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On nonlinear discrete weakly singular inequalities and applications to Volterra-type difference equations

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Abstract
Some nonlinear discrete weakly singular inequalities, which generalize some known results are discussed. Under suitable parameters, prior bounds on solutions to nonlinear Volterra-type difference equations are obtained. Two examples are presented to show the applications of our results in boundedness and uniqueness of solutions of difference equations, respectively.

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1 Introduction

As an important branch of the Gronwall-Bellman inequality, various weakly singular integral inequalities and their discrete analogues have attracted more and more attention and play a fundamental role in the study of the theory of singular differential equations and integral equations (for example, see [1–8] and [9]). When many problems such as the behavior, the perturbation and the numerical treatment of the solution for the Volterra type weakly singular integral equation are studied, they often involve some certain integral inequalities and discrete inequalities. Dixon and McKee [10] investigated the convergence of discretization methods for the Volterra integral and integro-differential equations, using the following inequalities

$$x(t) \leq \psi(t) + M\int_0^t \frac{x(s)}{(t-s)^\alpha}\,ds, \quad m > 0, 0 < \alpha < 1$$

and

$$x_i \leq \psi_i + Mh^{1-\alpha}\sum_{j=0}^{i-1}\frac{x_j}{(i-j)^\alpha}, \quad i = 1, 2, \ldots, N, n > 0, Nh = T. \tag{1.1}$$

As for the second kind Abel-Volterra singular integral equation, Beesack [11] also discussed the inequalities

$$x(t) \leq \psi(t) + M\int_0^t \frac{s^\sigma x(s)}{(t^\beta - s^\beta)^\alpha}\,ds$$

and

$$x_i \leq \psi_i + Mh^{1+\sigma-\alpha\beta} \sum_{j=0}^{i-1} \frac{j^\sigma x_j}{(i^\beta - j^\beta)^\alpha}, \tag{1.2}$$

where $0 < \alpha < 1$, $\beta \geq 1$ and $\sigma \geq \beta - 1$. It should be noted that the above mentioned results are based on the assumption that the mesh size is uniformly bounded. Unfortunately, such technique leads to weak results, which do not reflect the true order of consistency of the scheme and may not even yield a convergence result at all. To avoid the shortcoming of these results, Norbuy and Stuart [12, 13] presented some new inequalities to describe the numerical method for weakly singular Volterra integral equations, which is based on a variable mesh.

Another purpose of studying weakly singular integral inequalities and their discrete versions is related to the theory of the parabolic equation (for example, see [14–18] and the references therein). Consider the weakly singular integral inequality

$$x(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s)\omega\bigl(x(s)\bigr)\, ds, \quad 0 < \beta < 1,$$

and the corresponding discrete inequality of multi-step,

$$x_n \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k \omega(x_k), \tag{1.3}$$

where $t_0 = 0$, $\tau_k = t_{k+1} - t_k$, $\sup_{k\in\mathbb{N}} \tau_k = \tau$ and $\lim_{t\to\infty} t_k = \infty$. Henry [16] and Slodicka [19] discussed the linear case $\omega(\xi) = \xi$ of the two inequalities above and obtained the estimate of the solution. Furthermore, Medveď [20, 21] studied the general nonlinear case. However, his results are based on the '(q) condition': (1) ω satisfies $e^{-qt}[\omega(u)]^q \leq R(t)\omega(e^{-qt})u^q$; (2) there exists $c > 0$ such that $a_n e^{-\tau t_n} \leq c$. Later, Yang and Ma [22] generalized the results to a new case

$$x_n^\alpha \leq a_n + \sum_{k=0}^{n-1} (t_n - t_k)^{\beta-1} \tau_k b_k x_k^\lambda. \tag{1.4}$$

In this paper, we are concerned with the following weakly singular inequality on a variable mesh

$$x_n^\mu \leq a_n + \sum_{k=0}^{n-1} \bigl(t_n^\alpha - t_k^\alpha\bigr)^{\beta-1} t_k^{\gamma-1} \tau_k b_k x_k^\lambda, \tag{1.5}$$

where $\mu > 0$, $\lambda > 0$, and $0 < \beta, \gamma \leq 1$. Our proposed method can avoid the so-called '*q-condition*,' and under a new assumption, the more concise results are derived. Moreover, to show the application of the more general inequality to a Volterra-type difference equation, some examples are presented.

2 Preliminaries

In what follows, we denote $\mathbb{R}$ by the set of real numbers. Let $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$. $C(X, Y)$ denotes the collection of continuous functions from the set X to the set Y. As usual, the empty sum is taken to be 0.

Lemma 2.1 (Discrete Jensen inequality, see [22]) *Let $A_1, A_2, \ldots, A_n$ be nonnegative real numbers, and let $r > 1$ be a real number. Then*

$$(A_1 + A_2 + \cdots + A_n)^r \leq n^{r-1}\left(A_1^r + A_2^r + \cdots + A_n^r\right).$$

Lemma 2.2 (Discrete Hölder inequality, see [22]) *Let a_i, b_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers, and p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ (or $p = 1$, $q = \infty$). Then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Definition 2.1 (See [5]) Let $[x, y, z]$ be an ordered parameter group of nonnegative real numbers. The group is said to belong to the first-class distribution and is denoted by $[x, y, z] \in I$ if conditions $x \in (0, 1]$, $y \in (1/2, 1)$, $z \geq 3/2 - y$ and $z > y$ are satisfied; it is said to belong to the second-class distribution and is denoted by $[x, y, z] \in II$ if conditions $x \in (0, 1]$, $y \in (0, 1/2]$ and $z > \frac{1-2y^2}{1-y^2}$ are satisfied.

Lemma 2.3 (See [5]) *Let α, β, γ and p be positive constants. Then*

$$\int_0^t \left(t^\alpha - s^\alpha\right)^{p(\beta-1)} s^{p(\gamma-1)}\, ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+,$$

where $B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\, ds$ $(\operatorname{Re}\xi > 0, \operatorname{Re}\eta > 0)$ is the well-known B-function and $\theta = p[\alpha(\beta-1) + \gamma - 1] + 1$.

Lemma 2.4 *Suppose that $\omega(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\omega(u) > 0$ for $u > 0$. Let a_n, c_n be nonnegative and nondecreasing in n. If y_n is nonnegative such that*

$$y_n \leq a_n + c_n \sum_{k=0}^{n-1} b_k \omega(y_k), \quad n \in \mathbb{N}.$$

Then

$$y_n \leq \Omega^{-1}\left[\Omega(a_n) + c_n \sum_{k=0}^{n-1} b_k\right], \quad 0 \leq n \leq M,$$

where $\Omega(v) = \int_{v_0}^{v} \frac{1}{\omega(s)}\, ds$, $v \geq v_0$, Ω^{-1} is the inverse function of Ω, and M is defined by

$$M = \sup\left\{i : \Omega(a_i) + c_i \sum_{k=0}^{i-1} b_k \in \operatorname{Dom}\left(\Omega^{-1}\right)\right\}.$$

Remark 2.1 Martyniuk *et al.* [23] studied the inequality $y_n \leq c + \sum_{k=0}^{n-1} b_k \omega(y_k)$, $n \in \mathbb{N}$. Obviously, our result is a more general case of the nonlinear difference inequality.

Lemma 2.5 *If* $[\alpha,\beta,\gamma] \in I$, *then* $p_1 = \frac{1}{\beta}$; *if* $[\alpha,\beta,\gamma] \in II$, *then* $p_2 = \frac{1+4\beta}{1+3\beta}$. *Furthermore, for sufficiently small* τ_k, *we have*

$$\begin{aligned}
&\sum_{k=0}^{n-1}\bigl(t_n^\alpha - t_k^\alpha\bigr)^{p_i(\beta-1)} t_k^{p_i(\gamma-1)}\tau_k \\
&\quad \le \int_0^{t_n}\bigl(t_n^\alpha - s^\alpha\bigr)^{p_i(\beta-1)} s^{p_i(\gamma-1)}\,ds \\
&\quad = \frac{t_n^{\theta_i}}{\alpha} B\biggl[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\biggr]
\end{aligned} \tag{2.1}$$

for $i = 1,2$, *where* $\theta_i = p_i[\alpha(\beta-1)+\gamma-1]+1$.

Proof By the definition of θ_i, $\theta_i \ge 0$. For its proof, see [5]. On one hand, when $[\alpha,\beta,\gamma] \in I$, it follows from Definition 2.1 that $\gamma > \beta$. On the other hand, when $[\alpha,\beta,\gamma] \in II$, that is, $\alpha \in (0,1]$, $\beta \in (0,\frac{1}{2}]$, we have that

$$\gamma > \frac{1-2\beta^2}{1-\beta^2} > \frac{1}{2} \ge \beta \tag{2.2}$$

holds, since

$$\frac{1-2\beta^2}{1-\beta^2} > \frac{1}{2} \quad\Leftrightarrow\quad 2-4\beta^2 > 1-\beta^2 \quad\Leftrightarrow\quad 1-3\beta^2 > 0. \tag{2.3}$$

According to the condition $\beta \in (0,\frac{1}{2}]$, $1-3\beta^2 > 0$ holds, which yields (2.2) holds directly. Thus, when $[\alpha,\beta,\gamma] \in I$ or $[\alpha,\beta,\gamma] \in II$, we have

$$\gamma > \beta.$$

Next, we consider the integrated function in the B-function in (2.1).

$$\begin{aligned}
B\biggl[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\biggr] &= \int_0^1 (1-s)^{p_i(\beta-1)+1-1} s^{\frac{p_i(\gamma-1)+1}{\alpha}-1}\,ds \\
&:= \int_0^1 (1-s)^{n_2-1} s^{n_1-1}\,ds.
\end{aligned} \tag{2.4}$$

Denote $f(s) := (1-s)^{n_2-1}s^{n_1-1}$ for $s \in (0,1)$, where $n_1 = \frac{p_i(\gamma-1)+1}{\alpha}$ and $n_2 = p_i(\beta-1)+1$. If $n_2 = n_1$, then $f(s)$ is symmetric about $s = \frac{1}{2}$. In fact, because of $\alpha \in (0,1]$, we get

$$1 > n_1 = \frac{p_i(\gamma-1)+1}{\alpha} > p_i(\gamma-1)+1 > p_i(\beta-1)+1 = n_2 > 0,$$

i.e.,

$$0 > n_1 - 1 > n_2 - 1 > -1. \tag{2.5}$$

Moreover, we can obtain the zero-point of $f'(s)$ as follows

$$s_0 = \frac{n_1-1}{n_1+n_2-2} < \frac{1}{2}. \tag{2.6}$$

Therefore, the function $f(s)$ is decreasing on the interval $(0, s_0]$ while increasing sharply on the interval $[s_0, 1)$. So, for some given sufficiently small τ_k, by the properties of the left-rectangle integral formula, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\left(1^\alpha - t_k^\alpha\right)^{n_2-1} t_k^{n_1-1}\tau_k &\leq \int_0^1 \left(1-s^\alpha\right)^{n_2-1} s^{n_1-1}\,ds \\ &= B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right],\end{aligned} \tag{2.7}$$

where $0 = t_0 < t_1 < \cdots < t_n = 1$.

As for the general interval $(0, t_n]$, we can easily obtain the corresponding result (2.1), which is similar to (2.7). We omit the details here. □

3 Main result

To state our result conveniently, we fist introduce the following function

$$\Omega(u) = \int_{u_0}^{u} \frac{1}{s^{\frac{\lambda}{\mu}}}\,ds, \quad u \geq u_0 > 0, \mu > 0, \lambda > 0.$$

Thus, we have

$$\Omega(u) = \begin{cases} \ln \frac{u}{u_0}, & \mu = \lambda, u_0 > 0, \\ \frac{\mu}{\mu-\lambda} u^{\frac{\mu-\lambda}{\mu}}, & \mu \neq \lambda, u_0 = 0 \end{cases}$$

and

$$\Omega^{-1}(\xi) = \begin{cases} u_0 \exp \xi, \\ (\frac{\mu-\lambda}{\mu}\xi)^{\frac{\mu}{\mu-\lambda}}. \end{cases}$$

Denote $\tilde{a}_n = \max_{0\leq k\leq n, k\in\mathbb{N}} a_k$, where $\tau = \max_{0\leq k\leq n-1, k\in\mathbb{N}} \tau_k$.

Theorem 3.1 *Suppose that a_n, b_n are nonnegative functions for $n \in \mathbb{N}$. Let $\mu > 0$, $\lambda > 0$, $\mu \neq \lambda$. If x_n is nonnegative function such that* (1.5), *then for some sufficiently small τ_k:*

(1) $[\alpha, \beta, \gamma] \in I$, *letting $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$, we have*

$$x_n \leq \left[\left(2^{\frac{\beta}{1-\beta}} a_{\tilde{n}}^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}} + \frac{\mu-\lambda}{\mu} 2^{\frac{\beta}{1-\beta}} \tau \left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}} \mathcal{B}_1^{\frac{\beta}{1-\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}}\right]^{\frac{1-\beta}{\mu-\lambda}} \tag{3.1}$$

for $n \in \mathbb{N}$ ($\mu > \lambda$) or $0 \leq n \leq N_1$ ($\mu < \lambda$), where

$$\theta_1 = p_1\left[\alpha(\beta-1)+\gamma-1\right]+1 = \frac{\alpha(\beta-1)+\gamma-1}{\beta}+1,$$

$$\mathcal{B}_1 = B\left[\frac{\gamma-1+\beta}{\alpha\beta}, \frac{2\beta-1}{\beta}\right],$$

and N_1 is the largest integer number such that,

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{\beta}{1-\beta}}a_{\tilde{n}}^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{\beta}{1-\beta}}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}}\mathcal{B}_1^{\frac{\beta}{1-\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1}{1-\beta}}>0; \tag{3.2}$$

(2) $[\alpha,\beta,\gamma]\in II$, *letting* $p_2=\frac{1+4\beta}{1+3\beta}$, $q_2=\frac{1+4\beta}{\beta}$, *we have*

$$x_n\leq\left[\left(2^{\frac{1+3\beta}{\beta}}a_{\tilde{n}}^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+\frac{\mu-\lambda}{\mu}2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}\mathcal{B}_2^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}\right]^{\frac{1+4\beta}{\beta(\mu-\lambda)}} \tag{3.3}$$

for $n\in\mathbb{N}$ $(\mu>\lambda)$ *or* $0\leq n\leq N_2$ $(\mu<\lambda)$, *where*

$$\theta_2=\frac{1+4\beta}{1+3\beta}\left[\alpha(\beta-1)+\gamma-1\right]+1,$$
$$\mathcal{B}_2=B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)},\frac{4\beta^2}{1+3\beta}\right],$$

and N_2 is the largest integer number such that

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{1+3\beta}{\beta}}a_{\tilde{n}}^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}(\mathcal{B}_2)^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}>0. \tag{3.4}$$

Proof By the definition of $\tilde{a}_n$, obviously, $\tilde{a}_n$ is nonnegative and nondecreasing, that is, $\tilde{a}_n\geq a_n$. It follows from (1.5) that

$$x_n^{\mu}\leq\tilde{a}_n+\sum_{k=0}^{n-1}\left(t_n^{\alpha}-t_k^{\alpha}\right)^{\beta-1}t_k^{\gamma-1}\tau_k b_k x_k^{\lambda}, \tag{3.5}$$

where τ_k is the variable time step.

Next, for convenience, we take the indices p_i, q_i. Denote that if $[\alpha,\beta,\gamma]\in I$, let $p_1=\frac{1}{\beta}$ and $q_1=\frac{1}{1-\beta}$; if $[\alpha,\beta,\gamma]\in II$, let $p_2=\frac{1+4\beta}{1+3\beta}$, and let $q_2=\frac{1+4\beta}{\beta}$. Then $\frac{1}{p_i}+\frac{1}{q_i}=1$ holds for $i=1,2$.

Using Lemma 2.2 with indices p_i, q_i in (3.5), we have

$$\begin{aligned}x_n^{\mu}&\leq\tilde{a}_n+\sum_{k=0}^{n-1}\left(t_n^{\alpha}-t_k^{\alpha}\right)^{\beta-1}t_k^{\gamma-1}\tau_k^{\frac{1}{p_i}}\tau_k^{\frac{1}{q_i}}b_k x_k^{\lambda}\\&\leq\tilde{a}_n+\tau^{\frac{1}{q_i}}\sum_{k=0}^{n-1}\left(t_n^{\alpha}-t_k^{\alpha}\right)^{\beta-1}t_k^{\gamma-1}\tau_k^{\frac{1}{p_i}}b_k x_k^{\lambda}\\&\leq\tilde{a}_n+\tau^{\frac{1}{q_i}}\left[\sum_{k=0}^{n-1}\left(t_n^{\alpha}-t_k^{\alpha}\right)^{p_i(\beta-1)}t_k^{p_i(\gamma-1)}\tau_k\right]^{\frac{1}{p_i}}\left(\sum_{k=0}^{n-1}b_k^{q_i}x_k^{q_i\lambda}\right)^{\frac{1}{q_i}}.\end{aligned} \tag{3.6}$$

By Lemma 2.1, the inequality above can be rewritten as

$$x_n^{q_i\mu}\leq 2^{q_i-1}\tilde{a}_n^{q_i}+2^{q_i-1}\tau\left[\sum_{k=0}^{n-1}\left(t_n^{\alpha}-t_k^{\alpha}\right)^{p_i(\beta-1)}t_k^{p_i(\gamma-1)}\tau_k\right]^{\frac{q_i}{p_i}}\left(\sum_{k=0}^{n-1}b_k^{q_i}x_k^{q_i\lambda}\right). \tag{3.7}$$

By Lemma 2.5,

$$\sum_{k=0}^{n-1}\left(t_n^\alpha - t_k^\alpha\right)^{p_i(\beta-1)} t_k^{p_i(\gamma-1)}\tau_k \le \int_0^{t_n}\left(t_n^\alpha - s^\alpha\right)^{p_i(\beta-1)} s^{p_i(\gamma-1)}\,ds$$
$$= \frac{t_n^{\theta_i}}{\alpha} B\left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1\right] = \frac{t_n^{\theta_i}}{\alpha}\mathcal{B}_i, \tag{3.8}$$

where $\mathcal{B}_i = B[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1]$, we get

$$x_n^{q_i\mu} \le 2^{q_i-1}\tilde{a}_n^{q_i} + 2^{q_i-1}\tau\left(\frac{t_n^{\theta_i}}{\alpha}\right)^{\frac{q_i}{p_i}}(\mathcal{B}_i)^{\frac{q_i}{p_i}}\left(\sum_{k=0}^{n-1} b_k^{q_i} x_k^{q_i\lambda}\right), \tag{3.9}$$

and θ_i is given in Lemma 2.3 for $i = 1, 2$.

Let $y_n = x_n^{q_i\mu}$. Then $x_k^{q_i\lambda} = x_k^{q_i\mu\frac{\lambda}{\mu}} = y_k^{\frac{\lambda}{\mu}}$. It follows from (3.9) that

$$y_n \le 2^{q_i-1}\tilde{a}_n^{q_i} + 2^{q_i-1}\tau\left(\frac{t_n^{\theta_i}}{\alpha}\right)^{\frac{q_i}{p_i}}(\mathcal{B}_i)^{\frac{q_i}{p_i}}\left(\sum_{k=0}^{n-1} b_k^{q_i} y_k^{\frac{\lambda}{\mu}}\right). \tag{3.10}$$

According to Lemma 2.4, we have

$$y_n \le \Omega^{-1}\left[\Omega\left(2^{q_i-1}\tilde{a}_n^{q_i}\right) + 2^{q_i-1}\tau\left(\frac{t_n^{\theta_i}}{\alpha}\right)^{\frac{q_i}{p_i}}(\mathcal{B}_i)^{\frac{q_i}{p_i}}\sum_{k=0}^{n-1} b_k^{q_i}\right] \tag{3.11}$$

for $0 \le n \le M$, where M is the largest integer number such that

$$\Omega\left(2^{q_i-1}\tilde{a}_n^{q_i}\right) + 2^{q_i-1}\tau\left(\frac{t_n^{\theta_i}}{\alpha}\right)^{\frac{q_i}{p_i}}(\mathcal{B}_i)^{\frac{q_i}{p_i}}\sum_{k=0}^{n-1} b_k^{q_i} \in \operatorname{Dom}\left(\Omega^{-1}\right).$$

(1) For $[\alpha, \beta, \gamma] \in I$, $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$. By the definitions of Ω and Ω^{-1}, we can compute that

$$\begin{aligned}
&\Omega\left(2^{q_1-1}\tilde{a}_n^{q_1}\right) = \frac{\mu}{\mu-\lambda}\left(2^{\frac{\beta}{1-\beta}} a_{\tilde{n}}^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}}, \\
&\theta_1 = p_1\left[\alpha(\beta-1)+\gamma-1\right]+1 = \frac{\alpha(\beta-1)+\gamma-1}{\beta}+1, \\
&\mathcal{B}_1 = B\left[\frac{\gamma-1+\beta}{\alpha\beta}, \frac{2\beta-1}{\beta}\right].
\end{aligned} \tag{3.12}$$

Then

$$\begin{aligned}
y_n &\le \Omega^{-1}\left[\Omega\left(2^{q_1-1}\tilde{a}_n^{q_1}\right) + 2^{q_1-1}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{q_1}{p_1}}\mathcal{B}_1^{\frac{q_1}{p_1}}\sum_{k=0}^{n-1} b_k^{q_1}\right] \\
&\le \Omega^{-1}\left[\frac{\mu}{\mu-\lambda}\left(2^{\frac{\beta}{1-\beta}}\tilde{a}_n^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}} + 2^{\frac{\beta}{1-\beta}}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}}\mathcal{B}_1^{\frac{\beta}{1-\beta}}\sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}}\right] \\
&\le \left[\left(2^{\frac{\beta}{1-\beta}}\tilde{a}_n^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}} + \frac{\mu-\lambda}{\mu}2^{\frac{\beta}{1-\beta}}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}}\mathcal{B}_1^{\frac{\beta}{1-\beta}}\sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}}\right]^{\frac{\mu}{\mu-\lambda}}.
\end{aligned} \tag{3.13}$$

Observe the second formula in (3.13). To ensure that

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{\beta}{1-\beta}}\tilde{a}_n^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{\beta}{1-\beta}}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}}\mathcal{B}_1^{\frac{\beta}{1-\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1}{1-\beta}}\in\operatorname{Dom}\left(\Omega^{-1}\right),$$

we may take $M=\infty$ for $\mu>\lambda$ and $M=N_1$ for $\mu<\lambda$, respectively, where N_1 is the largest integer number such that

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{\beta}{1-\beta}}\tilde{a}_n^{\frac{1}{1-\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{\beta}{1-\beta}}\tau\left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}}\mathcal{B}_1^{\frac{\beta}{1-\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1}{1-\beta}}>0.$$

Since $x_n\le y_n^{\frac{1}{q_1\mu}}=y_n^{\frac{1-\beta}{\mu}}$, substituting it into (3.13), we can get (3.1).

(2) For $[\alpha,\beta,\gamma]\in II$, $p_2=\frac{1+4\beta}{1+3\beta}$, $q_2=\frac{1+4\beta}{\beta}$. Similarly to the computation above, we have

$$\begin{aligned}
&\Omega\left(2^{q_2-1}\tilde{a}_n^{q_2}\right)=\frac{\mu}{\mu-\lambda}\left(2^{\frac{1+3\beta}{\beta}}\tilde{a}_n^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}},\\
&\theta_2=\frac{1+4\beta}{1+3\beta}\left[\alpha(\beta-1)+\gamma-1\right]+1, \qquad (3.14)\\
&\mathcal{B}_2=B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)},\frac{4\beta^2}{1+3\beta}\right].
\end{aligned}$$

Then

$$\begin{aligned}
y_n&\le\Omega^{-1}\left[\Omega\left(2^{q_2-1}\tilde{a}_n^{q_2}\right)+2^{q_2-1}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{q_2}{p_2}}\mathcal{B}_2^{\frac{q_2}{p_2}}\sum_{k=0}^{n-1}b_k^{q_2}\right]\\
&\le\Omega^{-1}\left[\frac{\mu}{\mu-\lambda}\left(2^{\frac{1+3\beta}{\beta}}\tilde{a}_n^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}\mathcal{B}_2^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}\right]\\
&\le\left[\left(2^{\frac{1+3\beta}{\beta}}\tilde{a}_n^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+\frac{\mu-\lambda}{\mu}2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}\mathcal{B}_2^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}\right]^{\frac{\mu}{\mu-\lambda}}. \qquad (3.15)
\end{aligned}$$

Observe the second formula in (3.15). To ensure that

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{1+3\beta}{\beta}}\tilde{a}_n^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}\mathcal{B}_2^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}\in\operatorname{Dom}\left(\Omega^{-1}\right),$$

we take $M=\infty$ for $\mu>\lambda$ and $M=N_2$ for $\mu<\lambda$, respectively, where N_2 is the largest integer number such that

$$\frac{\mu}{\mu-\lambda}\left(2^{\frac{1+3\beta}{\beta}}\tilde{a}_n^{\frac{1+4\beta}{\beta}}\right)^{\frac{\mu-\lambda}{\mu}}+2^{\frac{1+3\beta}{\beta}}\tau\left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}}\mathcal{B}_2^{\frac{1+3\beta}{\beta}}\sum_{k=0}^{n-1}b_k^{\frac{1+4\beta}{\beta}}>0.$$

Because $x_n\le y_n^{\frac{1}{q_2\mu}}=y_n^{\frac{1-\beta}{\mu}}$, substituting it into (3.15), we can get (3.3). This completes the proof. □

For the case that $\mu = \lambda$, let $y_n = x_n^{\mu}$, we obtain from (1.5)

$$y_n \le a_n + \sum_{k=0}^{n-1} \left(t_n^{\alpha} - t_k^{\alpha}\right)^{\beta-1} t_k^{\gamma-1} \tau_k b_k y_k \tag{3.16}$$

and derive the estimation of the upper bound as follows.

Theorem 3.2 *Let a_n, b_n be defined as in Theorem* 3.1. *If y_n is nonnegative function such that* (3.16), *then for some sufficiently small τ_k*:

(1) $[\alpha, \beta, \gamma] \in I$, $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$, *we have*

$$y_n \le \left[2^{\frac{\beta}{1-\beta}} \tilde{a}_n^{\frac{1}{1-\beta}} \exp\left\{ 2^{\frac{\beta}{1-\beta}} \tau \left(\frac{t_n^{\theta_1}}{\alpha} \right)^{\frac{\beta}{1-\beta}} \mathcal{B}_1^{\frac{\beta}{1-\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}} \right\} \right]^{1-\beta} \tag{3.17}$$

for $n \in \mathbb{N}$, where θ_1, $\mathcal{B}_1$ are defined in Theorem 3.1;

(2) $[\alpha, \beta, \gamma] \in II$, $p_2 = \frac{1+4\beta}{1+3\beta}$, $q_2 = \frac{1+4\beta}{\beta}$, *we have*

$$y_n \le \left[2^{\frac{1+3\beta}{\beta}} \tilde{a}_n^{\frac{1+4\beta}{\beta}} \exp\left\{ 2^{\frac{1+3\beta}{\beta}} \tau \left(\frac{t_n^{\theta_2}}{\alpha} \right)^{\frac{1+3\beta}{\beta}} \mathcal{B}_2^{\frac{1+3\beta}{\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1+4\beta}{\beta}} \right\} \right]^{\frac{\beta}{1+4\beta}} \tag{3.18}$$

for $n \in \mathbb{N}$, where θ_2, $\mathcal{B}_2$ are defined in Theorem 3.1.

Proof In the proof of Theorem 3.1, before we apply Lemma 2.4, it is independent of the comparison of μ and λ. Hence, taking $y_n = x_n^{\mu} = x_n^{\lambda}$ in (3.9), we have

$$y_n^{q_i} \le 2^{q_i-1} \tilde{a}_n^{q_i} + 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \left(\sum_{k=0}^{n-1} b_k^{q_i} y_k^{q_i} \right). \tag{3.19}$$

We denote $z_n = y_n^{q_i}$ and get

$$z_n \le 2^{q_i-1} \tilde{a}_n^{q_i} + 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \left(\sum_{k=0}^{n-1} b_k^{q_i} z_k \right).$$

By Lemma 2.4 and the definitions of Ω and Ω^{-1} for $\mu = \lambda$, we have the following result

$$\begin{aligned}
z_n &\le \Omega^{-1} \left[\Omega\left(2^{q_i-1} \tilde{a}_n^{q_i}\right) + 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \sum_{k=0}^{n-1} b_k^{q_i} \right] \\
&= \Omega^{-1} \left[\ln \frac{(2^{q_i-1} \tilde{a}_n^{q_i})}{u_0} + 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \sum_{k=0}^{n-1} b_k^{q_i} \right] \\
&= \exp\left\{ \left[\ln\left(2^{q_i-1} \tilde{a}_n^{q_i}\right) + 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \sum_{k=0}^{n-1} b_k^{q_i} \right] \right\} \\
&= 2^{q_i-1} a_{\tilde{n}}^{q_i} \exp\left\{ 2^{q_i-1} \tau \left(\frac{t_n^{\theta_i}}{\alpha} \right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \sum_{k=0}^{n-1} b_k^{q_i} \right\},
\end{aligned} \tag{3.20}$$

which yields

$$y_n \leq \left[2^{q_i-1}\tilde{a}_n^{q_i} \exp\left\{ 2^{q_i-1}\tau \left(\frac{t_n^{\theta_i}}{\alpha}\right)^{\frac{q_i}{p_i}} \mathcal{B}_i^{\frac{q_i}{p_i}} \sum_{k=0}^{n-1} b_k^{q_i} \right\} \right]^{\frac{1}{q_i}}. \tag{3.21}$$

Finally, considering two situations for $i = 1, 2$ and using paremeters α, β, γ to denote p_i, q_i, $\mathcal{B}_i$ and θ_i in (3.21), we can obtain the estimations, respectively. We omit the details here. □

Remark 3.1 Henry [16] and Slodicka [19] discussed the special case of Theorem 3.2, that is, $\alpha = 1$ and $\gamma = 1$. Moreover, our result is simpler and has a wider range of applications.

Remark 3.2 Although Medveď [20, 21] investigated the more general nonlinear case, his result is under the assumption that '$w(u)$ satisfies the condition (q).' In our result, the '(q) condition' is eliminated.

Letting $\mu = 2$ and $\lambda = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.1 *Suppose that a_n, b_n are nonnegative functions for $n \in \mathbb{N}$, and u_n is nondecreasing such that*

$$x_n^2 \leq a_n + \sum_{k=0}^{n-1} \left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1} \tau_k b_k x_k, \tag{3.22}$$

then for some sufficiently small τ_k:

(1) *when $[\alpha, \beta, \gamma] \in I$, $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$, we have*

$$x_n \leq \left[\left(2^{\frac{\beta}{1-\beta}} a_{\tilde{n}}^{\frac{1}{1-\beta}}\right)^{\frac{1}{2}} + 2^{\frac{2\beta-1}{1-\beta}} \tau \left(\frac{t_n^{\theta_1}}{\alpha}\right)^{\frac{\beta}{1-\beta}} \mathcal{B}_1^{\frac{\beta}{1-\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}} \right]^{1-\beta} \tag{3.23}$$

for $n \in \mathbb{N}$, where θ_1, $\mathcal{B}_1$ are defined in Theorem 3.1;

(2) *when $[\alpha, \beta, \gamma] \in II$, $p_2 = \frac{1+4\beta}{1+3\beta}$, $q_1 = \frac{1+4\beta}{\beta}$, we have*

$$x_n \leq \left[\left(2^{\frac{1+3\beta}{\beta}} a_{\tilde{n}}^{\frac{1+4\beta}{\beta}}\right)^{\frac{1}{2}} + 2^{\frac{1+2\beta}{\beta}} \tau \left(\frac{t_n^{\theta_2}}{\alpha}\right)^{\frac{1+3\beta}{\beta}} \mathcal{B}_2^{\frac{1+3\beta}{\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1+4\beta}{\beta}} \right]^{\frac{1+4\beta}{\beta}} \tag{3.24}$$

for $n \in \mathbb{N}$, where θ_2, $\mathcal{B}_2$ are defined in Theorem 3.1.

Remark 3.3 Inequality (3.22) is the extension of the well-known Ou-Iang-type inequality. Clearly, our inequality enriches the results for such an inequality.

4 Applications

In this section, we apply our results to discuss the boundedness and uniqueness of solutions of a Volterra-type difference equation with a weakly singular kernel.

Example 1 Suppose that x_n satisfies the equation

$$x_n^2 = \frac{1}{2} + \sum_{k=0}^{n-1} (t_n - t_k)^{-\frac{1}{3}} t_k^{-\frac{1}{6}} \tau_k b_k x_k \tag{4.1}$$

for $n \in \mathbb{N}$. Then we get

$$|x_n|^2 \le \frac{1}{2} + \sum_{k=0}^{n-1} (t_n - t_k)^{-\frac{1}{3}} t_k^{-\frac{1}{6}} \tau_k b_k |x_k|. \tag{4.2}$$

Letting $|x_n| = y_n$ changes (4.2) into

$$y_n^2 \le \frac{1}{2} + \sum_{k=0}^{n-1} (t_n - t_k)^{-\frac{1}{3}} t_k^{-\frac{1}{6}} \tau_k b_k y_k. \tag{4.3}$$

From (4.3), we can see that

$$a_n = \frac{1}{2}, \qquad \alpha = 1, \qquad \beta = \frac{2}{3}, \qquad \gamma = \frac{5}{6}, \quad \gamma > \beta.$$

Obviously, $[\alpha, \beta, \gamma] \in I$. Letting $p_1 = \frac{3}{2}$, $q_1 = 3$, we have

$$\tilde{a}_n = \frac{1}{2}, \qquad b_k = 1,$$
$$\theta_1 = \frac{3}{2}\left[\left(\frac{3}{2} - 1\right) + \frac{5}{6} - 1\right] + 1 = \frac{1}{4}, \qquad \mathcal{B}_1 = B\left[\frac{3}{4}, \frac{1}{2}\right].$$

Using Corollary 3.1, we get

$$|x_n| \le \left[\frac{\sqrt{2}}{2} + 2n\tau t_n^{\frac{1}{2}} B^2\left[\frac{3}{4}, \frac{1}{2}\right]\right]^{\frac{1}{3}}, \quad n \in \mathbb{N}, \tag{4.4}$$

which implies that x_n in (4.1) is upper bounded.

Example 2 Consider the linear weakly singular difference equation

$$x_n \le a_n + \sum_{k=0}^{n-1} \left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1} \tau_k b_k x_k \tag{4.5}$$

and

$$y_n \le c_n + \sum_{k=0}^{n-1} \left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1} \tau_k b_k y_k, \tag{4.6}$$

where $|a_n - c_n| < \epsilon$, ϵ is an arbitrary positive number, and $[\alpha, \beta, \gamma] \in I$ or $[\alpha, \beta, \gamma] \in II$. From (4.5) and (4.6), we get

$$|x_n - y_n| \le |a_n - c_n| + \sum_{k=0}^{n-1} \left(t_n^\alpha - t_k^\alpha\right)^{\beta-1} t_k^{\gamma-1} \tau_k b_k |x_k - y_k|, \tag{4.7}$$

which is the form of inequality (3.16). Applying Theorem 3.2, we have

$$|x_n - y_n| \le \left[2^{\frac{\beta}{1-\beta}} \epsilon^{\frac{1}{1-\beta}} \exp\left\{ 2^{\frac{\beta}{1-\beta}} \tau \left(\frac{t_n^{\theta_1}}{\alpha} \right)^{\frac{\beta}{1-\beta}} \mathcal{B}_1^{\frac{\beta}{1-\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1}{1-\beta}} \right\} \right]^{1-\beta}$$

or

$$|x_n - y_n| \le \left[2^{\frac{1+3\beta}{\beta}} \epsilon^{\frac{1+4\beta}{\beta}} \exp\left\{ 2^{\frac{1+3\beta}{\beta}} \tau \left(\frac{t_n^{\theta_2}}{\alpha} \right)^{\frac{1+3\beta}{\beta}} \mathcal{B}_2^{\frac{1+3\beta}{\beta}} \sum_{k=0}^{n-1} b_k^{\frac{1+4\beta}{\beta}} \right\} \right]^{\frac{\beta}{1+4\beta}}$$

for $n \in \mathbb{N}$. If $a_n = c_n$, let $\epsilon \to 0$, and we obtain the uniqueness of the solution of equation (4.5).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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