3 Recurrence formulae for mixed Apostol-Bernoulli and Apostol-Euler polynomials

We next give a similar formula to Theorem 2.3, which is involving the mixed Apostol-Bernoulli and Apostol-Euler polynomials. As in the proof of Theorem 2.3, we need the following formula concerning the mixed Apostol-Bernoulli and Apostol-Euler polynomials.

Theorem 3.1 *Let* k, m, n *be any non-negative integers. Then*

$$\begin{aligned}&\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{i+j}(x;\mu)\beta_{k+m+n-i-j}(y;\lambda\mu)\\&\quad=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{E}_{m-i}\left(x;\frac{1}{\lambda}\right)\beta_{k+n+i}(x+y;\lambda\mu)-\frac{1}{2}\mathcal{E}_m(y;\lambda)\mathcal{E}_{k+n}(x+y;\mu)\\&\qquad-\delta_{1,\lambda\mu}\frac{(-1)^m m!(k+n)!}{(k+m+n+1)!}\mathcal{E}_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).\end{aligned}\tag{3.1}$$

Proof Multiplying both sides of the identity

$$\frac{1}{\lambda e^u+1}\frac{1}{\mu e^v+1}=\left(\frac{\lambda e^u}{\lambda e^u+1}-\frac{1}{\mu e^v+1}\right)\frac{1}{\lambda\mu e^{u+v}-1}\tag{3.2}$$

by $2e^{xv+y(u+v)}$ yields

$$\frac{2e^{xv}}{\mu e^v+1}\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}=\lambda\frac{2e^{(1-x)u}}{\lambda e^u+1}\frac{e^{(x+y)(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{1}{2}\frac{2e^{yu}}{\lambda e^u+1}\frac{2e^{(x+y)v}}{\mu e^v+1},\tag{3.3}$$

which means

$$\begin{aligned}&\frac{2e^{xv}}{\mu e^v+1}\left(\frac{e^{y(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right)\\&\quad=\lambda\frac{2e^{(1-x)u}}{\lambda e^u+1}\left(\frac{e^{(x+y)(u+v)}}{\lambda\mu e^{u+v}-1}-\frac{\delta_{1,\lambda\mu}}{u+v}\right)-\frac{1}{2}\frac{2e^{yu}}{\lambda e^u+1}\frac{2e^{(x+y)v}}{\mu e^v+1}\\&\qquad+\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{2e^{(1-x)u}}{\lambda e^u+1}-\frac{2e^{xv}}{\mu e^u+1}\right).\end{aligned}\tag{3.4}$$

In a similar consideration to (2.12), we have

$$\begin{aligned}\lambda\left(\frac{2e^{(1-x)u}}{\lambda e^{u}+1}\right) &= \lambda\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{\mathcal{E}_n(1-x;\lambda)}{n!}\binom{n}{k}(u+v)^k(-v)^{n-k}\\ &= \lambda\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\frac{\mathcal{E}_n(1-x;\lambda)}{n!}\binom{n}{k+1}(u+v)^{k+1}(-v)^{n-(k+1)}\\ &\quad+\lambda\sum_{n=0}^{\infty}\mathcal{E}_n(1-x;\lambda)\frac{(-v)^n}{n!}. \end{aligned}\tag{3.5}$$

So from the symmetric distributions for the Apostol-Euler polynomials $\lambda\mathcal{E}_n(1-x;\lambda) = (-1)^n\mathcal{E}_n(x;\frac{1}{\lambda})$, $n \geq 0$ (see, *e.g.*, [7]), we get

$$\begin{aligned}&\frac{\delta_{1,\lambda\mu}}{u+v}\left(\lambda\frac{2e^{(1-x)u}}{\lambda e^{u}+1}-\frac{2e^{xv}}{\mu e^{u}+1}\right)\\ &\quad= \delta_{1,\lambda\mu}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}(-1)^{k+1}\frac{\mathcal{E}_n(x;\frac{1}{\lambda})}{n!}\binom{n}{k+1}(u+v)^k v^{n-(k+1)}\\ &\quad= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\sum_{n=k+1}^{\infty}(-1)^{k+1}\frac{\mathcal{E}_n(x;\frac{1}{\lambda})}{n!}\binom{n}{k+1}\binom{k}{m}u^m v^{n-m-1}\\ &\quad= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=m+1}^{\infty}(-1)^{m+1}\frac{\mathcal{E}_n(x;\frac{1}{\lambda})}{n!}u^m v^{n-m-1}\\ &\quad= \delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+1}\frac{m!n!\mathcal{E}_{m+n+1}(x;\frac{1}{\lambda})}{(m+n+1)!}\frac{u^m}{m!}\frac{v^n}{n!}. \end{aligned}\tag{3.6}$$

Applying (2.5) and (3.6) to (3.4), and then making k-times derivative with respect to v, we obtain

$$\begin{aligned}&\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{n=0}^{\infty}\mathcal{E}_{n+j}(x;\mu)\frac{v^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{m+n+k-j}(y,\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\ &\quad= \left(\sum_{m=0}^{\infty}(-1)^m\mathcal{E}_m\left(x;\frac{1}{\lambda}\right)\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\beta_{k+m+n}(x+y;\lambda\mu)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\ &\qquad-\frac{1}{2}\left(\sum_{m=0}^{\infty}\mathcal{E}_m(y;\lambda)\frac{u^m}{m!}\right)\left(\sum_{n=0}^{\infty}\mathcal{E}_{k+n}(x+y;\mu)\frac{v^n}{n!}\right)\\ &\qquad-\delta_{1,\lambda\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^m\frac{m!(k+n)!\mathcal{E}_{k+m+n+1}(x;\frac{1}{\lambda})}{(k+m+n+1)!}\frac{u^m}{m!}\frac{v^n}{n!}, \end{aligned}\tag{3.7}$$

which together with the Cauchy product gives the desired result by comparing the coefficients of $u^m v^n/m!n!$. □

It follows that we show a special of Theorem 3.1. We have the following formulae of products of the Apostol-Euler polynomials due to He and Wang [17].

Corollary 3.2 *Let m, n be any non-negative integers. Then*

$$\begin{aligned}\mathcal{E}_m(x;\lambda)\mathcal{E}_n(y;\mu) &= 2\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{E}_{m-i}\left(y-x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+1+i}(y;\lambda\mu)}{n+1+i}\\ &\quad-2\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{n-i}(y-x;\mu)\frac{\mathcal{B}_{m+1+i}(x;\lambda\mu)}{m+1+i}\\ &\quad-2\delta_{1,\lambda\mu}\frac{(-1)^m m!n!}{(m+n+1)!}\mathcal{E}_{m+n+1}\left(y-x;\frac{1}{\lambda}\right).\end{aligned} \tag{3.8}$$

Proof Setting $k=0$ in Theorem 3.1, we get

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{n-i}(x;\mu)\frac{\mathcal{B}_{m+1+i}(y;\lambda\mu)}{m+1+i}\\ &\quad=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{E}_{m-i}\left(x;\frac{1}{\lambda}\right)\frac{\mathcal{B}_{n+1+i}(x+y;\lambda\mu)}{n+1+i}-\frac{1}{2}\mathcal{E}_m(y;\lambda)\mathcal{E}_n(x+y;\mu)\\ &\qquad-\delta_{1,\lambda\mu}\frac{(-1)^m m!n!}{(m+n+1)!}\mathcal{E}_{m+n+1}\left(x;\frac{1}{\lambda}\right).\end{aligned} \tag{3.9}$$

Thus, the desired result follows by replacing x by $y-x$ and y by x in (3.9). □

Now we apply Theorem 3.1 to give the following another recurrence formula for the mixed Apostol-Bernoulli and Apostol-Euler polynomials.

Theorem 3.3 *Let k, m, n be any non-negative integers. Then for* $x+y+z=1$,

$$\begin{aligned}&(-1)^k\sum_{i=0}^{k}\binom{k}{i}\mathcal{E}_{m+i}(x;\mu)\beta_{k+n-i}(y;\lambda\mu)+\frac{(-1)^m}{2\mu}\sum_{i=0}^{m}\binom{m}{i}\mathcal{E}_{n+i}(y;\lambda)\mathcal{E}_{k+m-i}\left(z;\frac{1}{\mu}\right)\\ &\quad+\frac{(-1)^n}{\lambda\mu}\sum_{i=0}^{n}\binom{n}{i}\beta_{k+i}\left(z;\frac{1}{\lambda\mu}\right)\mathcal{E}_{m+n-i}\left(x;\frac{1}{\lambda}\right)\\ &\quad=-\delta_{1,\lambda\mu}\frac{(-1)^{k+n}k!n!}{(k+n+1)!}\mathcal{E}_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).\end{aligned} \tag{3.10}$$

Proof We firstly prove that for any non-negative integers k, m, n,

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{k+i}(x;\mu)\beta_{m+n-i}(y;\lambda\mu)\\ &\quad=-\frac{1}{2}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{E}_{n+i}(x+y;\mu)\mathcal{E}_{k+m-i}(y;\lambda)\\ &\qquad+\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\beta_{n+i}(x+y;\lambda\mu)\mathcal{E}_{k+m-i}\left(x;\frac{1}{\lambda}\right)\\ &\qquad-\delta_{1,\lambda\mu}\frac{(-1)^m m!n!}{(m+n+1)!}\mathcal{E}_{k+m+n+1}\left(x;\frac{1}{\lambda}\right).\end{aligned} \tag{3.11}$$

The proof is similar to that of (2.22), and, therefore, we leave out some of the more obvious details. Clearly, the case $k = 0$ in (3.11) is complete. Next, consider the case $k \geq 1$ in (3.11). Assume that (3.11) holds for all positive integers being less than k. In light of (3.1), we have

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{k+i}(x;\mu)\beta_{m+n-i}(y;\lambda\mu) \\
&\quad=\sum_{i=0}^{k+m}\binom{m}{i-k}(-1)^{k+m-i}\mathcal{E}_{k+m-i}\left(x;\frac{1}{\lambda}\right)\beta_{n+i}(x+y;\lambda\mu) \\
&\qquad-\frac{1}{2}\mathcal{E}_m(y;\lambda)\mathcal{E}_{k+n}(x+y;\mu)-\delta_{1,\lambda\mu}\frac{(-1)^m m!(k+n)!}{(k+m+n+1)!}\mathcal{E}_{k+m+n+1}\left(x;\frac{1}{\lambda}\right) \\
&\qquad-\sum_{j=0}^{k-1}\binom{k}{j}\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{i+j}(x;\mu)\beta_{k+m+n-i-j}(y;\lambda\mu).
\end{aligned}\tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{k+i}(x;\mu)\beta_{m+n-i}(y;\lambda\mu) \\
&\quad=\sum_{i=0}^{k+m}\binom{m}{i-k}(-1)^{k+m-i}\mathcal{E}_{k+m-i}\left(x;\frac{1}{\lambda}\right)\beta_{n+i}(x+y;\lambda\mu) \\
&\qquad-\frac{1}{2}\mathcal{E}_m(y;\lambda)\mathcal{E}_{k+n}(x+y;\mu)-\delta_{1,\lambda\mu}\frac{(-1)^m m!(k+n)!}{(k+m+n+1)!}\mathcal{E}_{k+m+n+1}\left(x;\frac{1}{\lambda}\right) \\
&\qquad-\sum_{i=0}^{k+m}(-1)^{k+m-i}\beta_{n+i}(x+y;\lambda\mu)\mathcal{E}_{k+m-i}\left(x;\frac{1}{\lambda}\right)\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{k+m-j}{i} \\
&\qquad+\frac{1}{2}\sum_{i=0}^{k+m}(-1)^i\mathcal{E}_{n+i}(x+y;\mu)\mathcal{E}_{k+m-i}(y;\lambda)\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{j}{i} \\
&\qquad+\delta_{1,\lambda\mu}\frac{(-1)^{k+m}\mathcal{E}_{k+m+n+1}(x;\frac{1}{\lambda})}{n+1}\sum_{j=0}^{k-1}(-1)^j\frac{\binom{k}{j}}{\binom{k+m+n+1-j}{n+1}}.
\end{aligned}\tag{3.13}$$

Hence, applying (2.28), (2.29) and (2.32) to (3.13), we conclude the induction step. Thus, by setting $x + y + z = 1$ in (3.11), and applying the symmetric distributions of Apostol-Euler polynomials, we complete the proof of Theorem 3.3 by replacing n by k, k by m and m by n. □

We next give some special cases of Theorem 3.3. We have the following formula products of the mixed Apostol-Bernoulli and Apostol-Euler polynomials due to He and Wang [17].

Corollary 3.4 *Let m be any non-negative integer. Then for positive integer n,*

$$\begin{aligned}
\mathcal{E}_m(x;\lambda)\mathcal{B}_n(y;\mu)&=\frac{n}{2}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{E}_{m-i}\left(y-x;\frac{1}{\lambda}\right)\mathcal{E}_{n+i-1}(y;\lambda\mu) \\
&\quad+\sum_{i=0}^{n}\binom{n}{i}\mathcal{B}_{n-i}(y-x;\mu)\mathcal{E}_{m+i}(x;\lambda\mu).
\end{aligned}\tag{3.14}$$

Proof Setting $k = 0$, and substituting $1 - y$ for y, $y - x$ for z, λ for μ and $\frac{1}{\lambda\mu}$ for λ in Theorem 3.3, by the symmetric distributions of the Apostol-Bernoulli and Apostol-Euler polynomials, the desired result follows immediately. □

We next apply Theorem 3.3 to give the following quadratic recurrence formulae for the mixed Apostol-Bernoulli and Apostol-Euler polynomials presented in [6].

Corollary 3.5 *Let k, m, n be any non-negative integers. Then*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{k+i}(x;\mu)\mathcal{B}_{m+n-i}(y;\lambda\mu)\\
&\quad=\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{B}_{n+i}(x+y;\lambda\mu)\mathcal{E}_{k+m-i}\left(x;\frac{1}{\lambda}\right)\\
&\qquad-\frac{1}{2}\sum_{i=0}^{k+m}(-1)^{k-i}\left\{n\binom{k}{i}-m\binom{k}{i-1}\right\}\mathcal{E}_{n-1+i}(x+y;\mu)\mathcal{E}_{k+m-i}(y;\lambda).
\end{aligned}\tag{3.15}$$

Proof Substituting n for k, k for m and m for n and making the derivative operation $\partial/\partial y$ in both sides of Theorem 3.3, and then using the difference equation and symmetric distributions of the Apostol-Bernoulli polynomials gives the desired result. □

Corollary 3.6 *Let k, m, n be any non-negative integers. Then*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}\mathcal{E}_{k+i}(x;\mu)\mathcal{E}_{m+n-i}(y;\lambda\mu)\\
&\quad=2\delta_{1,\lambda}\frac{k!m!}{(k+m+1)!}\mathcal{E}_{k+m+n+1}(x+y;\lambda\mu)\\
&\qquad-2\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\mathcal{E}_{n+i}(x+y;\lambda\mu)\frac{\mathcal{B}_{k+m+1-i}(x;\frac{1}{\lambda})}{k+m+1-i}\\
&\qquad-2\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\mathcal{E}_{n+i}(x+y;\mu)\frac{\mathcal{B}_{k+m+1-i}(y;\lambda)}{k+m+1-i}.
\end{aligned}\tag{3.16}$$

Proof Substituting μ for λ, $\frac{1}{\lambda\mu}$ for μ, x for y and y for z in Theorem 3.3, by applying the symmetric distributions of the Apostol-Euler polynomials, the desired result follows immediately. □

Remark 3.7 We also mention that Theorem 3.3 above can also be used to obtain the formulae of products of the mixed Apostol-Bernoulli and Apostol-Euler polynomials. For example, setting $m = 0$, and substituting $y-x$ for x, x for y, n for k and m for n in Theorem 3.3, with the help of the symmetric distributions of the Apostol-Bernoulli polynomials, Corollary 3.2 follows immediately. For some corresponding applications of Corollaries 3.2, 3.4, 3.5 and 3.6, one is referred to [6, 17, 18].

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author would like to express her gratitude to the referees for their helpful comments and suggestions in improving this paper.

Received: 16 May 2013 Accepted: 31 July 2013 Published: 14 August 2013

References

1. Nielsen, N: Traité élémentaire des nombres de Bernoulli, pp. 75-78. Gauthier-Villars, Paris (1923)
2. Nörlund, NE: Vorlesungen über Differnzenrechnung. Springer, Berlin (1924)
3. Srivastava, HM, Choi, J: Series Associated with the Zeta and Related Functions. Kluwer Academic, Dordrecht (2001)
4. Agoh, T, Dilcher, K: Convolution identities and lacunary recurrences for Bernoulli numbers. J. Number Theory **124**, 105-122 (2007)
5. He, Y, Zhang, WP: A convolution formula for Bernoulli polynomials. Ars Comb. **108**, 97-104 (2013)
6. He, Y, Wang, CP: Recurrence formulae for Apostol-Bernoulli and Apostol-Euler polynomials. Adv. Differ. Equ. **2012**, 209 (2012)
7. Luo, QM, Srivastava, HM: Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. J. Math. Anal. Appl. **308**, 290-302 (2005)
8. Luo, QM: Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions. Taiwan. J. Math. **10**, 917-925 (2006)
9. Apostol, TM: On the Lerch zeta function. Pac. J. Math. **1**, 161-167 (1951)
10. Srivastava, HM: Some formulas for the Bernoulli and Euler polynomials at rational arguments. Math. Proc. Camb. Philos. Soc. **129**, 77-84 (2000)
11. Boyadzhiev, KN: Apostol-Bernoulli functions, derivative polynomials and Eulerian polynomials. Adv. Appl. Discrete Math. **1**, 109-122 (2008)
12. Garg, M, Jain, K, Srivastava, HM: Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions. Integral Transforms Spec. Funct. **17**, 803-815 (2006)
13. Luo, QM, Srivastava, HM: Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials. Comput. Math. Appl. **51**, 631-642 (2006)
14. Luo, QM: Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials. Math. Comput. **78**, 2193-2208 (2009)
15. Luo, QM: The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order. Integral Transforms Spec. Funct. **20**, 377-391 (2009)
16. Srivastava, HM, Todorov, PG: An explicit formula for the generalized Bernoulli polynomials. J. Math. Anal. Appl. **130**, 509-513 (1988)
17. He, Y, Wang, CP: Some formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials. Discrete Dyn. Nat. Soc. **2012**, Article ID 927953 (2012)
18. He, Y, Zhang, WP: Some sum relations involving Bernoulli and Euler polynomials. Integral Transforms Spec. Funct. **22**, 207-215 (2011)

doi:10.1186/1687-1847-2013-247
Cite this article as: Wang: New recurrence formulae for the Apostol-Bernoulli and Apostol-Euler polynomials. *Advances in Difference Equations* 2013 **2013**:247.