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# Boundary value problems for a new class of three-point nonlocal Riemann-Liouville integral boundary conditions

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## Abstract

In this paper we investigate the existence and uniqueness of solutions for a new class of fractional boundary value problems involving three-point nonlocal Riemann-Liouville integral boundary conditions. Some new results are obtained by using fixed point theorems. Illustrative examples of our results are also presented.

**MSC:** 26A33; 34A08

**Keywords:** fractional differential equations; nonlocal boundary conditions; fixed point theorems

## 1 Introduction

Fractional order differential equations have been of great interest recently because they play a vital role in describing many phenomena related to physics, chemistry, mechanics, control systems, flow in porous media, electrical networks, mathematical biology and viscoelasticity. For a reader interested in the systematic development of the topic, we refer to the books [1–7]. A variety of results on initial and boundary value problems of fractional differential equations and inclusions can easily be found in the literature on the topic. For some recent results, we can refer to [8–19] and references cited therein.

Bai [11] discussed the existence of positive solutions for the following three-point fractional boundary value problem:

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \quad 1 < q \leq 2, t \in (0, 1), \\ u(0) &= 0, \quad u(1) = \alpha \beta u(\eta), \quad 0 < \eta < 1, \end{aligned} \tag{1.1}$$

where  $D^q$  denotes the Riemann-Liouville fractional derivative, and  $0 < \beta \eta^{q-1} < 1$ . Some existence results for at least one positive solution are obtained by the use of fixed point index theory.

In paper [12] the authors studied the following boundary value problem of fractional order differential equations with three-point fractional integral boundary conditions:

$$\begin{aligned} {}^c D^q u(t) &= f(t, u(t)), \quad 1 < q \leq 2, t \in (0, 1), \\ u(0) &= 0, \quad u(1) = \alpha I^p u(\eta), \quad 0 < \eta < 1, \end{aligned} \tag{1.2}$$

where  ${}^c D^q$  denotes the Caputo fractional derivative of order  $q$ ,  $I^p$  is the Riemann-Liouville fractional integral of order  $p$ ,  $f \in C([0, 1] \times \mathbb{R})$  and  $\alpha \in \mathbb{R}$ ,  $\alpha \neq \Gamma(p + 2)/\eta^{p+1}$ . Existence and uniqueness results are proved via Banach's contraction principle and Schaefer's fixed point theorem. The results of [12] are completed in [13], by existence results via Krasnosel'skii's fixed point theorem and Leray-Schauder's degree theory, and extended to cover the multivalued case.

In [14] existence and uniqueness results are obtained for a single and multivalued case for the following boundary value problem of fractional order differential equations with nonlocal and fractional integral boundary conditions:

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \quad 1 < q \leq 2, t \in (0, 1), \\ u(0) &= u_0 + g(u), \quad u(1) = \alpha I^\alpha u(\eta), \quad 0 < \eta < 1, \end{aligned} \tag{1.3}$$

where  $g : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}$ .

Recently, in [20] the following boundary value problem of fractional differential equations with a fractional integral condition:

$$\begin{cases} D^q u(t) = f(t, u(t), D^p u(t)), & 0 < t < 1, 1 < q \leq 2, 0 < p < 1, \\ u(0) = 0, & u'(1) = \alpha I^p u(1), \end{cases} \tag{1.4}$$

was studied. Existence and uniqueness results are proved via Banach's contraction principle and Leray-Schauder's nonlinear alternative.

In [16] existence and uniqueness results are investigated for the following boundary value problem of fractional order differential equations with four-point nonlocal Riemann-Liouville fractional integral boundary conditions:

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \quad 1 < q \leq 2, t \in (0, 1), \\ u(0) &= a I^\beta u(\eta), \quad u(1) = b I^\alpha u(\sigma), \end{aligned} \tag{1.5}$$

where  $0 < \alpha, \beta \leq 1$  and  $0 < \eta, \sigma < 1$ .

In this paper, we study a new class of three-point boundary value problems of fractional order differential equations with nonlocal Riemann-Liouville fractional integral boundary conditions. More precisely, we consider the nonlinear fractional differential equation

$$D^\alpha u(t) = f(t, u(t)), \quad 1 < \alpha \leq 2, 0 < t < T, \tag{1.6}$$

subject to nonlocal fractional integral conditions

$$u(\eta) = 0, \quad I^\nu u(T) \equiv \int_0^T \frac{(T-s)^{\nu-1}}{\Gamma(\nu)} u(s) ds = 0, \tag{1.7}$$

where  $\eta \in (0, T)$  is a given constant. The novelty of this boundary value problem lies in the fact that instead of the value  $u(0)$ , which appeared in all the above mentioned boundary value problems, we have the value  $u(\eta)$  for some  $\eta \in (0, T)$ .

The paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we prove our main results. Some examples to illustrate our results are presented in Section 4.

## 2 Preliminaries

In this section we introduce some notations, definitions of fractional calculus [3, 4] and present preliminary results needed in our proofs later.

**Definition 2.1** The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $g \in L^1((0, T), \mathbb{R})$  is defined by

$$I^\alpha g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds,$$

where  $\Gamma$  is the gamma function.

**Definition 2.2** The Riemann-Liouville fractional derivative of order  $\alpha > 0$  of a continuous function  $g : (0, \infty) \rightarrow \mathbb{R}$  is defined by

$$D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds,$$

where  $n = [\alpha] + 1$ ,  $[\alpha]$  denotes the integral part of real number  $\alpha$ , provided the right-hand side is point-wise defined on  $(0, \infty)$ .

**Lemma 2.1** (see [4]) *Let  $\alpha > 0$  and  $y \in C(0, T) \cap L(0, T)$ . Then the fractional differential equation  $D^\alpha y(t) = 0$  has a unique solution*

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where  $c_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$  and  $n = [\alpha] + 1$ .

**Lemma 2.2** *Suppose that  $\eta \neq \frac{(\alpha-1)T}{\nu+\alpha-1}$ ,  $1 < \alpha \leq 2$ ,  $\nu > 0$  and  $h \in AC[0, T]$ . Then the problem*

$$D^\alpha x(t) = h(t), \quad 0 < t < T, \tag{2.1}$$

$$x(\eta) = 0, \quad I^\nu x(T) = 0, \quad \eta \in (0, T), \tag{2.2}$$

can be written as an integral equation

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds \\ & - \frac{(\nu + \alpha - 1)t^{\alpha-1} - T(\alpha - 1)t^{\alpha-2}}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)\Omega} \int_0^\eta (\eta - s)^{\alpha-1} h(s) ds \\ & + \frac{t^{\alpha-1} - \eta t^{\alpha-2}}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)\Omega} \int_0^T (T - s)^{\nu+\alpha-1} h(s) ds, \end{aligned} \tag{2.3}$$

where

$$\Omega = \eta - \frac{(\alpha - 1)T}{\nu + \alpha - 1}. \tag{2.4}$$

*Proof* From (2.1) and Lemma 2.1, we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds. \tag{2.5}$$

The first condition of (2.2) implies

$$c_1 \eta^{\alpha-1} + c_2 \eta^{\alpha-2} = - \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds. \tag{2.6}$$

Using the Riemann-Liouville fractional integral of order  $\nu > 0$  to (2.5) and applying Dirichlet's formula [1, p.56], we obtain

$$\begin{aligned} I^\nu x(t) &= \frac{c_1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} s^{\alpha-1} ds + \frac{c_2}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} s^{\alpha-2} ds \\ &\quad + \frac{1}{\Gamma(\nu)\Gamma(\alpha)} \int_0^t \int_0^s (t-s)^{\nu-1} (s-\rho)^{\alpha-1} h(\rho) d\rho ds \\ &= c_1 \frac{\Gamma(\alpha)}{\Gamma(\nu+\alpha)} t^{\nu+\alpha-1} + c_2 \frac{\Gamma(\alpha-1)}{\Gamma(\nu+\alpha-1)} t^{\nu+\alpha-2} \\ &\quad + \frac{1}{\Gamma(\nu+\alpha)} \int_0^t (t-s)^{\nu+\alpha-1} h(s) ds. \end{aligned}$$

The second condition of (2.2) implies

$$c_1 \frac{\Gamma(\alpha)}{\Gamma(\nu+\alpha)} T^{\nu+\alpha-1} + c_2 \frac{\Gamma(\alpha-1)}{\Gamma(\nu+\alpha-1)} T^{\nu+\alpha-2} = - \int_0^T \frac{(T-s)^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)} h(s) ds. \tag{2.7}$$

Solving the linear equations (2.6)-(2.7) for unknown constants  $c_1$  and  $c_2$ , we have

$$\begin{aligned} c_1 &= - \frac{1}{\eta^{\alpha-2}\Gamma(\alpha)\Omega} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds \\ &\quad + \frac{\Gamma(\nu+\alpha-1)}{T^{\nu+\alpha-2}\Gamma(\alpha-1)\Omega} \int_0^T \frac{(T-s)^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)} h(s) ds \end{aligned}$$

and

$$\begin{aligned} c_2 &= \frac{T(\alpha-1)}{(\nu+\alpha-1)\eta^{\alpha-2}\Gamma(\alpha)\Omega} \int_0^\eta (\eta-s)^{\alpha-1} h(s) ds \\ &\quad - \frac{\eta\Gamma(\nu+\alpha-1)}{T^{\nu+\alpha-2}\Gamma(\alpha-1)\Omega} \int_0^T \frac{(T-s)^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)} h(s) ds, \end{aligned}$$

where constant  $\Omega$  is defined by (2.4). Substituting constants  $c_1$  and  $c_2$  in (2.5), we obtain (2.3). □

Let  $\mathcal{C} = C([0, T], \mathbb{R})$  denote the Banach space of all continuous functions from  $[0, T]$  to  $\mathbb{R}$  endowed with the norm defined by  $\|u\| = \sup_{t \in [0, T]} |u(t)|$ . As in Lemma 2.2, we define

an operator  $A : \mathcal{C} \rightarrow \mathcal{C}$  by

$$\begin{aligned}
 (Au)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds \\
 & - \frac{(\nu + \alpha - 1)t^{\alpha-1} - T(\alpha - 1)t^{\alpha-2}}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)\Omega} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds \\
 & + \frac{t^{\alpha-1} - \eta t^{\alpha-2}}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)\Omega} \int_0^T (T - s)^{\nu+\alpha-1} f(s, u(s)) ds. \tag{2.8}
 \end{aligned}$$

It should be noticed that problem (1.6)-(1.7) has solutions if and only if the operator  $A$  has fixed points.

### 3 Main results

We are in a position to establish our main results. In the following subsections, we prove existence as well as existence and uniqueness results for BVP (1.6)-(1.7) by using a variety of fixed point theorems.

#### 3.1 Existence and uniqueness results via Banach’s fixed point theorem

In this subsection we give first an existence and uniqueness result for BVP (1.6)-(1.7) by using Banach’s fixed point theorem.

**Theorem 3.1** *Assume that*

(H<sub>1</sub>) *there exists a constant  $L > 0$  such that  $|f(t, u) - f(t, v)| \leq L|u - v|$  for each  $t \in [0, T]$  and  $u, v \in \mathbb{R}$ .*

*If*

$$\begin{aligned}
 \Lambda := & LT^\alpha \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} \right. \\
 & \left. + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\} < 1, \tag{3.1}
 \end{aligned}$$

*then problem (1.6)-(1.7) has a unique solution in  $[0, T]$ .*

*Proof* We transform problem (1.6)-(1.7) into a fixed point problem,  $u = Au$ , where the operator  $A$  is defined by (2.8). Obviously, fixed points of the operator  $A$  are solutions of problem (1.6)-(1.7). Using the Banach contraction principle, we shall show that  $A$  has a fixed point.

Setting  $\sup_{t \in [0, T]} |f(t, 0)| = M < \infty$  and choosing  $r \geq \frac{\Lambda M}{(1-\Lambda)L}$ , we show that  $AB_r \subset B_r$ , where  $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ . For  $x \in B_r$ , we have

$$\begin{aligned}
 \|Au\| \leq & \sup_{t \in [0, T]} \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s))| ds \right. \\
 & + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} |f(s, u(s))| ds \\
 & \left. + \frac{T^{\alpha-2}(T + \eta)}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \int_0^T (T - s)^{\nu+\alpha-1} |f(s, u(s))| ds \right\}
 \end{aligned}$$

$$\begin{aligned}
 &\leq \sup_{t \in [0, T]} \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) ds \right. \\
 &\quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) ds \\
 &\quad + \frac{T + \eta}{(\nu + \alpha - 1)T^\nu \Gamma(\alpha - 1)|\Omega|} \\
 &\quad \left. \times \int_0^T (T - s)^{\nu+\alpha-1} (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) ds \right\} \\
 &\leq (Lr + M) \sup_{t \in [0, T]} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \right. \\
 &\quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} ds \\
 &\quad \left. + \frac{T + \eta}{(\nu + \alpha - 1)T^\nu \Gamma(\alpha - 1)|\Omega|} \int_0^T (T - s)^{\nu+\alpha-1} ds \right\} \\
 &\leq (Lr + M)T^\alpha \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} \right. \\
 &\quad \left. + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right) \\
 &= (Lr + M) \frac{\Lambda}{L} \leq r,
 \end{aligned}$$

which proves that  $AB_r \subset B_r$ .

Now let  $u, v \in \mathcal{C}$ . Then, for  $t \in [0, T]$ , we have

$$\begin{aligned}
 &|(Au)(t) - (Av)(t)| \\
 &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s)) - f(s, v(s))| ds \\
 &\quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\
 &\quad + \frac{T^{\alpha-2}(T + \eta)}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \\
 &\quad \times \int_0^T (T - s)^{\nu+\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\
 &\leq L \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |u(s) - v(s)| ds \\
 &\quad + \frac{LT^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} |u(s) - v(s)| ds \\
 &\quad + \frac{L(T + \eta)}{(\nu + \alpha - 1)T^\nu \Gamma(\alpha - 1)|\Omega|} \int_0^T (T - s)^{\nu+\alpha-1} |u(s) - v(s)| ds \\
 &\leq \frac{LT^\alpha}{\Gamma(\alpha + 1)} \|u - v\| + \frac{L\eta^2 T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\Gamma(\alpha + 1)|\Omega|} \|u - v\| \\
 &\quad + \frac{LT^\alpha(T + \eta)}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \|u - v\|
 \end{aligned}$$

$$= LT^\alpha \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\} \|u - v\|.$$

Therefore,

$$\|Au - Av\| \leq \Lambda \|u - v\|.$$

From (3.1),  $A$  is a contraction. As a consequence of Banach's fixed point theorem, we get that  $A$  has a fixed point which is a unique solution of problem (1.6)-(1.7).  $\square$

Now we give another existence and uniqueness result for BVP (1.6)-(1.7) by using Banach's fixed point theorem and Hölder's inequality.

**Theorem 3.2** *Suppose that the continuous function  $f$  satisfies the following assumption:*

(H<sub>2</sub>)  $|f(t, x_1) - f(t, x_2)| \leq m(t)|x_1 - x_2|$ , for  $t \in [0, T]$ ,  $x_i \in \mathbb{R}$ ,  $i = 1, 2$  and  $m \in L^{\frac{1}{\gamma}}([0, T], \mathbb{R}^+)$ ,  $\gamma \in (0, 1)$ .

Denote  $\|m\| = (\int_0^T |m(s)|^{\frac{1}{\gamma}} ds)^\gamma$ . If

$$\frac{\|m\|}{\Gamma(\alpha)} \left\{ T^{\alpha-\gamma} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{T^{\alpha-1}\eta^{2-\gamma}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{T^{\alpha-\gamma}(T + \eta)(\alpha - 1)}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\nu + \alpha - \gamma} \right)^{1-\gamma} \right\} < 1,$$

then boundary value problem (1.6)-(1.7) has a unique solution.

*Proof* For  $u, v \in C$  and for each  $t \in [0, T]$ , by Hölder's inequality, we have

$$\begin{aligned} & |(Au)(t) - (Av)(t)| \\ & \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s)) - f(s, v(s))| ds \\ & \quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta-s)^{\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\ & \quad + \frac{T^{\alpha-2}(T + \eta)}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \int_0^T (T-s)^{\nu+\alpha-1} |f(s, u(s)) - f(s, v(s))| ds \\ & \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) |u(s) - v(s)| ds \\ & \quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta-s)^{\alpha-1} m(s) |u(s) - v(s)| ds \\ & \quad + \frac{T^{\alpha-2}(T + \eta)}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \int_0^T (T-s)^{\nu+\alpha-1} m(s) |u(s) - v(s)| ds \\ & \leq \frac{1}{\Gamma(\alpha)} \left( \int_0^t ((t-s)^{\alpha-1})^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left( \int_0^t (m(s))^{1/\gamma} ds \right)^\gamma \|u - v\| \end{aligned}$$

$$\begin{aligned}
 & + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}|\Omega|\Gamma(\alpha)} \\
 & \times \left( \int_0^\eta ((\eta - s)^{\alpha-1})^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left( \int_0^\eta (m(s))^{1/\gamma} ds \right)^\gamma \|u - \nu\| \\
 & + \frac{T + \eta}{(\nu + \alpha - 1)T^\nu \Gamma(\alpha - 1)|\Omega|} \\
 & \times \left( \int_0^T ((T - s)^{\nu+\alpha-1})^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left( \int_0^T (m(s))^{1/\gamma} ds \right)^\gamma \|u - \nu\| \\
 & \leq \frac{\|m\|}{\Gamma(\alpha)} \left\{ T^{\alpha-\gamma} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{T^{\alpha-1}\eta^{2-\gamma}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} \right. \\
 & \left. + \frac{T^{\alpha-\gamma}(T + \eta)(\alpha - 1)}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\nu + \alpha - \gamma} \right)^{1-\gamma} \right\} \|u - \nu\|.
 \end{aligned}$$

It follows that  $A$  is a contraction mapping. Hence Banach's fixed point theorem implies that  $A$  has a unique fixed point which is the unique solution of problem (1.6)-(1.7). This completes the proof.  $\square$

### 3.2 Existence result via Krasnoselskii's fixed point theorem

**Lemma 3.1** (Krasnoselskii's fixed point theorem) [21] *Let  $M$  be a closed, bounded, convex and nonempty subset of a Banach space  $X$ . Let  $A, B$  be the operators such that (a)  $Ax + By \in M$  whenever  $x, y \in M$ ; (b)  $A$  is compact and continuous; (c)  $B$  is a contraction mapping. Then there exists  $z \in M$  such that  $z = Az + Bz$ .*

**Theorem 3.3** *Let  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying  $(H_1)$ . Moreover, we assume that*

$$(H_3) \quad |f(t, u)| \leq \mu(t), \quad \forall (t, u) \in [0, T] \times \mathbb{R}, \text{ and } \mu \in C([0, T], \mathbb{R}^+).$$

*Then boundary value problem (1.6)-(1.7) has at least one solution on  $[0, T]$  if*

$$LT^\alpha \left\{ \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\} < 1. \tag{3.2}$$

*Proof* Letting  $\sup_{t \in [0, T]} |\mu(t)| = \|\mu\|$ , we fix

$$\bar{r} \geq T^\alpha \|\mu\| \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\}$$

and consider  $B_{\bar{r}} = \{u \in C : \|u\| \leq \bar{r}\}$ . We define the operators  $\mathcal{P}$  and  $\mathcal{Q}$  on  $B_{\bar{r}}$  as

$$\begin{aligned}
 (\mathcal{P}u)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds, \\
 (\mathcal{Q}u)(t) &= -\frac{(\nu + \alpha - 1)t^{\alpha-1} - T(\alpha - 1)t^{\alpha-2}}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)\Omega} \int_0^\eta (\eta - s)^{\alpha-1} f(s, u(s)) ds \\
 & \quad + \frac{t^{\alpha-1} - \eta t^{\alpha-2}}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)\Omega} \int_0^T (T - s)^{\nu+\alpha-1} f(s, u(s)) ds.
 \end{aligned}$$

For  $u, v \in B_{\bar{r}}$ , we find that

$$\begin{aligned} & \| \mathcal{P}u + \mathcal{Q}v \| \\ & \leq T^\alpha \| \mu \| \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(v + 2(\alpha - 1))}{(v + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(v + \alpha - 1)(v + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\} \\ & \leq \bar{r}. \end{aligned}$$

Thus,  $\mathcal{P}u + \mathcal{Q}v \in B_{\bar{r}}$ . It follows from assumption (H<sub>3</sub>) together with (3.2) that  $\mathcal{Q}$  is a contraction mapping. Continuity of  $f$  implies that the operator  $\mathcal{P}$  is continuous. Also,  $\mathcal{P}$  is uniformly bounded on  $B_{\bar{r}}$  as

$$\| \mathcal{P}u \| \leq \frac{T^\alpha}{\Gamma(\alpha + 1)} \| \mu \|.$$

Now we prove the compactness of the operator  $\mathcal{P}$ .

We define  $\sup_{(t,u) \in [0,T] \times B_{\bar{r}}} |f(t,u)| = \bar{f} < \infty$ , and consequently we have

$$\begin{aligned} |(\mathcal{P}u)(t_2) - (\mathcal{P}u)(t_1)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, u(s)) ds \right. \\ & \quad \left. + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, u(s)) ds \right| \\ & \leq \frac{\bar{f}}{\Gamma(\alpha + 1)} |t_1^\alpha - t_2^\alpha|, \end{aligned}$$

which is independent of  $u$ . Thus,  $\mathcal{P}$  is equicontinuous. So  $\mathcal{P}$  is relatively compact on  $B_{\bar{r}}$ . Hence, by the Arzelá-Ascoli theorem,  $\mathcal{P}$  is compact on  $B_{\bar{r}}$ . Thus all the assumptions of Lemma 3.1 are satisfied. So the conclusion of Lemma 3.1 implies that boundary value problem (1.6)-(1.7) has at least one solution on  $[0, T]$ .  $\square$

### 3.3 Existence result via Leray-Schauder's nonlinear alternative

**Theorem 3.4** (Nonlinear alternative for single-valued maps) [22] *Let  $E$  be a Banach space,  $C$  be a closed, convex subset of  $E$ ,  $U$  be an open subset of  $C$  and  $0 \in U$ . Suppose that  $F : \bar{U} \rightarrow C$  is a continuous, compact (that is,  $F(\bar{U})$  is a relatively compact subset of  $C$ ) map. Then either*

- (i)  $F$  has a fixed point in  $\bar{U}$ , or
- (ii) there is a  $u \in \partial U$  (the boundary of  $U$  in  $C$ ) and  $\lambda \in (0, 1)$  with  $u = \lambda F(u)$ .

**Theorem 3.5** *Assume that*

(H<sub>4</sub>) *there exists a continuous nondecreasing function  $\psi : [0, \infty) \rightarrow (0, \infty)$  and a function  $p \in L^1([0, T], \mathbb{R}^+)$  such that*

$$|f(t, u)| \leq p(t)\psi(\|u\|) \quad \text{for each } (t, u) \in [0, T] \times \mathbb{R};$$

(H<sub>5</sub>) *there exists a constant  $M > 0$  such that*

$$\frac{M}{T^\alpha \psi(M) \|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(v + 2(\alpha - 1))}{(v + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(v + \alpha - 1)(v + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\}} > 1.$$

*Then boundary value problem (1.6)-(1.7) has at least one solution on  $[0, T]$ .*

*Proof* We show that  $A$  maps bounded sets (balls) into bounded sets in  $C([0, T], \mathbb{R})$ . For a positive number  $\rho$ , let  $B_\rho = \{u \in C([0, T], \mathbb{R}) : \|u\| \leq \rho\}$  be a bounded ball in  $C([0, T], \mathbb{R})$ . Then for  $t \in [0, T]$  we have

$$\begin{aligned}
 |(Au)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s))| ds \\
 &\quad + \frac{T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta-s)^{\alpha-1} |f(s, u(s))| ds \\
 &\quad + \frac{T^{\alpha-2}(T + \eta)}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha-1)|\Omega|} \int_0^T (T-s)^{\nu+\alpha-1} |f(s, u(s))| ds \\
 &\leq \frac{\psi(\|u\|)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) ds \\
 &\quad + \frac{\psi(\|u\|)T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta-s)^{\alpha-1} p(s) ds \\
 &\quad + \frac{\psi(\|u\|)(T + \eta)}{(\nu + \alpha - 1)T^\nu\Gamma(\alpha-1)|\Omega|} \int_0^T (T-s)^{\nu+\alpha-1} p(s) ds \\
 &\leq \frac{\psi(\|u\|)\|p\|_{L^1}}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \\
 &\quad + \frac{\psi(\|u\|)\|p\|_{L^1}T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta-s)^{\alpha-1} ds \\
 &\quad + \frac{\psi(\|u\|)\|p\|_{L^1}(T + \eta)}{(\nu + \alpha - 1)T^\nu\Gamma(\alpha-1)|\Omega|} \int_0^T (T-s)^{\nu+\alpha-1} ds \\
 &\leq \frac{T^\alpha\psi(\|u\|)\|p\|_{L^1}}{\Gamma(\alpha+1)} + \frac{\eta^2\psi(\|u\|)\|p\|_{L^1}T^{\alpha-1}(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)\Gamma(\alpha+1)|\Omega|} \\
 &\quad + \frac{T^\alpha\psi(\|u\|)\|p\|_{L^1}(T + \eta)}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha-1)|\Omega|} \\
 &= T^\alpha\psi(\|u\|)\|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha+1)|\Omega|} \right. \\
 &\quad \left. + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha-1)|\Omega|} \right\}.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \|Au\| &\leq T^\alpha\psi(\rho)\|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha+1)|\Omega|} \right. \\
 &\quad \left. + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha-1)|\Omega|} \right\}.
 \end{aligned}$$

Next we show that  $A$  maps bounded sets into equicontinuous sets of  $C([0, T], \mathbb{R})$ . Let  $t_1, t_2 \in [0, T]$  with  $t_1 < t_2$  and  $u \in B_\rho$ . Then we have

$$\begin{aligned}
 |(Au)(t_2) - (Au)(t_1)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} |f(s, u(s))| ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} |f(s, u(s))| ds \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{(\nu + \alpha - 1)|t_2^{\alpha-1} - t_1^{\alpha-1}| + T(\alpha - 1)|t_2^{\alpha-2} - t_1^{\alpha-2}|}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} |f(s, u(s))| ds \\
 & + \frac{\eta|t_1^{\alpha-2} - t_1^{\alpha-2}| + |t_2^{\alpha-1} - t_1^{\alpha-1}|}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \int_0^T (T - s)^{\nu+\alpha-1} |f(s, u(s))| ds \\
 \leq & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} p(s)\psi(\rho) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} p(s)\psi(\rho) ds \right| \\
 & + \frac{(\nu + \alpha - 1)|t_2^{\alpha-1} - t_1^{\alpha-1}| + T(\alpha - 1)|t_2^{\alpha-2} - t_1^{\alpha-2}|}{(\nu + \alpha - 1)\eta^{\alpha-2}\Gamma(\alpha)|\Omega|} \int_0^\eta (\eta - s)^{\alpha-1} p(s)\psi(\rho) ds \\
 & + \frac{\eta|t_1^{\alpha-2} - t_1^{\alpha-2}| + |t_2^{\alpha-1} - t_1^{\alpha-1}|}{(\nu + \alpha - 1)T^{\nu+\alpha-2}\Gamma(\alpha - 1)|\Omega|} \int_0^T (T - s)^{\nu+\alpha-1} p(s)\psi(\rho) ds.
 \end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of  $x \in B_\rho$  as  $t_2 - t_1 \rightarrow 0$ . As  $A$  satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that  $A : C([0, T], \mathbb{R}) \rightarrow C([0, T], \mathbb{R})$  is completely continuous.

Let  $u$  be a solution. Then, for  $t \in [0, T]$ , and following similar computations as in the first step, we have

$$\begin{aligned}
 |u(t)| \leq & T^\alpha \psi(\|u\|) \|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} \right. \\
 & \left. + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\}.
 \end{aligned}$$

Consequently, we have

$$\frac{\|u\|}{T^\alpha \psi(\|u\|) \|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2(\nu + 2(\alpha - 1))}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{T + \eta}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\}} \leq 1.$$

In view of (H<sub>5</sub>), there exists  $M$  such that  $\|u\| \neq M$ . Let us set

$$U = \{u \in C([0, T], \mathbb{R}) : \|u\| < M\}.$$

Note that the operator  $A : \bar{U} \rightarrow C([0, T], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $u \in \partial U$  such that  $u = \lambda Au$  for some  $\lambda \in (0, 1)$ . Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.4), we deduce that  $A$  has a fixed point  $u \in \bar{U}$  which is a solution of problem (1.6)-(1.7). This completes the proof.  $\square$

#### 4 Examples

**Example 4.1** Consider the following fractional integral boundary value problem:

$$D^{\frac{3}{2}} u(t) = \frac{e^{-t}}{4(1 + e^t)} \cdot \frac{|u(t)|}{1 + |u(t)|}, \quad t \in \left(0, \frac{8}{3}\right), \tag{4.1}$$

$$u\left(\frac{5}{3}\right) = 0, \quad I^{\frac{7}{2}} u\left(\frac{8}{3}\right) = 0. \tag{4.2}$$

Here  $\alpha = 3/2$ ,  $\nu = 7/2$ ,  $\eta = 5/3$ ,  $T = 8/3$  and  $f(t, u) = (e^{-t}/4(1 + e^t))(|u|/(1 + |u|))$  and  $\Omega = \eta - (\alpha - 1)T/(\nu + \alpha - 1) = 4/3 \neq 0$ . Since  $|f(t, u) - f(t, v)| \leq (1/8)|u - v|$ , then (H<sub>1</sub>) is satisfied

with  $L = 1/8$ . We can show that

$$\begin{aligned}
 &LT^\alpha \left\{ \frac{1}{\Gamma(\alpha + 1)} + \frac{\eta^2[\nu + 2(\alpha - 1)]}{(\nu + \alpha - 1)T\Gamma(\alpha + 1)|\Omega|} + \frac{\eta + T}{(\nu + \alpha - 1)(\nu + \alpha)\Gamma(\alpha - 1)|\Omega|} \right\} \\
 &= \frac{1}{8} \left( \frac{8}{3} \right)^{\frac{3}{2}} \left[ \frac{4}{3\sqrt{\pi}} + \frac{75}{64\sqrt{\pi}} + \frac{13}{80\sqrt{\pi}} \right] \approx 0.819268997 < 1.
 \end{aligned}$$

Hence, by Theorem 3.1, boundary value problem (4.1)-(4.2) has a unique solution on  $[0, 8/3]$ .

**Example 4.2** Consider the following fractional integral boundary value problem:

$$D^{\frac{5}{3}}x(t) = t^2 \tan^{-1} x(t), \quad t \in \left(0, \frac{1}{2}\right), \tag{4.3}$$

$$x\left(\frac{3}{8}\right) = 0, \quad I^{\frac{2}{3}}x\left(\frac{1}{2}\right) = 0. \tag{4.4}$$

Set  $\alpha = 5/3$ ,  $\nu = 2/3$ ,  $\eta = 3/8$ ,  $T = 1/2$ ,  $f(t, x) = t^2 \tan^{-1} x$  and choose  $\gamma = (1/12) \in (0, 1)$ . It is easy to see that  $\Omega = \eta - (\alpha - 1)T/(\nu + \alpha - 1) = 1/8 \neq 0$ . Since  $|f(t, x) - f(t, y)| = t^2 |\tan^{-1} x - \tan^{-1} y| \leq t^2 |x - y|$ , then  $(H_2)$  is satisfied with  $m(t) = t^2 \in L^{12}([0, 1/2], \mathbb{R}^+)$ . We can show that

$$\|m\| = \left( \int_0^{\frac{1}{2}} |s^2|^{12} ds \right)^{\frac{1}{12}} \approx 0.1804509$$

and

$$\begin{aligned}
 &\frac{\|m\|}{\Gamma(\alpha)} \left\{ T^{\alpha-\gamma} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{\eta^{2-\gamma} T^{\alpha-1} [\nu + 2(\alpha - 1)]}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} \right. \\
 &\quad \left. + \frac{T^{\alpha-\gamma} (\eta + T)(\alpha - 1)}{(\nu + \alpha - 1)|\Omega|} \left( \frac{1-\gamma}{\nu + \alpha - \gamma} \right)^{1-\gamma} \right\} \approx 0.2826838 < 1.
 \end{aligned}$$

Hence, by Theorem 3.2, boundary value problem (4.3)-(4.4) has a unique solution on  $[0, 1/2]$ .

**Example 4.3** Consider the following fractional integral boundary value problem:

$$D^{\frac{9}{5}}x(t) = \frac{t}{10\pi} \sin(\pi x) + \frac{(t+1)x^2}{1+x^2}, \quad t \in \left(0, \frac{3}{4}\right), \tag{4.5}$$

$$x\left(\frac{1}{2}\right) = 0, \quad I^{\frac{4}{5}}x\left(\frac{3}{4}\right) = 0. \tag{4.6}$$

Set  $\alpha = 9/5$ ,  $\nu = 4/5$ ,  $\eta = 1/2$ ,  $T = 3/4$ ,  $f(t, x) = (t/10\pi) \sin(\pi x) + ((t+1)x^2/(1+x^2))$ . It is easy to see that  $\Omega = \eta - (\alpha - 1)T/(\nu + \alpha - 1) = 1/8 \neq 0$ . Clearly,

$$|f(t, x)| = \left| \frac{t}{10\pi} \sin(\pi x) + \frac{(t+1)x^2}{1+x^2} \right| \leq (t+1) \left( \frac{\|x\|}{10} + 1 \right).$$

Choosing  $p(t) = t + 1$ ,  $\psi(\|x\|) = (\|x\|/10) + 1$ , we obtain

$$\frac{M}{T^\alpha \|p\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^2[v+2(\alpha-1)]}{(v+\alpha-1)T\Gamma(\alpha+1)|\Omega|} + \frac{T+\eta}{(v+\alpha-1)(v+\alpha)\Gamma(\alpha-1)|\Omega|} \right\}} > \frac{M}{10} + 1,$$

which implies that  $M > 4.495162241$ . Hence, by Theorem 3.5, boundary value problem (4.5)-(4.6) has at least one solution on  $[0, 3/4]$ .

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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