Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The Korteweg-de Vries equation and a Diophantine problem related to Bernoulli polynomials

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Dedicated to Professor Hari M Srivastava

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Abstract
Some Diophantine equations related to the soliton solutions of the Korteweg-de Vries equation are resolved. The main tools are the connection with Bernoulli polynomials and the application of certain computational number-theoretical results.
MSC: Primary 11D41; 14H45; secondary 11Y50

Keywords: Diophantine equations; curves of genus 2; Korteweg-de Vries equation

1 Introduction

In the paper [1], Fairlie and Veselov obtained a relation of the Bernoulli polynomials with the theory of the Korteweg-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

This equation has infinitely many conservation laws (that is, certain laws, which show that a particular measurable property of an isolated physical system, like mass, energy, momentum, *etc.*, does not change as the system evolves) of the form

$$I_m[u] = \int_{-\infty}^{\infty} P_m(u, u_x, u_{xx}, \dots, u_m)\, dx,$$

where P_m are some polynomials of the function u and its x-derivatives up to order m, see [2]. For example,

$$I_{-1}[u] = \int_{-\infty}^{\infty} u\, dx, \qquad I_0[u] = \int_{-\infty}^{\infty} u^2\, dx, \qquad I_1[u] = \int_{-\infty}^{\infty} \left(u_x^2 + 2u^3\right) dx$$

and

$$I_2[u] = \int_{-\infty}^{\infty} \left(u_{xx}^2 + 10uu_x^2 + 5u^4\right) dx.$$

The KdV equation possesses a remarkable family of so-called n-soliton solutions corresponding to the initial profile $u_n(x,0) = -2n(n+1)\operatorname{sech}^2 x$. For some recent generalizations and applications of the Korteweg-de Vries equation, we refer to [3, 4] and [5] and the references given therein.

Using the spectral theory of Schrödinger operators, see [6], Fairlie and Veselov [1] proved that there is a strong connection between the improver integrals related to the functions $u_n(x,0)$ above and the well-known discrete sums of power values, namely,

$$I_k[u_n] = \frac{(-1)^k 4^{k+2}}{2k+3} \sum_{i=1}^{n} i^{2k+3}$$

for $k = -1, 0, 1, \ldots$.

Now, let $k \neq l$ be fixed integers with $k, l \in \{-1, 0, 1, 2, \ldots\}$, and suppose that

$$\left|I_k[u_n]\right| = \left|I_l[u_m]\right|.$$

One can ask how often can these integrals be equal for given k and l? In other words, what is the cardinality of the set of solutions m, n to the equation

$$\frac{4^k}{2k+3} \sum_{i=1}^{n} i^{2k+3} = \frac{4^l}{2l+3} \sum_{i=1}^{m} i^{2l+3}, \tag{1}$$

where k and l are fixed distinct integers? Of course, one can consider the much more general problem, when k and l are also unknown integers; however, in this case, the solution of the corresponding equation seems beyond the reach of current techniques.

Applying some recent results by Rakaczki, see [7] and [8], it is not too hard to give some ineffective and effective finiteness statements for the solutions m and n to equation (1). However, the purpose of this note is to resolve (1) for certain values of m and n, including an infinite family of the parameters.

Theorem 1 *For $k = -1$ and $l \in \{0, 1, 2, 3\}$, equation* (1) *has only one solution, namely,* $(l, m, n) = (0, 24, 5)$.

Theorem 2 *Assume that $k = 0$ and l is a positive integer such that $2l + 3$ is prime. Then* (1) *has no solution in positive integers m and n.*

2 Auxiliary results

In our first lemma, we summarize some classical properties of Bernoulli polynomials. For the proofs of these results, we refer to [9].

Lemma 1 *Let $B_j(X)$ denote the jth Bernoulli polynomial and $B_j = B_j(0)$, $j = 1, 2, \ldots$. Further, let D_j be the denominator of B_j. Then we have*

(A) $B_j(X) = X^n + \sum_{i=1}^{j} \binom{j}{i} B_i X^{j-i}$,
(B) $S_j(x) = 1^j + 2^j + \cdots + (x-1)^j = \frac{1}{j+1}(B_{j+1}(x) - B_{j+1})$,
(C) $B_1 = -\frac{1}{2}$, $B_{2j+1} = 0$, $j = 1, 2, \ldots$,
(D) (*von Staudt-Clausen*) $D_{2j} = \prod_{p-1|2j, p \text{ prime}} p$,
(E) $X^2(X-1)^2 | B_{2j}(X) - B_{2j}$ (*in* $\mathbb{Q}[X]$),
(F) $B_j(X) = (-1)^j B_j(1-X)$.

Consider the hyperelliptic curve

$$\mathcal{C}: y^2 = F(x) := x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \tag{2}$$

where $b_i \in \mathbb{Z}$. Let α be a root of F, and let $J(\mathbb{Q})$ be the Jacobian of the curve $\mathcal{C}$. We have that

$$x - \alpha = \kappa\xi^2,$$

where $\kappa, \xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell-Weil group of the curve $\mathcal{C}$, it is possible to provide a method to compute such a finite set. To each coset representative $\sum_{i=1}^{m}(P_i - \infty)$ of $J(\mathbb{Q})/2J(\mathbb{Q})$, we associate

$$\kappa = \prod_{i=1}^{m}\left(\gamma_i - \alpha d_i^2\right),$$

where the set $\{P_1, \dots, P_m\}$ is stable under the action of Galois, all $y(P_i)$ are non-zero and $x(P_i) = \gamma_i/d_i^2$, where γ_i is an algebraic integer and $d_i \in \mathbb{Z}_{\geq 1}$. If P_i, P_j are conjugate, then we may suppose that $d_i = d_j$, and so, γ_i, γ_j are conjugate. We have the following lemma (Lemma 3.1 in [10]).

Lemma 2 *Let $\mathcal{K}$ be a set of κ values, associated as above to a complete set of coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$. Then $\mathcal{K}$ is a finite subset of $\mathcal{O}_K$, and if (x, y) is an integral point on the curve* (2), *then $x - \alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}$ and $\xi \in K$.*

As an application of his theory of lower bounds for linear forms in logarithms, Baker [11] gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see, *e.g.*, [12–18] and [19]).

In [10], an improved completely explicit upper bound were proved combining ideas from [15, 19–25]. Now we will state the theorem, which gives the improved bound. We introduce some notation. Let K be a number field of degree d, and let r be its unit rank, and let R be its regulator. For $\alpha \in K$, we denote by $\mathrm{h}(\alpha)$ the logarithmic height of the element α. Let

$$\partial_K = \begin{cases} \frac{\log 2}{d} & \text{if } d = 1, 2, \\ \frac{1}{4}\left(\frac{\log\log d}{\log d}\right)^3 & \text{if } d \geq 3, \end{cases}$$

and let

$$\partial_K' = \left(1 + \frac{\pi^2}{\partial_K^2}\right)^{1/2}.$$

Define the constants

$$c_1(K) = \frac{(r!)^2}{2^{r-1}d^r}, \qquad c_2(K) = c_1(K)\left(\frac{d}{\partial_K}\right)^{r-1},$$

$$c_3(K) = c_1(K)\frac{d^r}{\partial_K}, \qquad c_4(K) = rdc_3(K), \qquad c_5(K) = \frac{r^{r+1}}{2\partial_K^{r-1}}.$$

Let

$$\partial_{L/K} = \max\left\{[L:\mathbb{Q}], [K:\mathbb{Q}]\partial'_K, \frac{0.16[K:\mathbb{Q}]}{\partial_K}\right\},$$

where $K \subseteq L$ are number fields. Define

$$C(K,n) := 3 \cdot 30^{n+4} \cdot (n+1)^{5.5} d^2 (1 + \log d).$$

The following result will be used to get an upper bound for the size of the integral solutions of our equations. It is Theorem 3 in [10].

Lemma 3 *Let α be an algebraic integer of degree at least 3, and let κ be an integer belonging to K. Denote by α_1, α_2, α_3 distinct conjugates of α and by κ_1, κ_2, κ_3 the corresponding conjugates of κ. Let*

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2}), \qquad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3}), \qquad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3})$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3}).$$

In what follows R stands for an upper bound for the regulators of K_1, K_2 and K_3, and r denotes the maximum of the unit ranks of K_1, K_2, K_3. Let

$$c_j^* = \max_{1\le i\le 3} c_j(K_i),$$

and let

$$N = \max_{1\le i,j\le 3} \left|\mathrm{Norm}_{\mathbb{Q}(\alpha_i,\alpha_j)/\mathbb{Q}}\left(\kappa_i(\alpha_i - \alpha_j)\right)\right|^2,$$

and let

$$H^* = c_5^* R + \frac{\log N}{\min_{1\le i\le 3}[K_i:\mathbb{Q}]} + \mathrm{h}(\kappa).$$

Define

$$A_1^* = 2H^* \cdot C(L, 2r+1) \cdot \left(c_1^*\right)^2 \partial_{L/L} \cdot \left(\max_{1\le i\le 3} \partial_{L/K_i}\right)^{2r} \cdot R^2$$

and

$$A_2^* = 2H^* + A_1^* + A_1^* \log\left\{(2r+1) \cdot \max\left\{c_4^*, 1\right\}\right\}.$$

If $x \in \mathbb{Z}\setminus\{0\}$ satisfies $x - \alpha = \kappa\xi^2$ for some $\xi \in K$ then

$$\log|x| \le 8A_1^* \log\left(4A_1^*\right) + 8A_2^* + H^* + 20\log 2 + 13\mathrm{h}(\kappa) + 19\mathrm{h}(\alpha).$$

To obtain a lower bound for the possible unknown integer solutions, we are going to use the so-called Mordell-Weil sieve. The Mordell-Weil sieve has been successfully applied to prove the non-existence of rational points on curves (see, *e.g.*, [26–28] and [29]).

Let $C/\mathbb{Q}$ be a smooth projective curve (in our case a hyperelliptic curve) of genus $g \geq 2$. Let J be its Jacobian. We assume the knowledge of some rational point on C, so let D be a fixed rational point on C, and let $\jmath$ be the corresponding Abel-Jacobi map

$$\jmath : C \to J, \qquad P \mapsto [P - D].$$

Let W be the image in J of the known rational points on C and $D_1, \dots, D_r$ generators for the free part of $J(\mathbb{Q})$. By using the Mordell-Weil sieve, we are going to obtain a very large and smooth integer B such that

$$\jmath\bigl(C(\mathbb{Q})\bigr) \subseteq W + BJ(\mathbb{Q}).$$

Let

$$\phi : \mathbb{Z}^r \to J(\mathbb{Q}), \qquad \phi(a_1, \dots, a_r) = \sum a_i D_i,$$

so that the image of ϕ is the free part of $J(\mathbb{Q})$. The variant of the Mordell-Weil sieve explained in [10] provides a method to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^r$

$$B\mathbb{Z}^r = L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_k$$

such that

$$\jmath\bigl(C(\mathbb{Q})\bigr) \subset W + \phi(L_j)$$

for $j = 1, \dots, k$.

The next lemma [10, Lemma 12.1] gives a lower bound for the size of rational points, whose images are not in the set W.

Lemma 4 *Let W be a finite subset of $J(\mathbb{Q})$, and let L be a sublattice of $\mathbb{Z}^r$. Suppose that $\jmath(C(\mathbb{Q})) \subset W + \phi(L)$. Let μ_1 be a lower bound for $h - \hat{h}$ and*

$$\mu_2 = \max\bigl\{\sqrt{\hat{h}(w)} : w \in W\bigr\}.$$

Denote by M the height-pairing matrix for the Mordell-Weil basis $D_1, \dots, D_r$, and let $\lambda_1, \dots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min\{\sqrt{\lambda_j} : j = 1, \dots, r\},$$

and let $m(L)$ be the Euclidean norm of the shortest non-zero vector of L. Then, for any $P \in C(\mathbb{Q})$, either $\jmath(P) \in W$ or

$$h\bigl(\jmath(P)\bigr) \geq \bigl(\mu_3 m(L) - \mu_2\bigr)^2 + \mu_1.$$

The following lemma plays a crucial role in the proof of Theorem 1.

Lemma 5 *The integral solutions of the equation*

$$\mathcal{C}: Y^2 = X(X+20)^2\bigl(X^2+10X+400\bigr)+140{,}625 \tag{3}$$

are

$$(X,Y) \in \bigl\{(0,\pm 375),(-20,\pm 375)\bigr\}.$$

Proof of Lemma 5 Let $J(\mathbb{Q})$ be the Jacobian of the genus two curve (3). Using MAGMA, we determine a Mordell-Weil basis, which is given by

$$D_1 = (0,375) - \infty,$$
$$D_2 = (-20,375) - \infty.$$

Let $f = x(x+20)^2(x^2+10x+400)+140{,}625$, and let α be a root of f. We will choose for coset representatives of $J(\mathbb{Q})/2J(\mathbb{Q})$ the linear combinations $\sum_{i=1}^{2} n_i D_i$, where $n_i \in \{0,1\}$. Then

$$x - \alpha = \kappa \xi^2,$$

where $\kappa \in \mathcal{K}$, and $\mathcal{K}$ is constructed as described in Lemma 2. We have that $\mathcal{K} = \{1, -\alpha, -20-\alpha, \alpha(\alpha+20)\}$. By local arguments, it is possible to restrict the set $\mathcal{K}$ further (see, *e.g.*, [26, 30]). In our case, one can eliminate

$$\alpha(\alpha+20)$$

by local computations in $\mathbb{Q}_3$. We apply Lemma 3 to get a large upper bound for $\log|x|$ in the remaining cases. A MAGMA code was written to obtain the bounds that appeared in [10]; they can be found at http://www.warwick.ac.uk/~maseap/progs/intpoint/bounds.m. We obtain that these bounds are as in Table 1.

The set of known rational points on the curve (3) is $\{\infty,(0,\pm 375),(-20,\pm 375)\}$. Let W be the image of this set in $J(\mathbb{Q})$. Applying the Mordell-Weil sieve, implemented by Bruin and Stoll and explained in [10], we obtain that $\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})$, where

$$B = 2^8 \cdot 5^3 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 53 \cdot 59 \cdot 71 \cdot 79 \cdot 83 \cdot 89$$

that is

$$B = 46{,}128{,}223{,}306{,}000{,}188{,}203{,}435{,}897{,}312{,}000.$$

Table 1 Bounds

κ	**Bound for log|*x*|**
1	$6.27 \cdot 10^{307}$
$-\alpha$	$4.48 \cdot 10^{668}$
$-20-\alpha$	$1.89 \cdot 10^{612}$

Now, we use an extension of the Mordell-Weil sieve due to Samir Siksek to obtain a very long decreasing sequence of lattices in $\mathbb{Z}^2$. After that, we apply Lemma 4 to obtain a lower bound for possible unknown rational points. We get that if (x,y) is an unknown integral point, then

$$\log|x| \geq 2.216448 \times 10^{782}.$$

This contradicts the bound for $\log|x|$ that we obtained by Baker's method. □

3 Proofs of the theorems

Proof of Theorem 1 For $k = -1$ and $l \in \{0,1,2,3\}$, we have the Diophantine equations

$$\frac{n(n+1)}{2} = \frac{m^2(m+1)^2}{3}, \tag{4}$$

$$\frac{n(n+1)}{8} = \frac{1}{15}z^2(2z-1) \quad \text{with } z = m(m+1), \tag{5}$$

$$\frac{n(n+1)}{8} = \frac{2}{21}z^2\left(3z^2-4z+2\right) \quad \text{with } z = m(m+1) \tag{6}$$

and

$$\frac{1}{4}\sum_{i=1}^{n} i = \frac{64}{9}\sum_{i=1}^{m} i^9, \tag{7}$$

respectively. One can see that the first three equations are elliptic Diophantine equations, thus using the program package MAGMA, subroutines `IntegralPoints` or `IntegralQuarticPoints` are just a straightforward calculation to solve them. In these cases, the unique solution is $(l,m,n) = (0,24,5)$. The forth equation can be written as follows

$$(2n+1)^2 = \frac{128}{45}\left(m^2+m-1\right)\left(m^2+m\right)^2\left(2m^4+4m^3-m^2-3m+3\right)+1.$$

So, we easily obtain a hyperelliptic curve

$$Y^2 = X(X+20)^2\left(X^2+10X+400\right)+140{,}625,$$

where $Y = 375(2n+1)$ and $X = 20m^2+20m-20$. By Lemma 5, we have that $X = 0$ or -20. Therefore, we have that $m \in \{-1,0\}$, a contradiction and there is no solution in positive integers of (7). □

Proof of Theorem 2 Now $k = 0$ and $p = 2l+3 \geq 3$ is a prime. From (1), we get

$$p \cdot n^2(n+1)^2 = 3 \cdot 4^{l+1}\left(1^p+2^p+\cdots+m^p\right).$$

Let m and n be an arbitrary but fixed solution. An elementary number theoretical argument and Lemma 1 yield that $p|m(m+1)$ and

$$\operatorname{ord}_p\left(\frac{1^p+2^p+\cdots+m^p}{m^2(m+1)^2}\right) = \operatorname{ord}_p\frac{B_{p+1}(m+1)-B_{p+1}}{m^2(m+1)^2} \neq 0.$$

Suppose that $p|m$, and let d be the smallest positive integer such that $B_{p+1}(m+1) - B_{p+1} = \frac{1}{d}f(m)m^2(m+1)^2$, and let $f(X) \in \mathbb{Z}[X]$. Since $\binom{p+1}{k}$ is divisible by p for $k = 2, \ldots, p-1$ and $B_1 = -1/2$, we have that p is not a divisor of d. The constant term of the polynomial $f(X)$ is $d\binom{p+1}{p-1}B_{p-1}$, and, by von Staudt-Clausen theorem, it is not divisible by p. On the other hand, p is a divisor of m and $f(m)$, we have a contradiction. If $p|m+1$, then we can repeat the previous argument using the fact $f(X) = f(-X-1)$, *cf.* Lemma 1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made equal contributions to this manuscript. All authors read and approved the final version.

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Acknowledgements
The work is supported by the TÁMOP-4.2.2.C-11/1/KONV-2012-0010 project. The project is implemented through the New Hungary Development Plan, co-financed by the European Social Fund and the European Regional Development Fund. Research was supported in part by the Hungarian Academy of Sciences, OTKA grants K75566, K100339, NK101680, NK104208 (ÁP) and OTKA grants PD75264, NK104208, K100339 and János Bolyai Research Scholarship of the Hungarian Academy of Sciences (ST). The authors are grateful to the reviewers for their helpful remarks.

Received: 19 April 2013 Accepted: 29 July 2013 Published: 14 August 2013

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doi:10.1186/1687-1847-2013-245
Cite this article as: Pintér and Tengely: The Korteweg-de Vries equation and a Diophantine problem related to Bernoulli polynomials. *Advances in Difference Equations* 2013 **2013**:245.