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Determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$

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Abstract

During the last three decades, the problem of evaluation of the determinants of the Laplacians on Riemann manifolds has received considerable attention from many authors. The functional determinant for the k-dimensional unit sphere $\mathbf{S}^k$ with the standard metric has been computed in several ways. Here we aim at computing the determinants of the Laplacians on $\mathbf{S}^k$ ($k = 2n + 1$) by mainly using certain closed-form evaluations of the series involving zeta function.

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1 Introduction and preliminaries

During the last three decades, the problem of evaluation of the determinants of the Laplacians on Riemann manifolds has received considerable attention from many authors including (among others) D'Hoker and Phong [1, 2], Sarnak [3], and Voros [4], who computed the determinants of the Laplacians on compact Riemann surfaces of constant curvature in terms of special values of the Selberg zeta function. Although the first interest in the determinants of the Laplacians arose mainly for Riemann surfaces, it is also interesting and potentially useful to compute these determinants for classical Riemannian manifolds of higher dimensions, such as spheres. Here, we are particularly concerned with the evaluation of the functional determinant for the k-dimensional unit sphere $\mathbf{S}^k$ ($k = 2n + 1$) with the standard metric.

For this purpose we need the following definitions. Let $\{\lambda_n\}$ be a sequence such that

$$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots ; \quad \lambda_n \uparrow \infty \ (n \to \infty); \tag{1.1}$$

henceforth we consider only such nonnegative increasing sequences. Then we can show that

$$Z(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}, \tag{1.2}$$

which is known to converge absolutely in a half-plane $\Re(s) > \sigma$ for some $\sigma \in \mathbb{R}$.

Definition 1 (*cf.* Osgood *et al.* [5]) The determinant of the Laplacian Δ on the compact manifold M is defined to be

$$\det{}'\Delta := \prod_{\lambda_k \neq 0} \lambda_k, \tag{1.3}$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of the Laplacian Δ on M. The sequence $\{\lambda_k\}$ is known to satisfy the condition as in (1.1), but the product in (1.3) is always divergent; so, in order for the expression (1.3) to make sense, some sort of regularization procedure must be used (see, *e.g.*, [6]). It is easily seen that, formally, $e^{-Z'(0)}$ is the product of nonzero eigenvalues of Δ. This product does not converge, but $Z(s)$ can be continued analytically to a neighborhood of $s = 0$. Therefore, we can give a *meaningful* definition:

$$\det{}'\Delta := e^{-Z'(0)}, \tag{1.4}$$

which is called the *functional determinant of the Laplacian* Δ on M.

Definition 2 The *order* μ of the sequence $\{\lambda_k\}$ is defined by

$$\mu := \inf\left\{\alpha > 0 \;\Big|\; \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\alpha}} < \infty\right\}. \tag{1.5}$$

The analogous and shifted analogous Weierstrass canonical products $E(\lambda)$ and $E(\lambda, a)$ of the sequence $\{\lambda_k\}$ are defined, respectively, by

$$E(\lambda) := \prod_{k=1}^{\infty}\left\{\left(1 - \frac{\lambda}{\lambda_k}\right)\exp\left(\frac{\lambda}{\lambda_k} + \frac{\lambda^2}{2\lambda_k^2} + \cdots + \frac{\lambda^{[\mu]}}{[\mu]\lambda_k^{[\mu]}}\right)\right\} \tag{1.6}$$

and

$$E(\lambda, a) := \prod_{k=1}^{\infty}\left\{\left(1 - \frac{\lambda}{\lambda_k + a}\right)\exp\left(\frac{\lambda}{\lambda_k + a} + \cdots + \frac{\lambda^{[\mu]}}{[\mu](\lambda_k + a)^{[\mu]}}\right)\right\}, \tag{1.7}$$

where $[\mu]$ denotes the greatest integer part in the order μ of the sequence $\{\lambda_k\}$.

There exists the following relationship between $E(\lambda)$ and $E(\lambda, a)$ (see Voros [4]):

$$E(\lambda, a) = \exp\left(\sum_{m=1}^{[\mu]} \mathcal{R}_{m-1}(-a)\frac{\lambda^m}{m!}\right)\frac{E(\lambda - a)}{E(-a)}, \tag{1.8}$$

where, for convenience,

$$\mathcal{R}_{[\mu]}(\lambda - a) := \frac{d^{[\mu]+1}}{d\lambda^{[\mu]+1}}\left\{-\log E(\lambda, a)\right\}. \tag{1.9}$$

The shifted series $Z(s, a)$ of $Z(s)$ in (1.2) by a is given by

$$Z(s, a) := \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + a)^s}. \tag{1.10}$$

Formally, indeed, we have

$$Z'(0,-\lambda) = -\sum_{k=1}^{\infty} \log(\lambda_k - \lambda),$$

which, if we define

$$D(\lambda) := \exp\bigl[-Z'(0,-\lambda)\bigr], \tag{1.11}$$

immediately implies that

$$D(\lambda) = \prod_{k=1}^{\infty} (\lambda_k - \lambda).$$

In fact, Voros [4] gave the relationship between $D(\lambda)$ and $E(\lambda)$ as follows:

$$\begin{aligned} D(\lambda) = {} & \exp\bigl[-Z'(0)\bigr] \exp\left[-\sum_{m=1}^{[\mu]} \mathrm{FP}\, Z(m) \frac{\lambda^m}{m}\right] \\ & \cdot \exp\left[-\sum_{m=2}^{[\mu]} C_{-m} \left(\sum_{k=1}^{m-1} \frac{1}{k}\right) \frac{\lambda^m}{m!}\right] E(\lambda), \end{aligned} \tag{1.12}$$

where an empty sum is interpreted to be nil and the *finite part* prescription is applied (as usual) as follows (*cf.* Voros [4, p.446]):

$$\mathrm{FP} f(s) := \begin{cases} f(s) & \text{if } s \text{ is not a pole,} \\ \lim_{\epsilon \to 0} (f(s+\epsilon) - \frac{\text{Residue}}{\epsilon}) & \text{if } s \text{ is a simple pole} \end{cases} \tag{1.13}$$

and

$$Z(-m) = (-1)^m m! C_{-m}. \tag{1.14}$$

Now consider the sequence of eigenvalues on the standard Laplacian Δ_n on $\mathbf{S}^n$. It is known from the work of Vardi [7] (see also Terras [8]) that the standard Laplacian Δ_n ($n \in \mathbb{N}$) has eigenvalues

$$\mu_k := k(k+n-1) \tag{1.15}$$

with multiplicity

$$\begin{aligned} q_n(k) := {} & \binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \\ = {} & \frac{2k+n-1}{(n-1)!} \prod_{j=1}^{n-2} (k+j) \quad (k \in \mathbb{N}_0), \end{aligned} \tag{1.16}$$

where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. From now on we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ in (1.15) by $(\frac{n-1}{2})^2$ as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + \left(\frac{n-1}{2}\right)^2 = \left(k + \frac{n-1}{2}\right)^2 \tag{1.17}$$

with the same multiplicity as in (1.16).

We will exclude the zero mode, that is, start the sequence at $k = 1$ for later use. Furthermore, with a view to emphasizing n on $\mathbf{S}^n$, we choose the notations $Z_n(s)$, $Z_n(s,a)$, $E_n(\lambda)$, $E_n(\lambda,a)$, and $D_n(\lambda)$ instead of $Z(s)$, $Z(s,a)$, $E(\lambda)$, $E(\lambda,a)$, and $D(\lambda)$, respectively.

We readily observe from (1.11) that

$$D_n\left(\left(\frac{n-1}{2}\right)^2\right) = \det{}' \Delta_n, \tag{1.18}$$

where $\det{}' \Delta_n$ denotes the *determinants of the Laplacians* on $\mathbf{S}^n$ ($n \in \mathbb{N}$).

Several authors (see Choi [9], Kumagai [10], Vardi [7], and Voros [4]) used the theory of multiple gamma functions (see Barnes [11–14]) to compute the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ ($n \in \mathbb{N} := \{1,2,3,\ldots\}$). Quine and Choi [15] made use of zeta regularized products to compute $\det{}' \Delta_n$ and the determinant of the conformal Laplacian, $\det(\Delta_{\mathbf{S}^n} + n(n-2)/4)$. Choi and Srivastava [16, 17], Choi *et al.* [18], and Choi [19] made use of some closed-form evaluations of the series involving zeta function (see [20, Chapter 3]) for the computation of the determinants of the Laplacians on $\mathbf{S}^n$ ($n = 2,3,4,5,6,7,8,9$). In the sequel, here, we aim at presenting a general explicit formula for the determinants of the Laplacians on $\mathbf{S}^k$ ($k = 2n+1$; $n \in \mathbb{N}$) by mainly using a summation formula of the series involving zeta function.

2 The Stirling numbers $s(n,k)$ of the first kind

We begin by recalling the Stirling numbers $s(n,k)$ of the first kind defined by the generating functions (see, *e.g.*, [20, Section 1.5]; see also [21, Section 1.6])

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k) z^k \tag{2.1}$$

and

$$\left\{\log(1+z)\right\}^k = k! \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!} \quad \left(|z| < 1\right). \tag{2.2}$$

The following recurrence relations are satisfied by $s(n,k)$:

$$s(n+1,k) = s(n,k-1) - ns(n,k) \quad (n \geq k \geq 1); \tag{2.3}$$

$$\binom{k}{j} s(n,k) = \sum_{\ell=k-j}^{n-j} \binom{n}{\ell} s(n-\ell,j) s(\ell,k-j) \quad (n \geq k \geq j). \tag{2.4}$$

It is not difficult to see also that

$$\begin{aligned} & s(n,0) = \begin{cases} 1 & (n=0), \\ 0 & (n \in \mathbb{N}), \end{cases} \qquad s(n,n) = 1, \qquad s(n,k) = 0 \quad (k > n), \\ & s(n,1) = (-1)^{n+1}(n-1)!, \qquad s(n,n-1) = -\binom{n}{2} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & \sum_{k=1}^{n} s(n,k) = 0 \quad \bigl(n \in \mathbb{N} \setminus \{1\}\bigr); \qquad \sum_{k=0}^{n} (-1)^{n+k} s(n,k) = n!; \\ & \sum_{j=k}^{n} s(n+1,j+1) n^{j-k} = s(n,k). \end{aligned} \tag{2.6}$$

The Pochhammer symbol $(z)_n$ is defined (for $z \in \mathbb{C}$) by

$$\begin{aligned} (z)_n : &= \begin{cases} 1 & (n=0), \\ z(z+1)\cdots(z+n-1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(z+n)}{\Gamma(z)} \quad \bigl(n \in \mathbb{N}_0; z \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr), \end{aligned} \tag{2.7}$$

in terms of the gamma function Γ, and $\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}$.

From the definition (2.1) of $s(n,k)$, the Pochhammer symbol in (2.7) can be written in the form

$$(z)_n = z(z+1)\cdots(z+n-1) = \sum_{k=0}^{n} (-1)^{n+k} s(n,k) z^k, \tag{2.8}$$

where $(-1)^{n+k}s(n,k)$ denotes the number of permutations of n symbols, which has exactly k cycles.

For potential use, we observe the following simple properties related to $s(n,k)$ in the lemma below.

Lemma 1 *For $n \in \mathbb{N}$, let*

$$\left[\sum_{j=1}^{n} s(n,j) z^j\right]\left[\sum_{j=1}^{n} (-1)^{n+j} s(n,j) z^j\right] := \sum_{\ell=1}^{2n} C_\ell(n) z^\ell.$$

Then we have

$$C_\ell(n) = \sum_{j=1}^{\ell-1} (-1)^j s(n,j) s(n,\ell-j) \quad \textit{and} \quad C_{2\ell+1}(n) = 0 \quad (\ell \in \mathbb{N}_0). \tag{2.9}$$

Proof It is easy to see the first expression for $C_\ell(n)$. For the second one, it is enough to see that the defined product is an even function of z. □

For later use, we compute the first few values of $C_\ell(n)$ as in Lemma 2.

Lemma 2 *The values of* $C_{2\ell}(n)$ *when* $(\ell, n) = (1,1), (1,2), (2,2), (3,2)$, *are computed as follows*:

$$C_2(1) = -1; \quad C_2(2) = -1; \quad C_4(2) = 1; \quad C_2(3) = -4; \quad C_4(3) = 5;$$
$$C_6(3) = -1; \quad C_2(4) = -36; \quad C_4(4) = 49; \quad C_6(4) = -14; \quad C_8(4) = 1.$$

3 Series associated with the zeta functions

A rather classical (over two centuries old) theorem of Christian Goldbach (1690-1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700-1782), was revived in 1986 by Shallit and Zikan [22] as the following problem:

$$\sum_{\omega \in \mathcal{S}} (\omega - 1)^{-1} = 1, \tag{3.1}$$

where $\mathcal{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathcal{S} := \{ n^k \mid n, k \in \mathbb{N} \setminus \{1\} \}. \tag{3.2}$$

In terms of the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1), \end{cases} \tag{3.3}$$

Goldbach's theorem (3.1) assumes the elegant form (*cf.* Shallit and Zikan [22, p.403])

$$\sum_{k=2}^{\infty} \{ \zeta(k) - 1 \} = 1 \tag{3.4}$$

or, equivalently,

$$\sum_{k=2}^{\infty} \mathcal{F}(\zeta(k)) = 1, \tag{3.5}$$

where $\mathcal{F}(x) := x - [x]$ denotes the *fractional* part of $x \in \mathbb{R}$. As a matter of fact, it is fairly straightforward to observe also that

$$\sum_{k=2}^{\infty} (-1)^k \mathcal{F}(\zeta(k)) = \frac{1}{2}, \tag{3.6}$$

$$\sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k)) = \frac{3}{4} \quad \text{and} \quad \sum_{k=1}^{\infty} \mathcal{F}(\zeta(2k+1)) = \frac{1}{4}. \tag{3.7}$$

The *Hurwitz* (or *generalized*) *zeta function* $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \quad (\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{3.8}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. It is noted that both the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s,a)$ can be continued meromorphically to the whole complex s-plane except for a simple pole only at $s = 1$ with their residue 1. For easy reference, we recall some properties of $\zeta(s)$ and $\zeta(s,a)$ as in the following lemma.

Lemma 3 *Each of the following identities holds true.*

$$\zeta(s) = \zeta(s,1) = \left(2^s - 1\right)^{-1} \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s,2), \tag{3.9}$$

$$\zeta(s,a) = \zeta(s,n+a) + \sum_{k=0}^{n-1} (k+a)^{-s} \quad (n \in \mathbb{N}), \tag{3.10}$$

$$\zeta(s) = \zeta(s,n+1) + \sum_{k=1}^{n} k^{-s} \quad (n \in \mathbb{N}_0), \tag{3.11}$$

$$\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1} \quad (n \in \mathbb{N}_0), \tag{3.12}$$

where $B_n(a)$ *and* $B_n := B_n(0)$ *are the Bernoulli polynomials and numbers, respectively* (*see* [20, Section 1.6]; *see also* [20, p.85, Eq. (17)]). *It is said that* $s = -2n$ ($n \in \mathbb{N}$) *are the trivial zeros of* $\zeta(s)$:

$$\zeta(-2n) = 0 \quad (n \in \mathbb{N}). \tag{3.13}$$

It is known that

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N}_0) \tag{3.14}$$

and

$$\zeta'(-2n) = \begin{cases} (-1)^n \frac{(2n)!}{2(2\pi)^{2n}} \zeta(2n+1) & (n \in \mathbb{N}), \\ -\frac{1}{2} \log(2\pi) & (n = 0). \end{cases} \tag{3.15}$$

Employing the various methods and techniques used in the vast literature on the subject of the closed-form evaluations series associated with the zeta functions, Srivastava and Choi (see [20, Chapter 3], [21, Chapter 3], and see also the related references therein) presented a rather extensive collection of closed-form sums of series involving the zeta functions. For the use in the next section, we recall two general formulas as in the following lemma (see [21, p.254]).

Lemma 4 *The following identity holds true*:

$$\begin{aligned} \sum_{\ell=1}^{\infty} \frac{\zeta(2\ell,a)}{\ell+j} t^{2\ell+2j} = {} & \sum_{m=0}^{2j} \binom{2j}{m} \left[\zeta'(-m,a-t) + (-1)^m \zeta'(-m,a+t)\right] t^{2j-m} \\ & - \sum_{m=0}^{j-1} \frac{\zeta(-2m,a)}{j-m} t^{2j-2m} - 2\zeta'(-2j,a) \quad \left(j \in \mathbb{N}_0; |t| < |a|\right). \end{aligned} \tag{3.16}$$

By setting $t = n$ and $a = n + 1$ in (3.16) and using suitable formulas given in this section, we get a special case identity of (3.16) as in Lemma 5 below.

Lemma 5 *The following identity holds true*:

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, n+1)}{\ell + j} n^{2\ell+2j} \\
&\quad = n^{2j} \log\left[\frac{(2n)!}{2\pi}\right] - (-1)^j \frac{(2j)!}{(2\pi)^{2j}} \zeta(2j+1) \\
&\qquad + \sum_{m=1}^{j} \binom{2j}{2m} \left[(-1)^m \frac{(2m)!}{(2\pi)^{2m}} \zeta(2m+1) + \sum_{\ell=1}^{2n} \ell^{2m} \log \ell\right] n^{2j-2m} \\
&\qquad - \sum_{m=0}^{j-1} \binom{2j}{2m+1} \left(\sum_{\ell=1}^{2n} \ell^{2m+1} \log \ell\right) n^{2j-2m-1} - 2 \sum_{\ell=1}^{n} \ell^{2j} \log \ell \\
&\qquad + \frac{1}{j}\left(\frac{1}{2} + n\right) n^{2j} + \sum_{m=1}^{j-1} \frac{1}{j-m} \left(\sum_{\ell=1}^{n} \ell^{2m}\right) n^{2j-2m} \quad (n, j \in \mathbb{N}).
\end{aligned} \tag{3.17}$$

Further specialized formulas of the identity in (3.17), for later use, are obtained as in Lemma 6 below.

Lemma 6 *Each of the following formulas holds true*:

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell) - 1}{\ell + 1} = \frac{3}{2} - \log \pi, \tag{3.18}$$

which has been recorded and used in several places (*see*, *for example*, [23, p.388, Eq. (2.17)], [16, p.215, Eq. (2.18)], [20, p.219, Eq. (519)])

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 3)}{\ell + 1} 2^{2\ell+2} = 10 - 4 \log 2 + \log 3 - 4 \log \pi, \tag{3.19}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 3)}{\ell + 2} 2^{2\ell+4} = 40 - 16 \log 2 + \log 3 - 16 \log \pi - \frac{12}{\pi^2} \zeta(3), \tag{3.20}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 4)}{\ell + 1} 3^{2\ell+2} = \frac{63}{2} - 5 \log 2 - 9 \log 3 + 4 \log 5 - 9 \log \pi, \tag{3.21}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 4)}{\ell + 2} 3^{2\ell+4} = \frac{1{,}071}{4} - 29 \log 2 - 81 \log 3 + 16 \log 5 - 81 \log \pi - \frac{27\zeta(3)}{\pi^2}, \tag{3.22}$$

$$\begin{aligned}
\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 4)}{\ell + 3} 3^{2\ell+6} &= \frac{4{,}599}{2} - 125 \log 2 - 729 \log 3 + 64 \log 5 - 729 \log \pi \\
&\quad - \frac{1{,}215\zeta(3)}{2\pi^2} + \frac{405\zeta(5)}{2\pi^4},
\end{aligned} \tag{3.23}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 5)}{\ell + 1} 4^{2\ell+2} = -32 \log 2 - 13 \log 3 + \log 5 + 9 \log 7 - 16 \log \pi, \tag{3.24}$$

$$\begin{aligned}
\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, 5)}{\ell + 2} 4^{2\ell+4} &= 1{,}056 + 77{,}312 \log 2 + 11{,}215 \log 3 + 6{,}049 \log 5 \\
&\quad + 14{,}129 \log 7 - 256 \log \pi - \frac{48\zeta(3)}{\pi^2},
\end{aligned} \tag{3.25}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell,5)}{\ell+3} 4^{2\ell+6} = 15{,}648 - 8{,}192 \log 2 - 1{,}393 \log 3 + \log 5 + 729 \log 7 \\ - 4{,}096 \log \pi - \frac{1{,}920\zeta(3)}{\pi^2} + \frac{360\zeta(5)}{\pi^4}, \tag{3.26}$$

$$\sum_{\ell=1}^{\infty} \frac{\zeta(2\ell,5)}{\ell+4} 4^{2\ell+8} = 238{,}240 - 131{,}072 \log 2 - 12{,}865 \log 3 + \log 5 + 6{,}561 \log 7 \\ - 65{,}536 \log \pi - \frac{57{,}344\zeta(3)}{\pi^2} + \frac{26{,}880\zeta(5)}{\pi^4} - \frac{5{,}040\zeta(7)}{\pi^6}. \tag{3.27}$$

4 The determinants of the Laplacians on $\mathbf{S}^k$ ($k = 2n+1$)

Here, by using (1.18) and the results in the previous sections, we are ready to compute the determinants of the Laplacians on $\mathbf{S}^k$ ($k = 2n+1$) as in the following theorem.

Theorem *The determinants of the Laplacians on* $\mathbf{S}^k$ ($k = 2n+1$) *are given as follows*:

$$\det{}' \Delta_{2n+1} = \exp\bigl[-Z'_{2n+1}(0)\bigr] \exp\Biggl[-\sum_{m=1}^{n} Z_{2n+1}(m) \frac{n^{2m}}{m}\Biggr] \\ \cdot \exp\Biggl[-\sum_{m=2}^{n} \Omega_{-m}(n) H_{m-1} \frac{n^{2m}}{m!}\Biggr] E_{2n+1}\bigl(n^2\bigr), \tag{4.1}$$

where

$$Z_{2n+1}(-m) := (-1)^m m! \Omega_{-m}(n) \quad (m \in \mathbb{N}) \tag{4.2}$$

and H_m *are harmonic numbers defined by*

$$H_m := \sum_{k=1}^{m} \frac{1}{k} \quad (m \in \mathbb{N}). \tag{4.3}$$

Here we have

$$Z'_{2n+1}(0) = \frac{4(-1)^n}{(2n)!} \sum_{\ell=1}^{n} C_{2\ell}(n) \Biggl[(-1)^{\ell} \frac{(2\ell)!}{2(2\pi)^{2\ell}} \zeta(2\ell+1) + \sum_{j=1}^{n} j^{2\ell} \log j\Biggr], \tag{4.4}$$

$$Z_{2n+1}(m) = \frac{2(-1)^n}{(2n)!} \\ \cdot \Biggl[\sum_{\ell=1}^{m} C_{2\ell}(n) \Biggl((-1)^{m-\ell+1} \frac{(2\pi)^{2(m-\ell)}}{2(2m-2\ell)!} B_{2(m-\ell)} - \sum_{j=1}^{n} \frac{1}{j^{2m-2\ell}}\Biggr) \\ - \sum_{\ell=m+1}^{n} C_{2\ell}(n) \Biggl(\sum_{j=1}^{n} \frac{1}{j^{2m-2\ell}}\Biggr)\Biggr], \tag{4.5}$$

$$Z_{2n+1}(-m) = (-1)^m m! \Omega_{-m}(n) = \frac{2(-1)^{n+1}}{(2n)!} \sum_{\ell=1}^{n} C_{2\ell}(n) \Biggl(\sum_{j=1}^{n} j^{2m+2\ell}\Biggr) \tag{4.6}$$

and

$$\log E_{2n+1}\left(n^2\right) = \frac{2(-1)^{n+1}}{(2n)!}\sum_{j=1}^{n} C_{2j}(n)P(n,j), \tag{4.7}$$

where, for convenience,

$$\begin{aligned} P(n,j) &:= \sum_{\ell=n+1}^{\infty} \frac{\zeta(2\ell-2j,n+1)}{\ell} n^{2\ell} \\ &= \sum_{\ell=1}^{n-j}\left[(-1)^{\ell}\frac{(2\pi)^{2\ell}}{2(2\ell)!}B_{2\ell} + \sum_{k=1}^{n}\frac{1}{k^{2\ell}}\right]\frac{n^{2\ell+2j}}{\ell+j} + \sum_{\ell=1}^{\infty}\frac{\zeta(2\ell,n+1)}{\ell+j}n^{2\ell+2j}, \end{aligned} \tag{4.8}$$

of which the last series is expressed in a closed-form as given in Lemma 5.

Proof In view of (1.15), the sequence $\{\mu_k\}$ of eigenvalues on the standard Laplacian Δ_{2n+1} on $\mathbf{S}^{2n+1}$ is given as follows: $\mu_k := k(k+2n)$ with multiplicity $q_{2n+1}(k)$ as in (1.16). Here we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ by n^2 as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + n^2 = (k+n)^2 \tag{4.9}$$

with the same multiplicity $q_{2n+1}(k)$. It is noted that λ_k has the order

$$\mu = (2n+1)/2 = n + 1/2. \tag{4.10}$$

It follows from (1.18) and (1.12) with the aid of (4.9) and (4.10) that

$$\begin{aligned} \det{}' \Delta_{2n+1} = {} & \exp\left[-Z'_{2n+1}(0)\right]\exp\left[-\sum_{m=1}^{n}\mathrm{FP}\, Z_{2n+1}(m)\frac{n^{2m}}{m}\right] \\ & \cdot \exp\left[-\sum_{m=2}^{n}\Omega_{-m}(n)H_{m-1}\frac{n^{2m}}{m!}\right]E_{2n+1}\left(n^2\right), \end{aligned} \tag{4.11}$$

where $\Omega_{-m}(n)$ and H_m are given as in (4.2) and (4.3), respectively.

In order to express $Z_{2n+1}(s)$ in terms of certain known functions, we first consider the corresponding multiplicity $q_{2n+1}(k)$: From (1.16), we have

$$\begin{aligned} q_{2n+1}(k) &= \frac{2(n+k)}{(2n)!}\prod_{j=1}^{2n-1}(k+j) \\ &= \frac{2}{(2n)!}\cdot\left[(k+1)(k+2)\cdots(k+n)\right]\cdot\left[(k+n)(k+n+1)\cdots(k+2n-1)\right]. \end{aligned}$$

We find from (2.1) and (2.8) that

$$q_{2n+1}(k) = \frac{2}{(2n)!}\left[\sum_{j=1}^{n}s(n,j)(k+n)^j\right]\left[\sum_{j=1}^{n}(-1)^{n+j}s(n,j)(k+n)^j\right],$$

which, in view of Lemma 1 (see also (2.9)), is expressed as follows:

$$q_{2n+1}(k) = \frac{2(-1)^n}{(2n)!} \sum_{\ell=1}^{n} C_{2\ell}(n)(k+n)^{2\ell}. \tag{4.12}$$

Now it is easy to see that

$$Z_{2n+1}(s) := \sum_{k=1}^{\infty} \frac{q_{2n+1}(k)}{(k+n)^{2s}} = \frac{2(-1)^n}{(2n)!} \sum_{\ell=1}^{n} C_{2\ell}(n)\zeta(2s-2\ell, n+1), \tag{4.13}$$

where $C_\ell(n)$ is given in (2.9).

Since the Hurwitz zeta function $\zeta(s,a)$ has only a simple pole at $s = 1$ with its residue 1, it is seen that $Z_{2n+1}(s)$ has simple poles at $s = \ell + 1/2$ ($\ell = 1, 2, \ldots, n$) with their respective residue $(-1)^n C_{2\ell}(n)/(2n)!$. Therefore we find from the finite part prescription (1.13) that

$$\operatorname{FP} Z_{2n+1}(m) = Z_{2n+1}(m) \quad (m = 1, 2, \ldots, n). \tag{4.14}$$

It is seen that applying (4.14) to (4.11) proves (4.1). Employing (3.11), (3.13), (3.14), and (3.15) in (4.13), we obtain the expressions (4.4), (4.5), and (4.6).

In view of (1.6), (4.9), and (4.10), we obtain

$$\begin{aligned} E_{2n+1}(n^2) = \prod_{k=1}^{\infty} \Biggl\{ & \left(1 - \frac{n^2}{(k+n)^2}\right)^{q_{2n+1}(k)} \\ & \cdot \exp\left[q_{2n+1}(k) \left(\frac{n^2}{(k+n)^2} + \frac{n^4}{2(k+n)^4} + \cdots + \frac{n^{2n}}{n(k+n)^{2n}} \right) \right] \Biggr\}. \end{aligned} \tag{4.15}$$

Taking the logarithm of both sides in (4.15) and considering

$$\log(1-x) = -\sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell} \quad (-1 \leqq x < 1), \tag{4.16}$$

we obtain

$$\log E_{2n+1}(n^2) = -\sum_{\ell=n+1}^{\infty} \frac{n^{2\ell}}{\ell} \sum_{k=1}^{\infty} \frac{q_{2n+1}(k)}{(k+n)^{2\ell}}. \tag{4.17}$$

Applying (4.12) to (4.17), we get (4.7). Letting $\ell - j = \ell'$ in the definition of $P(n,j)$ in (4.8) and dropping the prime on ℓ, we obtain

$$\begin{aligned} P(n,j) &= \sum_{\ell=n+1-j}^{\infty} \frac{\zeta(2\ell, n+1)}{\ell + j} n^{2\ell+2j} \\ &= -\sum_{\ell=1}^{n-j} \frac{\zeta(2\ell, n+1)}{\ell + j} n^{2\ell+2j} + \sum_{\ell=1}^{\infty} \frac{\zeta(2\ell, n+1)}{\ell + j} n^{2\ell+2j} \\ &\quad (n, j \in \mathbb{N}; 1 \leqq j \leqq n), \end{aligned}$$

which, upon using (3.11) and (3.14) in the first finite series, yields (4.8). □

By setting $n = 1, 2, 3, 4$ in (4.1), we give $\det'\Delta_{2n+1}$ as in the following corollary.

Corollary *The determinants of the Laplacians on* $\mathbf{S}^k$ $(k = 3, 5, 7, 9)$ *are given as follows*:

$$\det'\Delta_3 = \pi \exp\left[\frac{\zeta(3)}{2\pi^2}\right], \tag{4.18}$$

$$\det'\Delta_5 = \frac{\pi}{2} \exp\left[\frac{197}{3} + \frac{23\zeta(3)}{24\pi^2} - \frac{\zeta(5)}{8\pi^4}\right], \tag{4.19}$$

$$\det'\Delta_7 = \frac{\pi}{3} \exp\left[-\frac{164{,}033}{8} + \frac{949\zeta(3)}{720\pi^2} - \frac{13\zeta(5)}{24\pi^4} + \frac{\zeta(7)}{32\pi^6}\right], \tag{4.20}$$

$$\begin{aligned} \det'\Delta_9 = {} & 2^{-\frac{506{,}669}{2{,}520}} 3^{-\frac{14{,}285}{504}} 5^{-\frac{147}{10}} 7^{-\frac{3{,}073}{90}} \pi \\ & \cdot \exp\left[\frac{24{,}546{,}672{,}397}{1{,}890} + \frac{16{,}399\zeta(3)}{10{,}080\pi^2}\right. \\ & \left. - \frac{2{,}087\zeta(5)}{1{,}920\pi^4} + \frac{31\zeta(7)}{128\pi^6} - \frac{\zeta(9)}{128\pi^8}\right]. \end{aligned} \tag{4.21}$$

Competing interests
The author declares that he has no competing interests.

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