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Identities involving harmonic and hyperharmonic numbers

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Abstract
In this paper, we give some new and interesting identities involving harmonic and hyperharmonic numbers which are derived from the transfer formula for the associated sequences.

1 Introduction

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbf{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbf{C} \right\}. \tag{1}$$

Suppose that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbf{C}$ and that $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$.

Let $f(t) \in \mathcal{F}$. Then we consider a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0) \text{ (see [1, 2])}. \tag{2}$$

From (1) and (2), we note that

$$\langle t^k|x^n \rangle = n!\delta_{n,k} \quad (n, k \geq 0) \text{ (see [1, 3–5])}, \tag{3}$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^n\rangle}{k!} t^k$. Then we see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $O(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 0$, then $f(t)$ is called an invertible series. If $O(f(t)) = 1$, then $f(t)$ is called a delta series. Let $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$ for $n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [1, 3, 6]). If $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$. By (3), we easily see that $\langle e^{yt}|p(x)\rangle = p(y)$.

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k \quad \text{(see [1, 6, 7])}. \tag{4}$$

From (4), we note that

$$p^{(k)}(0) = \langle t^k|p(x)\rangle, \qquad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{5}$$

By (5), we easily see that

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (k \geq 0) \text{ (see [2, 3, 6, 7])}. \tag{6}$$

Let $\phi_n(x)$ be exponential polynomials which are given by

$$\sum_{k=0}^{\infty} \frac{\phi_k(x)}{k!} t^k = e^{x(e^t-1)} \quad \text{(see [2, 6, 8])}. \tag{7}$$

Thus, by (7), we get

$$\phi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \sim \bigl(1, \log(1+t)\bigr), \tag{8}$$

where $S_2(n,k)$ is the Stirling number of the second kind.

The Stirling number of the first kind is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} S_1(n,k)x^k. \tag{9}$$

Thus, by (9), we get

$$S_1(n,k) = \frac{1}{k!}\langle t^k|(x)_n\rangle \quad \text{(see [2, 5])}. \tag{10}$$

Let $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$. Then the transfer formula for the associated sequences is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad \text{(see [2, 8])}. \tag{11}$$

The nth harmonic number is $H_n = \sum_{i=1}^{n} \frac{1}{i}$ $(n \geq 1)$ and $H_0 = 0$.

In general, the hyperharmonic number $H_n^{(r)}$ of order r is defined by

$$H_n^{(r)} = \begin{cases} 0 & \text{if } n \leq 0 \text{ or } r < 0, \\ \frac{1}{n} & \text{if } r = 0, n \geq 1, \\ \sum_{i=1}^{n} H_i^{(r-1)} & \text{if } r, n \geq 1 \end{cases} \qquad \text{(see [9, 10])}. \tag{12}$$

From (12), we note that $H_n^{(1)}$ is the ordinary harmonic number H_n. It is known that

$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}) \quad \text{(see [9, 10])}. \tag{13}$$

The generating functions of the harmonic and hyperharmonic numbers are given by

$$\sum_{n=1}^{\infty} H_n t^n = -\frac{\log(1-t)}{1-t} \tag{14}$$

and

$$\sum_{n=1}^{\infty} H_n^{(r)} t^n = -\frac{\log(1-t)}{(1-t)^r}, \quad \text{respectively.} \tag{15}$$

The purpose of this paper is to give some new and interesting identities involving harmonic and hyperharmonic numbers which are derived from the transfer formula for the associated sequences.

2 Identities involving harmonic and hyperharmonic numbers

From (7) and (8), we note that

$$\phi_n(x) = \sum_{j=0}^{n} S_2(n,j)x^j \sim \big(1, \log(1+t)\big) \tag{16}$$

and

$$(-1)^n \phi_n(-x) \sim \big(1, -\log(1-t)\big). \tag{17}$$

Let us assume that

$$q_n(x) \sim \big(1, t(1-t)^r\big). \tag{18}$$

From (11), (18) and $x^n \sim (1,t)$, we note that

$$\begin{aligned}
q_n(x) &= x\left(\frac{t}{t(1-t)^r}\right)^n x^{-1}x^n = x(1-t)^{-rn}x^{n-1} \\
&= x\sum_{k=0}^{n-1}\binom{-rn}{k}(-t)^k x^{n-1} = x\sum_{k=0}^{n-1}\binom{rn+k-1}{k}t^k x^{n-1} \\
&= x\sum_{k=0}^{n-1}\binom{rn+k-1}{k}(n-1)_k x^{n-1-k} = \sum_{k=1}^{n-1}\binom{rn+k-1}{k}(n-1)_k x^{n-k} \\
&= \sum_{k=1}^{n}\binom{rn+n-k-1}{n-k}(n-1)_{n-k}x^k.
\end{aligned} \tag{19}$$

Now, we use the following fact:

$$\sum_{n=1}^{\infty} H_n^{(r)} t^n = -\frac{\log(1-t)}{(1-t)r}. \tag{20}$$

For $n \geq 1$, by (11), (17) and (18), we get

$$\begin{aligned}
q_n(x) &= x\left(\frac{-\log(1-t)}{t(1-t)^r}\right)^n x^{-1}(-1)^n \phi_n(-x) \\
&= x\left(\sum_{l=0}^{\infty} H_{l+1}^{(r)} t^l\right)^n x^{-1}(-1)^n \sum_{j=1}^{n} S_2(n,j)(-x)^j \\
&= (-1)^n \sum_{j=1}^{n} S_2(n,j)(-1)^j x\left(\sum_{l=0}^{\infty} H_{l+1}^{(r)} t^l\right)^n x^{j-1} \\
&= (-1)^n \sum_{j=1}^{n} S_2(n,j)(-1)^j x\left(\sum_{l=0}^{j-1}\left(\sum_{l_1+\cdots+l_n=l} H_{l_1+1}^{(r)}\cdots H_{l_n+1}^{(r)}\right) t^l\right) x^{j-1} \\
&= (-1)^n \sum_{j=1}^{n}\sum_{l=0}^{j-1}\sum_{l_1+\cdots+l_n=l} S_2(n,j)(-1)^j H_{l_1+1}^{(r)}\cdots H_{l_n+1}^{(r)}(j-1)_l x^{j-l} \\
&= (-1)^n \sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{l_1+\cdots+l_n=j-k} S_2(n,j)(-1)^j H_{l_1+1}^{(r)}\cdots H_{l_n+1}^{(r)}(j-1)_{j-k} x^k \\
&= (-1)^n \sum_{k=1}^{n}\left\{\sum_{j=k}^{n}\sum_{l_1+\cdots+l_n=j-k} (-1)^j S_2(n,j) H_{l_1+1}^{(r)}\cdots H_{l_n+1}^{(r)}(j-1)_{j-k}\right\} x^k. \qquad (21)
\end{aligned}$$

Therefore, by comparing coefficients on both sides of (19) and (20), we obtain the following theorem.

Theorem 1 *For $n \geq 1$, $r \geq 1$, $1 \leq k \leq n$, we have*

$$\binom{rn+n-k-1}{n-k}(n-1)_{n-k} = (-1)^n \sum_{j=k}^{n}\sum_{l_1+\cdots+l_n=j-k} S_2(n,j)(-1)^j H_{l_1+1}^{(r)}\cdots H_{l_n+1}^{(r)}(j-1)_{j-k}.$$

We recall the following equation:

$$\left(\frac{\log(1+t)}{t}\right)^n = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_1(l+n,n) t^l. \qquad (22)$$

For $n \geq 1$, from (11), (17) and (18), we have

$$\begin{aligned}
q_n(x) &= x\left(\frac{-\log(1-t)}{t(1-t)^r}\right)^n x^{-1}(-1)^n \phi_n(-x) \\
&= x\left(\frac{\log(1-t)}{-t}\right)^n (1-t)^{-rn} x^{-1}(-1)^n \phi_n(-x) \\
&= (-1)^n \sum_{j=1}^{n} S_2(n,j)(-1)^j x\left(\frac{\log(1-t)}{-t}\right)^n (1-t)^{-rn} x^{j-1} \\
&= (-1)^n \sum_{j=1}^{n} S_2(n,j)(-1)^j x\left(\frac{\log(1-t)}{-t}\right)^n \sum_{l=0}^{j-1}\binom{rn+l-1}{l}(j-1)_l x^{j-1-l}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^n \sum_{j=1}^{n} S_2(n,j)(-1)^j \sum_{l=0}^{j-1} \binom{rn+l-1}{l} (j-1)_l x \sum_{m=0}^{j-1-l} \frac{n!}{(m+n)!} \\
&\quad \times S_1(m+n,n)(-t)^m x^{j-1-l} \\
&= (-1)^n \sum_{j=1}^{n} \sum_{l=0}^{j-1} \sum_{m=0}^{j-1-l} (-1)^{j+m} \binom{rn+l-1}{l} \frac{n!}{(m+n)!} \frac{(j-1)!}{(j-1-l-m)!} \\
&\quad \times S_1(m+n,n) S_2(n,j) x^{j-l-m} \\
&= (-1)^n \sum_{k=1}^{n} \Biggl\{ \sum_{j=k}^{n} \sum_{l=0}^{j-k} (-1)^{k+l} \binom{rn+l-1}{l} \frac{n!}{(j-l-k+n)!} \frac{(j-1)!}{(k-1)!} \\
&\quad \times S_1(j-l-k+n,n) S_2(n,j) \Biggr\} x^k.
\end{aligned} \tag{23}$$

Therefore, by (19) and (23), we obtain the following theorem.

Theorem 2 *For $r, n \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned}
&\binom{rn+n-k-1}{n-k} (n-1)_{n-k} \\
&\quad = (-1)^n \sum_{j=k}^{n} \sum_{l=0}^{j-k} (-1)^{k+l} \binom{rn+l-1}{l} \frac{n!}{(j-l-k+n)!} \frac{(j-1)!}{(k-1)!} \\
&\quad \times S_1(j-l-k+n,n) S_2(n,j).
\end{aligned}$$

Here we invoke the following identity:

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} m H_m^{(r)} \right) t^n = \frac{t(1 - r\log(1-t))}{(1-t)^{r+2}}. \tag{24}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \bigl(1, t(1-t)^{r+2}\bigr). \tag{25}$$

For $n \geq 1$, by (19) and (25), we get

$$q_n(x) = \sum_{k=1}^{n} \binom{(r+3)n-k-1}{n-k} (n-1)_{n-k} x^k. \tag{26}$$

Let us assume that

$$p_n(x) \sim \bigl(1, t\bigl(1 - r\log(1-t)\bigr)\bigr). \tag{27}$$

For $n \geq 1$, by (11), (27) and $x^n \sim (1,t)$, we get

$$\begin{aligned}
p_n(x) &= 7x \left(\frac{t}{t(1-r\log(1-t))} \right)^n x^{-1} x^n \\
&= x\bigl(1 - r\log(1-t)\bigr)^{-n} x^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= x\sum_{l=0}^{\infty}\binom{n+l-1}{l}r^l\bigl(\log(1-t)\bigr)^l x^{n-1} \\
&= x\sum_{l=0}^{n-1}\binom{n+l-1}{l}r^l\sum_{j=0}^{n-1-l}\frac{l!}{(j+l)!}S_1(j+l,l)t^{j+l}x^{n-1} \\
&= \sum_{l=0}^{n-1}\sum_{j=0}^{n-1-l}l!r^l\binom{n+l-1}{l}\binom{n-1}{j+l}S_1(j+l,l)x^{n-j-l} \\
&= \sum_{k=1}^{n}\left\{\sum_{l=0}^{n-k}l!r^l\binom{n+l-1}{l}\binom{n-1}{k-1}S_1(n-k,l)\right\}x^k. \qquad (28)
\end{aligned}
$$

For $n \geq 1$, from (11), (25) and (27), we can derive the following equation:

$$
\begin{aligned}
q_n(x) &= x\left(\frac{t(1-r\log(1-t))}{t(1-t)^{r+2}}\right)^n x^{-1}p_n(x) \\
&= x\left(\sum_{j=1}^{\infty}\left(\sum_{m=1}^{j}mH_m^{(r)}\right)t^{j-1}\right)^n\sum_{a=1}^{n}\left\{\sum_{l=0}^{n-a}l!r^l\binom{n+l-1}{l}\binom{n-1}{a-1}\right. \\
&\quad\left.\times S_1(n-a,l)\right\}x^{a-1} \\
&= \sum_{a=1}^{n}\left\{\sum_{l=0}^{n-a}l!r^l\binom{n+l-1}{l}\binom{n-1}{a-1}S_1(n-a,l)\right\} \\
&\quad\times x\left[\sum_{j=0}^{\infty}\left\{\sum_{j_1+\cdots+j_n=j}\left(\sum_{m_1=1}^{j_1+1}\cdots\sum_{m_n=1}^{j_n+1}m_1\cdots m_nH_{m_1}^{(r)}\cdots H_{m_n}^{(r)}\right)\right\}t^j\right]x^{a-1} \\
&= \sum_{a=1}^{n}\sum_{l=0}^{n-a}\sum_{k=1}^{a}\sum_{j_1+\cdots+j_n=a-k}\left(\sum_{m_1=1}^{j_1+1}\cdots\sum_{m_n=1}^{j_n+1}m_1\cdots m_nH_{m_1}^{(r)}\cdots H_{m_n}^{(r)}\right) \\
&\quad\times l!r^l\binom{n+l-1}{l}\binom{n-1}{a-1}S_1(n-a,l)(a-1)_{a-k}x^k \\
&= \sum_{k=1}^{n}\left\{\sum_{a=k}^{n}\sum_{l=0}^{n-a}\sum_{j_1+\cdots+j_n=a-k}\left(\sum_{m_1=1}^{j_1+1}\cdots\sum_{m_n=1}^{j_n+1}m_1\cdots m_nH_{m_1}^{(r)}\cdots H_{m_n}^{(r)}\right)\right. \\
&\quad\left.\times l!r^l\binom{n+l-1}{l}\binom{n-1}{a-1}S_1(n-a,l)(a-1)_{a-k}\right\}x^k. \qquad (29)
\end{aligned}
$$

Therefore, by (26) and (29), we obtain the following theorem.

Theorem 3 *For $n, r \geq 1$, $1 \leq k \leq n$, we have*

$$
\begin{aligned}
&\binom{(r+3)n-k-1}{n-k}(n-1)_{n-k} \\
&= \sum_{a=k}^{n}\sum_{l=0}^{n-a}\sum_{j_1+\cdots+j_n=a-k}\left(\sum_{m_1=1}^{j_1+1}\cdots\sum_{m_n=1}^{j_n+1}m_1\cdots m_nH_{m_1}^{(r)}\cdots H_{m_n}^{(r)}\right)l!r^l \\
&\quad\times\binom{n+l-1}{l}\binom{n-1}{a-1}S_1(n-a,l)(a-1)_{n-k}.
\end{aligned}
$$

Here we use the following identity:

$$\sum_{n=1}^{\infty} nH_n^{(r)} t^n = \frac{t(1 - r\log(1-t))}{(1-t)^{r+1}}. \tag{30}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \bigl(1, t(1-t)^{r+1}\bigr). \tag{31}$$

For $n \geq 1$, from (19) and (31), we have

$$q_n(x) = \sum_{k=1}^{n} \binom{(r+2)n-k-1}{n-k} (n-1)_{n-k} x^k. \tag{32}$$

Let us assume that

$$p_n(x) \sim \bigl(1, t\bigl(1 - r\log(1-t)\bigr)\bigr). \tag{33}$$

Then, from (28) and (33), we note that, for $n \geq 1$,

$$p_n(x) = \sum_{k=1}^{n} \left\{ \sum_{l=0}^{n-k} l! r^l \binom{n+l-1}{l} \binom{n-1}{k-1} S_1(n-k, l) \right\} x^k. \tag{34}$$

For $n \geq 1$, by (11), (32) and (33), we get

$$\begin{aligned}
q_n(x) &= x\left(\frac{t(1-r\log(1-t))}{t(1-t)^{r+1}}\right)^n x^{-1} p_n(x) \\
&= x\left(\sum_{j=1}^{\infty} jH_j^{(r)} t^{j-1}\right)^n x^{-1} \sum_{a=1}^{n} \left\{ \sum_{l=0}^{n-a} l! r^l \binom{n+l-1}{l} \binom{n-1}{a-1} S_1(n-a, l) \right\} x^a \\
&= \sum_{a=1}^{n} \left\{ \sum_{l=0}^{n-a} l! r^l \binom{n+l-1}{l} \binom{n-1}{a-1} S_1(n-a, l) \right\} \\
&\quad \times x \sum_{j=0}^{a-1} \left(\sum_{j_1+\cdots+j_n=j} (j_1+1)\cdots(j_n+1) H_{j_1+1}^{(r)} \cdots H_{j_n+1}^{(r)} \right) t^j x^{a-1} \\
&= \sum_{a=1}^{n} \sum_{l=0}^{n-a} \sum_{j=0}^{a-1} \left(\sum_{j_1+\cdots+j_n=j} (j_1+1)\cdots(j_n+1) H_{j_1+1}^{(r)} \cdots H_{j_n+1}^{(r)} \right) l! r^l \\
&\quad \times \binom{n+l-1}{l} \binom{n-1}{a-1} S_1(n-a, l)(a-1)_j x^{a-j} \\
&= \sum_{k=1}^{n} \left\{ \sum_{a=k}^{n} \sum_{l=0}^{n-a} \left(\sum_{j_1+\cdots+j_n=a-k} (j_1+1)\cdots(j_n+1) H_{j_1+1}^{(r)} \cdots H_{j_n+1}^{(r)} \right) l! r^l \right. \\
&\quad \left. \times \binom{n+l-1}{l} \binom{n-1}{a-1} S_1(n-a, l)(a-1)_{a-k} \right\} x^k.
\end{aligned} \tag{35}$$

Therefore, by (32) and (35), we obtain the following theorem.

Theorem 4 *For $n, r \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned}
&\binom{(r+2)n-k-1}{n-k}(n-1)_{n-k} \\
&\quad = \sum_{a=k}^{n}\sum_{l=0}^{n-a}\left(\sum_{j_1+\cdots+j_n=a-k}(j_1+1)\cdots(j_n+1)H_{j_1+1}^{(r)}\cdots H_{j_n+1}^{(r)}\right)l!r^l \\
&\qquad \times \binom{n+l-1}{l}\binom{n-1}{a-1}S_1(n-a,l)(a-1)_{a-k}.
\end{aligned}$$

Now, we utilize the following identity:

$$\sum_{n=1}^{\infty}(n+1)H_n t^n = \frac{t-\log(1-t)}{(1-t)^2}. \tag{36}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \bigl(1, t(1-t)^2\bigr). \tag{37}$$

For $n \geq 1$, from (19) and (37), we have

$$q_n(x) = \sum_{k=1}^{n}\binom{3n-k-1}{n-k}(n-1)_{n-k}x^k. \tag{38}$$

Let us assume that

$$p_n(x) \sim \bigl(1, t-\log(1-t)\bigr). \tag{39}$$

We observe that

$$t-\log(1-t) = t + \sum_{n=1}^{\infty}\frac{t^n}{n} = 2t + \sum_{n=2}^{\infty}\frac{t^n}{n}. \tag{40}$$

From (11), (39), (40) and $x^n \sim (1,t)$, we can derive the following equation:

$$\begin{aligned}
p_n(x) &= x\left(\frac{t}{2(t+\sum_{n=2}^{\infty}\frac{t^n}{n})}\right)^n x^{-1}x^n \\
&= 2^{-n}x\left(1+\sum_{n=2}^{\infty}\frac{t^{n-1}}{2n}\right)^{-n}x^{n-1} \\
&= 2^{-n}x\sum_{l=0}^{\infty}\binom{-n}{l}\left(\sum_{n=2}^{\infty}\frac{t^{n-1}}{2n}\right)^l x^{n-1} \\
&= 2^{-n}x\sum_{l=0}^{n-1}(-1)^l\binom{n+l-1}{l} \\
&\quad \times \sum_{m=0}^{n-1-l}\sum_{m_1+\cdots+m_l=m}\frac{1}{2^l(m_1+2)\cdots(m_l+2)}t^{m+l}x^{n-1}
\end{aligned}$$

$$= 2^{-n}\sum_{l=0}^{n-1}\sum_{m=0}^{n-1-l}\sum_{m_1+\cdots+m_l=m}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l}\frac{(n-1)_{m+l}}{(m_1+2)\cdots(m_l+2)}x^{n-l-m}$$
$$= 2^{-n}\sum_{k=1}^{n}\left\{\sum_{l=0}^{n-k}\sum_{m_1+\cdots+m_l=n-l-k}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l}\frac{(n-1)_{n-k}}{(m_1+2)\cdots(m_l+2)}\right\}x^k. \tag{41}$$

For $n \geq 1$, by (11), (37), (39) and (41), we get

$$\begin{aligned}
q_n(x) &= x\left(\frac{t-\log(1-t)}{t-(1-t)^2}\right)^n x^{-1}p_n(x) \\
&= x\left(\sum_{j=0}^{\infty}(j+2)H_{j+1}t^j\right)^n x^{-1}2^{-n}\sum_{a=1}^{n}\left\{\sum_{l=0}^{n-a}\sum_{m_1+\cdots+m_l=n-l-a}\left(-\frac{1}{2}\right)^l\right. \\
&\quad\left.\times\binom{n+l-1}{l}\frac{(n-1)_{n-a}}{(m_1+2)\cdots(m_l+2)}\right\}x^a \\
&= 2^{-n}\sum_{a=1}^{n}\left\{\sum_{l=0}^{n-a}\sum_{m_1+\cdots+m_l=n-l-a}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l}\right. \\
&\quad\left.\times\frac{(n-1)_{n-a}}{(m_1+2)\cdots(m_l+2)}\right\}x\sum_{j=0}^{a-1}\left(\sum_{j_1+\cdots+j_n=j}(j_1+2)\cdots(j_n+2)\right. \\
&\quad\left.\times H_{j_1+1}\cdots H_{j_n+1}\right)(a-1)_j x^{a-1-j} \\
&= 2^{-n}\sum_{a=1}^{n}\sum_{l=0}^{n-a}\sum_{k=1}^{a}\sum_{m_1+\cdots+m_l=n-l-a}\sum_{j_1+\cdots+j_n=a-k}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l} \\
&\quad\times\frac{(n-1)_{n-a}(a-1)_{a-k}}{(m_1+2)\cdots(m_l+2)}(j_1+2)\cdots(j_n+2)H_{j_1+1}\cdots H_{j_n+1}x^k \\
&= 2^{-n}\sum_{k=1}^{n}\left\{\sum_{a=k}^{n}\sum_{l=0}^{n-a}\sum_{m_1+\cdots+m_l=n-l-a}\sum_{j_1+\cdots+j_n=a-k}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l}\right. \\
&\quad\left.\times\frac{(n-1)_{n-a}(a-1)_{a-k}}{(m_1+2)\cdots(m_l+2)}(j_1+2)\cdots(j_n+2)H_{j_1+1}\cdots H_{j_n+1}\right\}x^k.
\end{aligned} \tag{42}$$

Therefore, by (38) and (42), we obtain the following theorem.

Theorem 5 *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned}
&\binom{3n-k-1}{n-k}(n-1)_{n-k} \\
&= 2^{-n}\sum_{a=k}^{n}\sum_{l=0}^{n-a}\sum_{m_1+\cdots+m_l=n-l-a}\sum_{j_1+\cdots+j_n=a-k}\left(-\frac{1}{2}\right)^l\binom{n+l-1}{l} \\
&\quad\times\frac{(n-1)_{n-a}(a-1)_{a-k}}{(m_1+2)\cdots(m_l+2)}(j_1+2)\cdots(j_n+2)H_{j_1+1}\cdots H_{j_n+1}.
\end{aligned}$$

Now, we recall the following identity:

$$\sum_{n=1}^{\infty} n^2 H_n t^n = \frac{t\{1+2t-(1+t)\log(1-t)\}}{(1-t)^3}. \tag{43}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \bigl(1, t(1-t)^3\bigr). \tag{44}$$

For $n \geq 1$, from (19) and (44), we can derive the following equation:

$$q_n(x) = \sum_{k=1}^{n} \binom{4n-k-1}{n-k} (n-1)_{n-k} x^k. \tag{45}$$

Let us assume that

$$p_n(x) \sim \bigl(1, t\bigl\{1+2t-(1+t)\log(1-t)\bigr\}\bigr). \tag{46}$$

We observe that

$$\begin{aligned}
1+2t-(1+t)\log(1-t) &= 1+2t+(1+t)\sum_{j=1}^{\infty}\frac{t^j}{j} \\
&= 1+2t+t+\sum_{j=2}^{\infty}\frac{t^j}{j}+\sum_{j=1}^{\infty}\frac{t^{j+1}}{j} \\
&= 1+3t+\sum_{j=0}^{\infty}\frac{t^{j+2}}{j+2}+\sum_{j=0}^{\infty}\frac{t^{j+2}}{j+1} \\
&= 1+3t+\sum_{j=0}^{\infty}\frac{2j+3}{(j+2)(j+1)}t^{j+2}.
\end{aligned} \tag{47}$$

For $n \geq 1$, by (11), (46), (47) and $x^n \sim (1, t)$, we get

$$\begin{aligned}
p_n(x) &= x\left(\frac{t}{t\{1+2t-(1+t)\log(1-t)\}}\right)^n x^{-1}x^n \\
&= x\left(1+3t+\sum_{j=0}^{\infty}\frac{2j+3}{(j+1)(j+2)}t^{j+2}\right)^{-n} x^{n-1} \\
&= x\sum_{l=0}^{n-1}(-1)^l\binom{n+l-1}{l}\left(3+\sum_{j=0}^{\infty}\frac{2j+3}{(j+1)(j+2)}t^{j+1}\right)^l t^l x^{n-1} \\
&= \sum_{l=0}^{n-1}\sum_{a=0}^{n-1-l}\sum_{k=1}^{n-a-l}\sum_{j_1+\cdots+j_a=n-a-k-l}(-1)^l\binom{n+l-1}{l}\binom{l}{a}3^{l-a}(n-1)_{n-k} \\
&\quad \times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)}\right)x^k
\end{aligned}$$

$$= \sum_{k=1}^{n} \left\{ \sum_{l=0}^{n-k} \sum_{a=0}^{n-k-l} \sum_{j_1+\cdots+j_a=n-a-k-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} (n-a)_{n-k} \times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \right\} x^k. \tag{48}$$

For $n \geq 1$, from (11), (44), (46) and (48), we have

$$\begin{aligned} q_n(x) &= x \left(\frac{t(1+2t-(1+t)\log(1-t))}{t(1-t)^3} \right)^n x^{-1} p_n(x) \\ &= \sum_{m=1}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} (n-1)_{n-m} \\ &\quad \times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) x \sum_{b=0}^{m-1} \sum_{b_1+\cdots+b_n=b} \left(\prod_{i=1}^{n} (b_i+1)^2 H_{b_i+1} \right) t^b x^{m-1} \\ &= \sum_{m=1}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} (n-1)_{n-m} \\ &\quad \times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \sum_{b=0}^{m-1} \sum_{b_1+\cdots+b_n=b} \prod_{i=1}^{n} (b_i+1)^2 H_{b_i+1} (m-1)_b x^{m-b} \\ &= \sum_{k=1}^{n} \left\{ \sum_{m=k}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} \sum_{b_1+\cdots+b_n=m-k} (-1)^l \binom{n+l-1}{l} \binom{l}{a} \right. \\ &\quad \left. \times 3^{l-a} (n-1)_{n-m} (m-1)_{m-k} \left(\frac{\prod_{i=1}^{a}(2j_i+3) \prod_{i=1}^{n}(b_i+1)^2 H_{b_i+1}}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \right\} x^k. \end{aligned} \tag{49}$$

Therefore, by (45) and (49), we obtain the following theorem.

Theorem 6 *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned} &\binom{4n-k-1}{n-k} (n-1)_{n-k} \\ &\quad = \sum_{m=k}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} \sum_{b_1+\cdots+b_n=m-k} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} \\ &\quad\quad \times (n-1)_{n-m} (m-1)_{m-k} \left(\frac{\prod_{i=1}^{a}(2j_i+3) \prod_{i=1}^{n}(b_i+1)^2 H_{b_i+1}}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right). \end{aligned}$$

Here we invoke the following identity:

$$\sum_{b=1}^{\infty} \left(\sum_{c=1}^{b} c^2 H_c \right) t^b = \frac{t\{1+2t-(1+t)\log(1-t)\}}{(1-t)^4}. \tag{50}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \big(1, t(1-t)^4\big). \tag{51}$$

From (19) and (51), we note that

$$q_n(x) = \sum_{k=1}^{n} \binom{5n-k-1}{n-k} (n-1)_{n-k} x^k. \tag{52}$$

Let us assume that

$$p_n(x) \sim \bigl(1, t\bigl(1 + 2t - (1+t)\log(1-t)\bigr)\bigr). \tag{53}$$

For $n \geq 1$, from (48) and (49), we have

$$\begin{aligned} p_n(x) = \sum_{k=1}^{n} \Biggl\{ &\sum_{l=0}^{n-k} \sum_{a=0}^{n-k-l} \sum_{j_1+\cdots+j_a=n-a-k-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} (n-1)_{n-k} \\ &\times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \Biggr\} x^k. \end{aligned} \tag{54}$$

For $n \geq 1$, from (11), (51), (53) and (50), we can derive the following identity:

$$\begin{aligned} q_n(x) &= x \left(\frac{t\{1 + 2t - (1+t)\log(1-t)\}}{t(1-t)^4} \right)^n x^{-1} p_n(x) \\ &= x \left(\sum_{b=0}^{\infty} \left(\sum_{c=1}^{b+1} c^2 H_c \right) t^b \right)^n x^{-1} p_n(x) \\ &= x \sum_{b=0}^{\infty} \sum_{b_1+\cdots+b_n=b} \left\{ \sum_{c_1=1}^{b_1+1} \cdots \sum_{c_n=1}^{b_n+1} c_1^2 \cdots c_n^2 H_{c_1} \cdots H_{c_n} \right\} t^b \\ &\quad \times \sum_{m=1}^{n} \Biggl\{ \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} \\ &\quad \times (n-1)_{n-m} \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \Biggr\} x^{m-1} \\ &= \sum_{m=1}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} (-1)^l \binom{n+l-1}{l} \binom{l}{a} 3^{l-a} (n-1)_{n-m} \\ &\quad \times \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \sum_{b=0}^{m-1} \sum_{b_1+\cdots+b_n=b} \left\{ \sum_{c_1=1}^{b_1+1} \cdots \sum_{c_n=1}^{b_n+1} c_1^2 \cdots c_n^2 H_{c_1} \cdots H_{c_n} \right\} \\ &\quad \times (m-1)_b x^{m-b} \\ &= \sum_{k=1}^{n} \Biggl\{ \sum_{m=k}^{n} \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{j_1+\cdots+j_a=n-a-m-l} \sum_{b_1+\cdots+b_n=m-k} (-1)^l \binom{n+l-1}{l} \\ &\quad \times \binom{l}{a} 3^{l-a} (n-1)_{n-m} (m-1)_{m-k} \left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)} \right) \\ &\quad \times \sum_{c_1=1}^{b_1+1} \cdots \sum_{c_n=1}^{b_n+1} \prod_{i=1}^{n} c_i^2 H_{c_i} \Biggr\} x^k. \end{aligned} \tag{55}$$

Therefore, by (52) and (55), we obtain the following theorem.

Theorem 7 *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned}
&\binom{5n-k-1}{n-k}(n-1)_{n-k} \\
&\quad = \sum_{m=k}^{n}\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\sum_{j_1+\cdots+j_a=n-a-m-l}\sum_{b_1+\cdots+b_n=m-k}(-1)^l\binom{n+l-1}{l}\binom{l}{a}3^{l-a} \\
&\qquad \times (n-1)_{n-m}(m-1)_{m-k}\left(\frac{\prod_{i=1}^{a}(2j_i+3)}{\prod_{i=1}^{a}(j_i+1)(j_i+2)}\right)\sum_{c_1=1}^{b_1+1}\cdots\sum_{c_n=1}^{b_n+1}\prod_{i=1}^{n}c_i^2 H_{c_i}.
\end{aligned}$$

Here we use the following identity:

$$\sum_{n=1}^{\infty} n(2n+1)H_n t^n = \frac{t\{3(1+t)-(t+3)\log(1-t)\}}{(1-t)^3}. \tag{56}$$

Let us consider the following associated sequence:

$$q_n(x) \sim \left(1, t(1-t)^3\right). \tag{57}$$

By (19) and (57), we get

$$q_n(x) = \sum_{k=1}^{n}\binom{4n-k-1}{n-k}(n-1)_{n-k}x^k \quad (n \geq 1). \tag{58}$$

Let us assume that

$$p_n(x) \sim \left(1, t\left\{3(1+t)-(t+3)\log(1-t)\right\}\right). \tag{59}$$

We see that

$$3(1+t)-(t+3)\log(1-t) = 3+6t+\sum_{n=1}^{\infty}\frac{4n+1}{n(n+1)}t^{n+1}. \tag{60}$$

For $n \geq 1$, from (11), (59), (60) and $x^n \sim (1,t)$, we have

$$\begin{aligned}
p_n(x) &= x\left(\frac{t}{t\{3(1+t)-(t+3)\log(1-t)\}}\right)^n x^{-1}x^n \\
&= x\left(3(1+t)-(t+3)\log(1-t)\right)^{-n}x^{n-1} \\
&= x\left(3+6t+\sum_{j=1}^{\infty}\frac{4j+1}{j(j+1)}t^{j+1}\right)^{-n}x^{n-1}.
\end{aligned} \tag{61}$$

From (61), by the same method of (48), we get

$$\begin{aligned}
p_n(x) &= 3^{-n}\sum_{k=1}^{n}\left\{\sum_{l=0}^{n-k}\sum_{a=0}^{n-k-l}\sum_{j_1+\cdots+j_a=n-a-l-k}(-1)^l\binom{n+l-1}{l}\binom{l}{a}2^{l-a}\right. \\
&\qquad \left.\times (n-1)_{n-k}\left(\prod_{i=1}^{a}\frac{(4j_i+5)}{3(j_i+1)(j_i+2)}\right)\right\}x^k.
\end{aligned} \tag{62}$$

For $n \geq 1$, by (11), (56), (57), (59) and (62), we get

$$\begin{aligned} q_n(x) &= x\left(\frac{t\{3(1+t)-(t+3)\log(1-t)\}}{t(1-t)^3}\right)^n x^{-1}p_n(x) \\ &= x\left(\sum_{b=0}^{\infty}(b+1)(2b+3)H_{b+1}t^b\right)^n x^{-1}p_n(x) \\ &= x\sum_{b=0}^{\infty}\left(\sum_{b_1+\cdots+b_n=b}\left(\prod_{i=1}^{b}(b_i+1)(2b_i+3)H_{b_i+1}\right)t^b\right) \\ &\quad \times 3^{-n}\sum_{m=1}^{n}\left\{\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\sum_{j_1+\cdots+j_a=n-a-l-m}(-1)^l\binom{n+l-1}{l}\binom{l}{a}2^{l-a}\right. \\ &\quad \left.\times (n-1)_{n-m}\prod_{i=1}^{a}\frac{(4j_i+5)}{3(j_i+1)(j_i+2)}\right\}x^{m-1} \\ &= 3^{-n}\sum_{m=1}^{n}\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\sum_{j_1+\cdots+j_a=n-a-l-m}(-1)^l\binom{n+l-1}{l}\binom{l}{a}2^{l-a}(n-1)_{n-m} \\ &\quad \times\left(\prod_{i=1}^{a}\frac{(4j_i+5)}{3(j_i+1)(j_i+2)}\right)\sum_{b=0}^{m-1}\sum_{b_1+\cdots+b_n=b}\left(\prod_{i=1}^{n}(b_i+1)(2b_i+3)H_{b_i+1}\right) \\ &\quad \times (m-1)_b x^{m-b}. \end{aligned} \tag{63}$$

By the same method, we can derive the following identity from (63):

$$\begin{aligned} q_n(x) &= 3^{-n}\sum_{k=1}^{n}\left\{\sum_{m=k}^{n}\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\sum_{j_1+\cdots+j_a=n-a-l-m}\sum_{b_1+\cdots+b_n=m-k}(-1)^l\right. \\ &\quad \times\binom{n+l-1}{l}\binom{l}{a}2^{l-a}(n-1)_{n-m}(m-1)_{m-k}\left(\prod_{i=1}^{a}\frac{(4j_i+5)}{3(j_i+1)(j_i+2)}\right) \\ &\quad \left.\times\prod_{i=1}^{n}(b_i+1)(2b_i+3)H_{b_i+1}\right\}x^k. \end{aligned} \tag{64}$$

By comparing coefficients on both sides of (58) and (64), we get

$$\begin{aligned} &\binom{4n-k-1}{n-k}(n-1)_{n-k} \\ &\quad = 3^{-n}\sum_{m=k}^{n}\sum_{l=0}^{n-m}\sum_{a=0}^{n-m-l}\sum_{j_1+\cdots+j_a=n-a-l-m}\sum_{b_1+\cdots+b_n=m-k}(-1)^l\binom{n+l-1}{l}\binom{l}{a} \\ &\quad\quad \times 2^{l-a}(n-1)_{n-m}(m-1)_{m-k}\left(\prod_{i=1}^{a}\frac{(4j_i+5)}{3(j_i+1)(j_i+2)}\right) \\ &\quad\quad \times\left(\prod_{i=1}^{n}(b_i+1)(2b_i+3)H_{b_i+1}\right). \end{aligned} \tag{65}$$

Remark Recently, several authors have studied the q-extension of harmonic and hyper-harmonic numbers (see [11–13]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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