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On power and non-power asymptotic behavior of positive solutions to Emden-Fowler type higher-order equations

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Abstract

For the equation

$$y^{(n)} = y^k, \quad k > 1, n = 12, 13, 14,$$

the existence of positive solutions with non-power asymptotic behavior is proved, namely

$$y = (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad \alpha = \frac{n}{k-1}, x < x^*,$$

where x^* is an arbitrary point, h is a positive periodic non-constant function on $\mathbf{R}$.

To prove this result, the Hopf bifurcation theorem is used.

Keywords: asymptotic behavior; Emden-Fowler higher-order equations

Introduction

For the equation

$$y^{(n)} = p\left(x, y, y', \dots, y^{(n-1)}\right)|y|^k \operatorname{sgn} y, \quad n \geq 2, k > 1, \tag{1}$$

Kiguradze posed the problem on the asymptotic behavior of its positive solutions such that

$$\lim_{x \to x^* - 0} y(x) = \infty. \tag{2}$$

He found an asymptotic formula for these solutions to (1) with $n = 2$ (see [1]) and supposed all such solutions to have power asymptotic behavior for other n, too. The problem was solved for $n = 3$ and $n = 4$ [2]. For these n, it was proved that all such solutions behave as

$$y(x) = C\left(x^* - x\right)^{-\alpha}\left(1 + o(1)\right), \quad x \to x^* - 0, \tag{3}$$

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with

$$\alpha = \frac{n}{k-1}, \qquad C = \left(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\right)^{\frac{1}{k-1}}, \tag{4}$$

$p_0 = \text{const} > 0$ - is a limit of $p(x, y_0, \dots, y_{n-1})$ as $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$.

So, the hypothesis of Kiguradze was confirmed in this case.

The existence of solutions satisfying (3) was proved for arbitrary $n \geq 2$. For $2 \leq n \leq 11$, an $(n-1)$-parametric family of such solutions to equation (1) was proved to exist (see [2], [3], Ch.**I**(5.1)).

For the equation

$$y^{(n)} = y^k, \quad k > 1, \tag{5}$$

a negative answer to the conjecture of Kiguradze for large n was obtained. It was proved [4] that for any N and $K > 1$, there exist an integer $n > N$ and $k \in \mathbf{R}$, $1 < k < K$, such that equation (5) has a solution

$$y = (x^* - x)^{-\alpha} h\big(\log(x^* - x)\big), \tag{6}$$

where α is defined by (4), h is a positive periodic non-constant function on $\mathbf{R}$.

Still, it was not clear how large n should be for the existence of that type of solutions.

Preliminary results

Suppose the following conditions hold:

(A) The continuous positive function $p(x, y_0, \dots, y_{n-1})$ has a limit $p_0 = \text{const} > 0$ as $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$, and for some $\gamma > 0$, it holds

$$p(x, y_0, \dots, y_{n-1}) - p_0 = O\left((x^* - x)^{\gamma} + \sum_{j=0}^{n-1} y_j^{-\gamma} \right). \tag{7}$$

(B) For some $K_1 > 0$ and $\mu > 0$ in a neighborhood of x^* for sufficiently large $y_0, \dots, y_{n-1}$, $z_0, \dots, z_{n-1}$, it holds

$$\big| p(x, y_0, \dots, y_{n-1}) - p(x, z_0, \dots, z_{n-1}) \big| \leq K_1 \max_j \big| y_j^{-\mu} - z_j^{-\mu} \big|. \tag{8}$$

Then equation (1) can be transformed (see [2] or [3], Ch.**I**(5.1)) by using the substitution

$$x^* - x = e^{-t}, \qquad y = (C + v) e^{\alpha t}, \tag{9}$$

where C and α are defined by (4). The derivatives $y^{(j)}$, $j = 0, 1, \dots, n-1$, become

$$e^{(\alpha+j)t} \cdot L_j\big(v, v', \dots, v^{(j)}\big),$$

where $v^{(j)} = \frac{d^j v}{dt^j}$ and L_j is a linear function with

$$L_j(0, 0, \dots, 0) = C\alpha(\alpha+1)\cdots(\alpha+j-1) \neq 0$$

and the coefficient of $v^{(j)}$ equal to 1.

Thus (1) is transformed into

$$e^{(\alpha+n)t} \cdot L_n(v, v', \ldots, v^{(n)}) = (C+v)^k e^{\alpha kt} \tilde{p}(t, v, v', \ldots, v^{(n-1)}), \tag{10}$$

where the function $\tilde{p}(t, v_0, \ldots, v_{n-1})$ is obtained from $p(x, y_0, \ldots, y_{n-1})$ with $x, y_0, \ldots, y_{n-1}$ properly expressed in terms of $t, v_0, \ldots, v_{n-1}$. This function tends to p_0 as $t \to \infty$, $v \to 0, \ldots, v^{(n-1)} \to 0$.

Due to condition (8) for the function $p(x, y_0, \ldots, y_{n-1})$, we obtain the following inequalities for sufficiently large t and sufficiently small $v_0, \ldots, v_{n-1}$, $w_0, \ldots, w_{n-1}$:

$$\begin{aligned} &\left|\tilde{p}(t, v_0, \ldots, v_{n-1}) - \tilde{p}(t, w_0, \ldots, w_{n-1})\right| \\ &\quad \le K_1 \max_j e^{-\mu(\alpha+j)t} \left|L_j^{-\mu}(v_0, \ldots, v_{n-1}) - L_j^{-\mu}(w_0, \ldots, w_{n-1})\right|. \end{aligned}$$

Since $L_j(0, 0, \ldots, 0) \ne 0$, the function $L_j^{-\mu}$ is a C^∞ one in a neighborhood of 0 and

$$\left|\tilde{p}(t, v_0, \ldots, v_{n-1}) - \tilde{p}(t, w_0, \ldots, w_{n-1})\right| \le K_2 e^{-\mu\alpha t} \max_j |v_j - w_j|$$

for some $K_2 > 0$.

Solving (10) for $v^{(n)}$ and using formulae (4), we obtain the equation

$$v^{(n)} = (C+v)^k \tilde{p}(t, v, v', \ldots, v^{(n-1)}) - p_0 C^k - \sum_{j=0}^{n-1} a_j v^{(j)}, \tag{11}$$

where a_j are the coefficients of the linear function L_n. Equation (11) can be written as

$$v^{(n)} = kC^{k-1} p_0 v - \sum_{j=0}^{n-1} a_j v^{(j)} + f(v) + g(t, v, v', \ldots, v^{(n-1)}), \tag{12}$$

where

$$\begin{aligned} &f(v) = p_o\left((C+v)^k - C^k - kC^{k-1}v\right) = O(v^2) \quad \text{as } v \to 0, \\ &f'(v) = O(v) \quad \text{as } v \to 0, \\ &g(t, v_0, \ldots, v_{n-1}) = (C+v_0)^k \left(\tilde{p}(t, v_0, \ldots, v_{n-1}) - p_0\right) \\ &\qquad = O\left(e^{-\gamma t} + \sum_{j=0}^{n-1} e^{-\gamma(\alpha+j)t}\right) = O\left(\exp\left(-\gamma \min(\alpha, 1)t\right)\right) \\ &\qquad \text{as } t \to \infty, v_0 \to 0, \ldots, v_{n-1} \to 0. \end{aligned}$$

Besides, for sufficiently large t and sufficiently small $v_0, \ldots, v_{n-1}$, $w_0, \ldots, w_{n-1}$, it holds

$$\begin{aligned} &\left|g(t, v_0, \ldots, v_{n-1}) - g(t, w_0, \ldots, w_{n-1})\right| \\ &\quad \le \left|(C+v_0)^k - (C+w_0)^k\right| \cdot \left|\tilde{p}(t, v_0, \ldots, v_{n-1}) - p_0\right| \\ &\qquad + (C+w_0)^k \left|\tilde{p}(t, v_0, \ldots, v_{n-1}) - \tilde{p}(t, w_0, \ldots, w_{n-1})\right| \\ &\quad \le K_3 \max_j |w_j - v_j| e^{-\min(\gamma,\mu)\cdot\min(\alpha,1)t}. \end{aligned}$$

Suppose that V is the vector with coordinates $V_j = v^{(j)}$, $j = 0, \dots, n-1$. Then equation (12) can be written as

$$\frac{dV}{dt} = AV + F(V) + G(t, V), \tag{13}$$

where A is a constant $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\tilde{a}_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{pmatrix},$$

with

$$\begin{aligned} -\tilde{a}_0 &= a_0 - kc^{k-1}p_0 = a_0 - k\alpha(\alpha+1)\cdots(\alpha+n-1) \\ &= a_0 - (\alpha+1)\cdots(\alpha+n-1)(\alpha+n) \end{aligned}$$

and eigenvalues satisfying the equation

$$\begin{aligned} 0 &= \det(A - \lambda E) = (-1)^{n+1}\bigl(-\tilde{a}_0 - a_1\lambda - \cdots - a_{n-1}\lambda^{n-1} - \lambda^n\bigr) \\ &= (-1)^{n+1}\bigl((\alpha+1)(\alpha+2)\cdots(\alpha+n) - (\lambda+\alpha)\cdots(\lambda+\alpha+n-1)\bigr), \end{aligned}$$

which is equivalent to

$$\prod_{j=0}^{n-1}(\lambda + \alpha + j) = \prod_{j=0}^{n-1}(1 + \alpha + j). \tag{14}$$

The mappings $F : \mathbf{R}^n \to \mathbf{R}^n$ and $G : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ satisfy the following estimates as $t \to \infty$:

$$\begin{cases} \|F(V)\| = O(\|V\|^2), \\ \|F'_V(V)\| = O(\|V\|), \\ \|G(t, V)\| = O(e^{-2\beta t}), \\ \|G(t, V) - G(t, W)\| \le K\|V - W\|e^{-2\beta t} \end{cases} \tag{15}$$

with some constants $\beta > 0$, $K > 0$.

Lemma 1 [3] *Suppose that* (15) *holds and* A *is an arbitrary constant* $n \times n$ *matrix. Then there exists a solution* $V(t)$ *to equation* (13) *tending to zero as* $t \to \infty$.

Lemma 2 [3] *Let the conditions of Lemma* 1 *hold. If equation* (14) *has* m *roots with negative real part, then there exists an* m*-parametric family of solutions* $V(t)$ *to equation* (13) *tending to zero as* $t \to \infty$.

If equation (13) has a solution $V(t)$ tending to 0 as $t \to \infty$ and $V_0(t)$ is its first coordinate, then the function

$$y(x) = \left(V_0\left(-\log\left(x^* - x\right)\right) + C\right) \cdot \left(x^* - x\right)^{-\alpha}$$

with C and α defined by (4) is a solution to (1) such that (2) and (3) hold.

Theorem 1 [2, 3] *Suppose that conditions* (A) *and* (B) *are satisfied. Then for such* x^* *there exists a solution to* (1) *with power asymptotic behavior* (3).

Investigating the signs of the real parts of the roots of equation (14), by the Routh-Hurwitz criterion, we can prove the following theorem.

Theorem 2 [2, 3] *Suppose that* $3 \le n \le 11$ *and conditions* (A) *and* (B) *are satisfied. Then there exists an* $(n-1)$*-parametric family of solutions to equation* (1) *with power asymptotic behavior* (3).

Theorem 3 [2, 3, 5] *Suppose that* $n = 3$ *or* $n = 4$ *in equation* (1), *the continuous positive function* $p(x, y_0, \dots, y_{n-1})$ *is Lipschitz continuous in* $y_0, \dots, y_{n-1}$ *and has a limit* $p_0 > 0$ *as* $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. *Then any positive solution to this equation with a vertical asymptote* $x = x^*$ *has asymptotic behavior* (3).

To prove the main results of this article, we use the Hopf bifurcation theorem [6].

Theorem (Hopf) *Consider the* α*-parameterized dynamical system* $\dot{x} = L_\alpha x + Q_\alpha(x)$ *in a neighborhood of* $0 \in \mathbf{R}^n$ *with linear operators* L_α *and smooth enough functions* $Q_\alpha(x) = O(|x|^2)$ *as* $x \to 0$. *Let* λ_α *and* $\bar{\lambda}_\alpha$ *be simple complex conjugated eigenvalues of the operators* L_α. *Suppose that* $\operatorname{Re}\lambda_{\tilde{\alpha}} = \operatorname{Re}\bar{\lambda}_{\tilde{\alpha}} = 0$ *for some* $\tilde{\alpha}$ *and the operator* $L_{\tilde{\alpha}}$ *has no other eigenvalues with zero real part.*

If $\operatorname{Re}\frac{d\lambda_\alpha}{d\alpha}(\tilde{\alpha}) \neq 0$, *then there exist continuous mappings* $\varepsilon \mapsto \alpha(\varepsilon) \in \mathbf{R}$, $\varepsilon \mapsto T(\varepsilon) \in \mathbf{R}$, *and* $\varepsilon \mapsto b(\varepsilon) \in \mathbf{R}^n$ *defined in a neighborhood of* 0 *and such that* $\alpha(0) = \tilde{\alpha}$, $T(0) = 2\pi / \operatorname{Im}\lambda_{\tilde{\alpha}}$, $b(0) = 0$, $b(\varepsilon) \neq 0$ *for* $\varepsilon \neq 0$, *and the solutions to the problems*

$$\dot{x} = L_{\alpha(\varepsilon)}x + Q_{\alpha(\varepsilon)}(x), \quad x(0) = b(\varepsilon)$$

are $T(\varepsilon)$*-periodic and non-constant.*

Main results

In this section, the result about the existence of solutions with non-power asymptotic behavior is proved for equation (5) with $n = 12, 13, 14$.

Theorem 4 *For* $n = 12, 13, 14$, *there exists* $k > 1$ *such that equation* (5) *has a solution* $y(x)$ *with*

$$y^{(j)}(x) = \left(x^* - x\right)^{-\alpha - j} h_j\left(\log\left(x^* - x\right)\right),$$
$$j = 0, 1, \dots, n-1,$$

where α *is defined by* (4) *and* h_j *are periodic positive non-constant functions on* $\mathbf{R}$.

Proof To apply the Hopf bifurcation theorem, we investigate equation (13) with $G(t, V) \equiv 0$ corresponding to the case of the constant function p and the roots of the algebraic equation (14). F is a vector function with all zero components $F(V) = (0, \dots, 0, F_{n-1}(V))$, $V = (V_0, \dots, V_{n-1})$, and

$$F_{n-1}(V) = \bigl((C + V_0)^k - C^k - kC^{k-1}V_0\bigr) = O\bigl({V_0}^2\bigr), \quad V_0 \to 0,$$
$$\frac{d}{dV}F_{n-1}(V) = O\bigl(|V_0|\bigr), \quad V_0 \to 0.$$

If equation (14) has a pair of pure imaginary roots, we have to check other conditions of this theorem and then apply it.

Proposition 1 *For any integer $n > 11$, there exist $\alpha > 0$ and $q > 0$ such that*

$$\prod_{j=0}^{n-1}(qi + \alpha + j) = \prod_{j=0}^{n-1}(1 + \alpha + j) \tag{16}$$

with $i^2 = -1$.

Remark 1 In the particular case $n = 12$, this result was obtained by Vyun [7].

Proof Consider the positive functions $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ defined for all $\alpha > 0$ via the equations

$$\prod_{j=0}^{n-1}\bigl(\rho_n(\alpha)^2 + (\alpha + j)^2\bigr) = \prod_{j=0}^{n-1}(1 + \alpha + j)^2 \tag{17}$$

and

$$\sum_{j=0}^{n-1}\arg\bigl(\sigma_n(\alpha)i + \alpha + j\bigr) = 2\pi \tag{18}$$

supposing $\arg z \in [0, 2\pi)$ for all $z \in \mathbb{C} \setminus \{0\}$.

First, we prove the functions to be well defined for all $\alpha > 0$.

The product $\prod_{j=0}^{n-1}(q^2 + (\alpha + j)^2)$ is continuous and strictly increasing as a function of $q > 0$.

It tends to $\prod_{j=0}^{n-1}(\alpha + j)^2 < \prod_{j=0}^{n-1}(1 + \alpha + j)^2$ as $q \to 0$ and to $+\infty$ as $q \to +\infty$. Hence, for any $\alpha > 0$, there exists a unique $q > 0$ such that $\prod_{j=0}^{n-1}(q^2 + (\alpha + j)^2) = \prod_{j=0}^{n-1}(1 + \alpha + j)^2$.

In the same way, for any $\alpha > 0$, the sum $\sum_{j=0}^{n-1}\arg(qi + \alpha + j)$ is a continuous function of $q > 0$ strictly increasing from 0 to $\frac{\pi n}{2} > 2\pi$. So, there exists a unique $q > 0$ such that the sum is equal to 2π.

Since both the product and the sum considered are C^1-functions with positive partial derivative in $q > 0$, the implicit function theorem provides both $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ to be C^1-functions, too.

Now it is sufficient to prove the existence of $\alpha > 0$ such that $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ are equal to the same value q, which makes the two sides of (16) be equal.

Compare the functions $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ near the boundaries of their common domain.

Equation (17) defining the function $\rho_n(\alpha)$ may be written as

$$\prod_{j=0}^{n-1}\left(1+\frac{2j}{\alpha}+\frac{j^2}{\alpha^2}+\left(\frac{\rho_n(\alpha)}{\alpha}\right)^2\right)=\prod_{j=0}^{n-1}\left(1+\frac{j+1}{\alpha}\right)^2.$$

This shows that $\frac{\rho_n(\alpha)}{\alpha}\to 0$ as $\alpha\to+\infty$.

Equation (18) defining the function $\sigma_n(\alpha)$ may be written as

$$\sum_{j=0}^{n-1}\arctan\frac{\frac{\sigma_n(\alpha)}{\alpha}}{1+\frac{j}{\alpha}}=2\pi.$$

This shows that $\frac{\sigma_n(\alpha)}{\alpha}\to\tan\frac{2\pi}{n}>0$ as $\alpha\to+\infty$. Thus, $\rho_n(\alpha)<\sigma_n(\alpha)$ for sufficiently large α.

Now, to prove Proposition 1, it is sufficient to show that $\rho_n(\alpha)>\sigma_n(\alpha)$ for sufficiently small α. To compare the functions $\rho_n(\alpha)$ and $\sigma_n(\alpha)$ for small $\alpha>0$, we need some lemmas.

Lemma 3 *For all* $\alpha>0$, *it holds* $\rho_n(\alpha)^2<2(\alpha+n)-1$.

Proof Suppose that $\rho_n(\alpha)^2\geq 2(\alpha+n)-1$ for some $\alpha>0$. Then

$$\begin{aligned}\prod_{j=0}^{n-1}\left(\rho_n(\alpha)^2+(\alpha+j)^2\right)&\geq\prod_{j=0}^{n-1}\left(2(\alpha+n)-1+(\alpha+j)^2\right)\\&>\prod_{j=0}^{n-1}\left(2(\alpha+j+1)-1+(\alpha+j)^2\right)=\prod_{j=0}^{n-1}\left(1+(\alpha+j)\right)^2.\end{aligned}$$

This contradiction with the definition of $\rho_n(\alpha)$ completes the proof of Lemma 3. □

Lemma 4 *For all* $\alpha>0$, *it holds* $\rho_{n+1}(\alpha)>\rho_n(\alpha)$.

Proof According to the definition of $\rho_n(\alpha)$ by (17) and Lemma 3, we have

$$\begin{aligned}\prod_{j=0}^{n}\left(\rho_n(\alpha)^2+(\alpha+j)^2\right)&=\prod_{j=0}^{n-1}(1+\alpha+j)^2\cdot\left(\rho_n(\alpha)^2+(\alpha+n)^2\right)\\&<\prod_{j=0}^{n-1}(1+\alpha+j)^2\cdot\left(2(\alpha+n)-1+(\alpha+n)^2\right)<\prod_{j=0}^{n}(1+\alpha+j)^2.\end{aligned}$$

In order to make the first and the last products be equal, we have to replace $\rho_n(\alpha)$ in the first one by a greater value. This means that $\rho_{n+1}(\alpha)>\rho_n(\alpha)$ and Lemma 4 is proved. □

Lemma 5 *For all* $\alpha>0$, *it holds* $\sigma_{n+1}(\alpha)<\sigma_n(\alpha)$.

Proof According to the definition of $\sigma_n(\alpha)$ by (18), we have

$$\sum_{j=0}^{n}\arg\left(\sigma_n(\alpha)i+\alpha+j\right)=2\pi+\arg\left(\sigma_n(\alpha)i+\alpha+n\right)>2\pi.$$

In order to make the sum equal 2π, we have to replace $\sigma_n(\alpha)$ by a smaller value. So, $\sigma_{n+1}(\alpha) < \sigma_n(\alpha)$ and Lemma 5 is proved. □

Due to Lemmas 3, 4, 5 proved, it is sufficient now for the proof of Proposition 1 to show that $\rho_{12}(\alpha) > \sigma_{12}(\alpha)$ for sufficiently small $\alpha > 0$.

Lemma 6 *It holds* $\rho_{12}(\alpha) > 2$ *for all sufficiently small* $\alpha > 0$.

Proof Straightforward exact calculations show that

$$\lim_{\alpha\to 0}\prod_{j=0}^{11}\bigl(2^2 + (\alpha + j)^2\bigr) = \prod_{j=0}^{11}\bigl(4 + j^2\bigr) = 192{,}175{,}659{,}520{,}000{,}000 < 2\cdot 10^{17}$$

and

$$\lim_{\alpha\to 0}\prod_{j=0}^{11}(1 + \alpha + j)^2 = (12!)^2 = 229{,}442{,}532{,}802{,}560{,}000 > 2\cdot 10^{17}.$$

So, for sufficiently small $\alpha > 0$, we have

$$\prod_{j=0}^{11}\bigl(2^2 + (\alpha + j)^2\bigr) < 2\cdot 10^{17} < \prod_{j=0}^{11}(1 + \alpha + j)^2.$$

Hence, for these α, in order to avoid contradiction with the definition of $\rho_{12}(\alpha)$, the inequality $\rho_{12}(\alpha)^2 > 2^2$ is necessary. Lemma 6 is proved. □

Lemma 7 *It holds* $\sigma_{12}(\alpha) < 2$ *for sufficiently small* $\alpha > 0$.

Proof Consider the limit

$$\begin{aligned}
&\lim_{\alpha\to 0}\sum_{j=0}^{11}\arg(2i + \alpha + j)\\
&\quad = \arg 2i + \arctan 2 + \arctan 1 + \arctan\frac{2}{3} + \arctan\frac{1}{2} + \sum_{j=5}^{11}\arctan\frac{2}{j}\\
&\quad = \frac{5\pi}{4} + \arctan\frac{2}{3} + \sum_{j=5}^{11}\arctan\frac{2}{j}\\
&\quad = \frac{5\pi}{4} + \arctan\frac{\frac{2}{3}+\frac{2}{5}}{1-\frac{2}{3}\cdot\frac{2}{5}} + \arctan\frac{\frac{2}{6}+\frac{2}{7}}{1-\frac{2}{6}\cdot\frac{2}{7}} + \arctan\frac{\frac{2}{8}+\frac{2}{9}}{1-\frac{2}{8}\cdot\frac{2}{9}} + \arctan\frac{\frac{2}{10}+\frac{2}{11}}{1-\frac{2}{10}\cdot\frac{2}{11}}\\
&\quad = \frac{5\pi}{4} + \arctan\frac{16}{11} + \arctan\frac{13}{19} + \arctan\frac{1}{2} + \arctan\frac{21}{53}\\
&\quad = \frac{5\pi}{4} + \arctan\frac{\frac{16}{11}+\frac{13}{19}}{1-\frac{16}{11}\cdot\frac{13}{19}} + \arctan\frac{\frac{1}{2}+\frac{21}{53}}{1-\frac{1}{2}\cdot\frac{21}{53}} = \frac{5\pi}{4} + \arctan 447 + \arctan\frac{19}{17}.
\end{aligned}$$

Note that

$$\tan\left(\arctan 447 + \arctan\frac{19}{17}\right) = \frac{447 + \frac{19}{17}}{1 - 447 \cdot \frac{19}{17}} = -\frac{3{,}809}{4{,}238}.$$

Hence, $\arctan 447 + \arctan\frac{19}{17} > \frac{3\pi}{4}$ and $\sum_{j=0}^{11} \arg(2i + \alpha + j) > 2\pi$ for sufficiently small $\alpha > 0$. Thus, for these α, we have $\sigma_{12}(\alpha) < 2$, which completes the proof of Lemma 7. □

Now Proposition 1 is also proved. □

Proposition 2 *For any* $\alpha > 0$ *and any integer* $n > 1$, *all roots* $\lambda \in \mathbb{C}$ *to equation* (14) *are simple*.

Proof Since we consider a polynomial equation of degree n, it is sufficient to prove the existence of n different roots to (14). We will show that for any integer m such that $-n < m \le n$, there exists $\mu_m \in \mathbb{C}$ satisfying

$$\prod_{j=0}^{n-1} |\mu_m + j| = \prod_{j=0}^{n-1} (1 + \alpha + j) \tag{19}$$

and

$$\sum_{j=0}^{n-1} \arg(\mu_m + j) = m\pi \tag{20}$$

with $\arg z$ denoting the principal value of the argument lying in the open-closed interval $(-\pi, \pi]$. Surely, all these $2n$ complex numbers μ_m are different. Those with even m generate, via the relation $\lambda_m + \alpha = \mu_m$, just n different roots λ_m to (14).

We begin to accomplish this plan by noting that the set of μ satisfying equation (20) with $m = 0$ is the real semi-axis $(0, +\infty)$ containing a single point satisfying (19), namely $\mu_0 = 1 + \alpha$.

Similarly, the set of μ satisfying equation (20) with $m = n$ is the real unbounded interval $(-\infty, 1 - n)$ containing a single point satisfying (19), namely $\mu_n = \alpha - n$.

Now consider the cases $0 < m < n$ and the upper complex half-plane. For any $\omega > 0$, the smooth function

$$\phi_\omega(r) = \sum_{j=0}^{n-1} \arg(r + \omega i + j) = \sum_{j=0}^{n-1} \operatorname{arccot}\frac{r + j}{\omega}$$

monotonically decreases from $n\pi$ to 0 as r increases from $-\infty$ to $+\infty$. So, for any $\omega > 0$ and $b \in (0, n\pi)$, there exists a unique value r such that $\phi_\omega(r) = b$. Due to the inequality $\frac{d\phi_\omega}{dr}(r) < 0$, the implicit function theorem provides the existence of the smooth functions $r_m(\omega)$ satisfying $\phi_\omega(r_m(\omega)) = m\pi$.

Note that if $r \le -m$, then $r + j < 0$ for all $j < m$ and $r + m \le 0$. Hence,

$$\lim_{\omega\to+0} \sum_{j=0}^{n-1} \operatorname{arccot}\frac{r + j}{\omega} \ge \lim_{\omega\to+0} \sum_{j=0}^{m-1} \operatorname{arccot}\frac{r + j}{\omega} + \lim_{\omega\to+0} \operatorname{arccot}\frac{0}{\omega} = m\pi + \frac{\pi}{2} > m\pi$$

and such r cannot be the value of $r_m(\omega)$ for sufficiently small $\omega > 0$.

Similarly, if $r \geq 1 - m$, then $r + j > 0$ for all $j > m - 1$ and $r + m - 1 \geq 0$. Hence,

$$\overline{\lim_{\omega \to +0}} \sum_{j=0}^{n-1} \operatorname{arccot} \frac{r+j}{\omega} \leq \overline{\lim_{\omega \to +0}} \sum_{j=0}^{m-2} \operatorname{arccot} \frac{r+j}{\omega} + \frac{\pi}{2} + \overline{\lim_{\omega \to +0}} \sum_{j=m}^{n-1} \operatorname{arccot} \frac{r+j}{\omega}$$
$$\leq (m-1)\pi + \frac{\pi}{2} + 0 < m\pi$$

and such r cannot be the value of $r_m(\omega)$ for sufficiently small $\omega > 0$.

So, if $\omega > 0$ is sufficiently small, then $r_m(\omega)$ satisfies the inequality $-m < r_m(\omega) < 1 - m$ and thereby is negative.

Consider the product $\prod_{j=0}^{n-1} |r_m(\omega) + \omega i + j|$ with $0 < m < n$ and investigate its behavior for small $\omega > 0$.

If $j \geq m$, then for sufficiently small $\omega > 0$, we have $|r_m(\omega) + j| = r_m(\omega) + j < j$ and

$$\prod_{j=m}^{n-1} \bigl|r_m(\omega) + j\bigr| \leq \prod_{j=m}^{n-1} j < \prod_{j=m}^{n-1} (1+j). \tag{21}$$

If $j \leq m - 1$, then for sufficiently small $\omega > 0$, we have $|r_m(\omega) + j| = -r_m(\omega) - j < m - j = 1 + (m - 1 - j)$

$$\prod_{j=0}^{m-1} \bigl|r_m(\omega) + j\bigr| \leq \prod_{j=0}^{m-1} \bigl|1 + (m-1-j)\bigr| = \prod_{J=0}^{m-1} (1+J), \quad J = m - 1 - j. \tag{22}$$

Combining (21) and (22), we obtain, for sufficiently small $\omega > 0$,

$$\prod_{j=0}^{n-1} \bigl|r_m(\omega) + j\bigr| < \prod_{j=0}^{n-1} (1+j),$$

and

$$\prod_{j=0}^{n-1} \bigl|r_m(\omega) + \omega i + j\bigr| < \prod_{j=0}^{n-1} (1 + \alpha + j).$$

As for large ω, the left-hand side of the above inequality evidently tends to $+\infty$ as $\omega \to +\infty$ and hence is greater than its right-hand side for sufficiently large ω. By continuity there exists $\omega_m > 0$ such that

$$\prod_{j=0}^{n-1} \bigl|r_m(\omega_m) + \omega_m i + j\bigr| = \prod_{j=0}^{n-1} (1 + \alpha + j).$$

Thus, we can take $\mu_m = r_m(\omega_m) + \omega_m i \in \mathbb{C}$ to satisfy (19) and (20) for $0 < m < n$. For $-n < m < 0$, we can take the conjugates $\mu_m = \overline{\mu_{-m}}$. Thus, the existence of all μ_m needed is proved. This completes the proof of Proposition 2. □

Lemma 8 *If* $12 \le n \le 14$, $\alpha > 0$, *and* $q > 0$ *satisfy the polynomial equation*

$$\prod_{j=0}^{n-1}\bigl((\alpha + j)^2 + q^2\bigr) = \prod_{j=0}^{n-1}(\alpha + j + 1)^2,$$

then $2\alpha + 4 < q^2 < 3\alpha + 5$.

Proof It can be proved in the same way for all n mentioned. We show this for $n = 12$.
First, compute the right-hand side of the equation:

$$\begin{aligned}
&\prod_{j=0}^{11}(\alpha + j + 1)^2 \\
&\quad = \alpha^{24} + 156\alpha^{23} + 11{,}518\alpha^{22} + 535{,}392\alpha^{21} + 17{,}581{,}135\alpha^{20} \\
&\qquad + 433{,}823{,}676\alpha^{19} + 8{,}353{,}410{,}208\alpha^{18} + 128{,}665{,}048{,}512\alpha^{17} \\
&\qquad + 1{,}612{,}229{,}817{,}055\alpha^{16} + 16{,}625{,}859{,}652{,}116\alpha^{15} + 142{,}196{,}061{,}481{,}318\alpha^{14} \\
&\qquad + 1{,}013{,}438{,}536{,}648{,}512\alpha^{13} + 6{,}032{,}418{,}472{,}347{,}265\alpha^{12} \\
&\qquad + 29{,}989{,}851{,}619{,}249{,}236\alpha^{11} \\
&\qquad + 124{,}253{,}074{,}219{,}885{,}468\alpha^{10} + 427{,}135{,}043{,}298{,}835{,}872\alpha^{9} \\
&\qquad + 1{,}209{,}806{,}045{,}835{,}003{,}760\alpha^{8} + 2{,}795{,}060{,}589{,}044{,}133{,}696\alpha^{7} \\
&\qquad + 5{,}194{,}030{,}186{,}679{,}450{,}688\alpha^{6} + 7{,}613{,}724{,}634{,}416{,}755{,}712\alpha^{5} \\
&\qquad + 8{,}564{,}233{,}279{,}835{,}510{,}784\alpha^{4} + 7{,}096{,}936{,}674{,}284{,}421{,}120\alpha^{3} \\
&\qquad + 4{,}059{,}952{,}667{,}309{,}260{,}800\alpha^{2} + 1{,}424{,}017{,}035{,}657{,}216{,}000\alpha \\
&\qquad + 229{,}442{,}532{,}802{,}560{,}000.
\end{aligned}$$

Now, estimate the left-hand side supposing $q^2 \ge 3\alpha + 5 > 0$:

$$\begin{aligned}
&\prod_{j=0}^{11}\bigl((\alpha + j)^2 + q^2\bigr) \\
&\quad \ge \prod_{j=0}^{11}\bigl((\alpha + j)^2 + 3\alpha + 5\bigr) \\
&\quad \ge \alpha^{24} + 168\alpha^{23} + 13{,}216\alpha^{22} + 647{,}658\alpha^{21} + 22{,}191{,}136\alpha^{20} \\
&\qquad + 565{,}650{,}624\alpha^{19} + 11{,}143{,}609{,}279\alpha^{18} + 174{,}022{,}752{,}156\alpha^{17} \\
&\qquad + 2{,}192{,}303{,}359{,}180\alpha^{16} + 22{,}557{,}120{,}652{,}044\alpha^{15} + 191{,}221{,}185{,}335{,}728\alpha^{14} \\
&\qquad + 1{,}343{,}463{,}278{,}373{,}840\alpha^{13} + 7{,}851{,}135{,}965{,}424{,}751\alpha^{12} \\
&\qquad + 38{,}226{,}775{,}470{,}470{,}448\alpha^{11} \\
&\qquad + 155{,}030{,}143{,}411{,}290{,}136\alpha^{10} + 522{,}520{,}458{,}095{,}057{,}994\alpha^{9} \\
&\qquad + 1{,}457{,}064{,}439{,}886{,}002{,}624\alpha^{8} + 3{,}337{,}255{,}633{,}900{,}992{,}816\alpha^{7}
\end{aligned}$$

$$+ 6{,}209{,}925{,}089{,}367{,}687{,}345\alpha^6 + 9{,}237{,}499{,}888{,}429{,}090{,}764\alpha^5 + 10{,}723{,}421{,}856{,}201{,}549{,}372\alpha^4 + 9{,}360{,}016{,}963{,}404{,}522{,}912\alpha^3 + 5{,}777{,}193{,}048{,}791{,}013{,}360\alpha^2 + 2{,}247{,}088{,}906{,}508{,}241{,}600\alpha + 413{,}920{,}896{,}501{,}672{,}000.$$

The difference of this polynomial and the previous one is equal to

$$\begin{aligned}
&\prod_{j=0}^{11}\bigl((\alpha + j)^2 + 3\alpha + 5\bigr) - \prod_{j=0}^{11}(\alpha + j + 1)^2 \\
&\quad = 12\alpha^{23} + 1{,}698\alpha^{22} + 112{,}266\alpha^{21} + 4{,}610{,}001\alpha^{20} + 131{,}826{,}948\alpha^{19} \\
&\qquad + 2{,}790{,}199{,}071\alpha^{18} + 45{,}357{,}703{,}644\alpha^{17} + 580{,}073{,}542{,}125\alpha^{16} \\
&\qquad + 5{,}931{,}260{,}999{,}928\alpha^{15} + 49{,}025{,}123{,}854{,}410\alpha^{14} + 330{,}024{,}741{,}725{,}328\alpha^{13} \\
&\qquad + 1{,}818{,}717{,}493{,}077{,}486\alpha^{12} + 8{,}236{,}923{,}851{,}221{,}212\alpha^{11} \\
&\qquad + 30{,}777{,}069{,}191{,}404{,}668\alpha^{10} + 95{,}385{,}414{,}796{,}222{,}122\alpha^{9} \\
&\qquad + 247{,}258{,}394{,}050{,}998{,}864\alpha^{8} + 542{,}195{,}044{,}856{,}859{,}120\alpha^{7} \\
&\qquad + 1{,}015{,}894{,}902{,}688{,}236{,}657\alpha^{6} + 1{,}623{,}775{,}254{,}012{,}335{,}052\alpha^{5} \\
&\qquad + 2{,}159{,}188{,}576{,}366{,}038{,}588\alpha^{4} + 2{,}263{,}080{,}289{,}120{,}101{,}792\alpha^{3} \\
&\qquad + 1{,}717{,}240{,}381{,}481{,}752{,}560\alpha^{2} + 823{,}071{,}870{,}851{,}025{,}600\alpha \\
&\qquad + 184{,}478{,}363{,}699{,}112{,}000,
\end{aligned}$$

which is positive for any $\alpha \geq 0$. This shows that the polynomial equation cannot be satisfied by $\alpha > 0$ and $q > 0$ with $q^2 \geq 3\alpha + 5$.

In the same way, compute

$$\begin{aligned}
&\prod_{j=0}^{11}(\alpha + j + 1)^2 - \prod_{j=0}^{11}\bigl((\alpha + j)^2 + 2\alpha + 4\bigr) \\
&\quad = 96\alpha^{22} + 13{,}156\alpha^{21} + 844{,}624\alpha^{20} + 33{,}778{,}316\alpha^{19} + 943{,}838{,}852\alpha^{18} \\
&\qquad + 19{,}590{,}096{,}240\alpha^{17} + 313{,}464{,}915{,}984\alpha^{16} + 3{,}960{,}996{,}926{,}744\alpha^{15} \\
&\qquad + 40{,}162{,}617{,}066{,}616\alpha^{14} + 330{,}203{,}929{,}721{,}796\alpha^{13} \\
&\qquad + 2{,}215{,}299{,}128{,}334{,}800\alpha^{12} \\
&\qquad + 12{,}163{,}303{,}361{,}220{,}828\alpha^{11} + 54{,}651{,}209{,}110{,}677{,}476\alpha^{10} \\
&\qquad + 200{,}323{,}721{,}839{,}107{,}240\alpha^{9} + 595{,}229{,}721{,}350{,}941{,}648\alpha^{8} \\
&\qquad + 1{,}419{,}051{,}246{,}703{,}474{,}880\alpha^{7} + 2{,}673{,}079{,}829{,}956{,}829{,}568\alpha^{6} \\
&\qquad + 3{,}889{,}993{,}689{,}940{,}050{,}432\alpha^{5} + 4{,}228{,}750{,}706{,}659{,}177{,}984\alpha^{4} \\
&\qquad + 3{,}257{,}831{,}645{,}648{,}401{,}920\alpha^{3} + 1{,}625{,}109{,}784{,}526{,}284{,}800\alpha^{2} \\
&\qquad + 437{,}271{,}322{,}981{,}376{,}000\alpha + 37{,}266{,}873{,}282{,}560{,}000.
\end{aligned}$$

Hence, $\prod_{j=0}^{11}(\alpha + j + 1)^2 > \prod_{j=0}^{11}((\alpha + j)^2 + q^2)$ if $2\alpha + 4 \geq q^2$.
This contradiction yields $2\alpha + 4 < q^2 < 3\alpha + 5$. So, Lemma 8 is proved. □

The condition $\operatorname{Re} \frac{d\lambda_\alpha}{d\alpha}(\tilde{\alpha}) \neq 0$ needed for the Hopf theorem, expressed explicitly by means of the implicit function theorem, looks like

$$\left[\sum_{j=0}^{n-1} \frac{\alpha + j}{q^2 + (\alpha + j)^2}\right]^2 + \left[\sum_{j=0}^{n-1} \frac{q}{q^2 + (\alpha + j)^2}\right]^2 \neq \sum_{j=0}^{n-1} \frac{\alpha + j}{q^2 + (\alpha + j)^2} \sum_{j=0}^{n-1} \frac{1}{1 + \alpha + j}.$$

Lemma 9 *If* $12 \leq n \leq 14$, $\alpha > 0$ *and* $0 < q^2 < 3\alpha + 5$, *then*

$$\left[\sum_{j=0}^{n-1} \frac{\alpha + j}{q^2 + (\alpha + j)^2}\right]^2 + \left[\sum_{j=0}^{n-1} \frac{q}{q^2 + (\alpha + j)^2}\right]^2 > \sum_{j=0}^{n-1} \frac{\alpha + j}{q^2 + (\alpha + j)^2} \sum_{j=0}^{n-1} \frac{1}{1 + \alpha + j}. \tag{23}$$

Proof Hereafter all sums and products with no limits indicated are over $j = 0, 1, \ldots, n - 1$.

Multiplying inequality (23) by $U_* = \prod(1 + \alpha + j)$ and then twice by $V_* = \prod[q^2 + (\alpha + j)^2]$, we obtain the following equivalent inequality provided $\alpha > 0$:

$$U_* \left[\left(\sum(\alpha + j)V_j\right)^2 + q^2\left(\sum V_j\right)^2\right] > V_* \sum(\alpha + j)V_j \sum U_j \tag{24}$$

with the polynomials $U_j = \frac{U_*}{1+\alpha+j}$ and $V_j = \frac{V_*}{q^2+(\alpha+j)^2}$.

Put $q^2 = \frac{3\alpha+5}{1+w}$, $w > 0$. Substituting this into inequality (24) and multiplying the result by $(1 + w)^{2n-1}$, we obtain another equivalent one:

$$U_* \left[(1 + w)\left(\sum(\alpha + j)P_j\right)^2 + (3\alpha + 5)\left(\sum P_j\right)^2\right] > P_* \cdot \sum(\alpha + j)P_j \cdot \sum U_j \tag{25}$$

with $P_* = \prod[3\alpha + 5 + (1 + w)(\alpha + j)^2]$ and $P_j = \frac{P_*}{3\alpha+5+(1+w)(\alpha+j)^2}$.

Both sides of inequality (25) are polynomials of α and w with non-negative integer coefficients. So, they can be computed exactly, with no rounding. This rather cumbersome computation gives the following result for the difference of the left- and right-hand sides of (25) expressed as

$$U_* \left[(1 + w)\left(\sum(\alpha + j)P_j\right)^2 + (3\alpha + 5)\left(\sum P_j\right)^2\right] - P_* \sum(\alpha + j)P_j \sum U_j = \sum_{j=0}^{5n-2} \Delta_j \alpha^j \tag{26}$$

with polynomials $\Delta_j \in \mathbb{R}[w]$. Straightforward though very cumbersome calculations show that $\Delta_{5n-2} = 0$, and all other Δ_j in (26) are polynomials with positive coefficients.

This completes the proof of Lemma 9. □

To apply the Hopf bifurcation theorem, we need to check that equation (14) cannot have more than a single pair of imaginary conjugated roots. It can be easily obtained by considering equation (16).

Now, the Hopf bifurcation theorem and the lemmas proved provide, for $n = 12, 13, 14$, the existence of a family $\alpha_\varepsilon > 0$ such that equation (14) with $\alpha = \alpha_0$ has imaginary roots $\lambda = \pm qi$ and for sufficiently small ε, system (13) with $\alpha = \alpha_\varepsilon$ has a periodic solution $V_\varepsilon(t)$ with period $T_\varepsilon \to T = \frac{2\pi}{q}$ as $\varepsilon \to 0$. In particular, the coordinate $V_{\varepsilon,0}(t) = v(t)$ of the vector $V_\varepsilon(t)$ is also a periodic function with the same period. Then, taking into account (9), we obtain

$$y(x) = \left(C + v\left(-\ln\left(x^* - x\right)\right)\right)\left(x^* - x\right)^{-\alpha}.$$

Put $h(s) = C + v(-s)$, which is a non-constant continuous periodic and positive for sufficiently small ε function and obtain the required equality

$$y(x) = \left(x^* - x\right)^{-\alpha} h\left(\ln\left(x^* - x\right)\right).$$

In the similar way, we obtain the related expressions for $y^{(j)}(x)$, $j = 0, 1, \ldots, n-1$.

Theorem 4 is proved. □

Conclusions, concluding remarks and open problems

1. Computer calculations give approximate values of α providing equation (14) to have a pure imaginary root λ. They are, with corresponding values of k, as follows:
   if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;
   if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;
   if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.
2. Note that equation (14) has no pure imaginary roots if $n \leq 11$. So, the Hopf bifurcation theorem cannot be applied, but it does not follow that Theorem 4 cannot be proved for some $n < 12$.
3. Equation (5) with $n = 3$ has solutions of type (6) with oscillatory h (see [3, 5]).
4. If $n \geq 15$, then the inequality needed for the Hopf bifurcation theorem $\operatorname{Re} \frac{d\lambda_\alpha}{d\alpha}(\tilde{\alpha}) \neq 0$ cannot be proved in the same way because the estimate $q^2 < 3\alpha + 5$ does not hold.

Competing interests
The author declares that she has no competing interests.

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