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Generalized statistical convergence of difference sequences

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Abstract

In this paper we define the $\lambda(u)$-statistical convergence that generalizes, in a certain sense, the notion of λ-statistical convergence. We find some relations with sets of sequences which are related to the notion of strong convergence.

MSC: 40A05; 40H05

Keywords: statistical convergence; λ-statistical convergence; difference sequences

1 Introduction and preliminaries

The notion of statistical convergence (see Fast [1]) has been studied in various setups, and its various generalizations, extensions and variants have been studied by various authors so far. For example, lacunary statistical convergence [2], A-statistical convergence [3, 4], statistical summability $(C,1)$ [5, 6], statistical λ-summability [7], statistical σ-convergence [8], statistical A-summability [9], λ-statistical convergence with order α [10], lacunary and λ-statistical convergence in a solid Riesz space [11, 12], lacunary statistical convergence and ideal convergence in random 2-normed spaces [13, 14], generalized weighted statistical convergence [15] *etc.* In this paper we define the notion of λ-statistical convergence as a matrix domain of a difference operator [16], which is obtained by replacing the sequence x by $u\Delta x$, where $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$ and $u = (u_k)_{k=1}^{\infty}$ is another sequence with $u_k \neq 0$ for all k. We find some relations with sets of sequences which are related to the notion of strong convergence [17].

Let K be a subset of the set of natural numbers $\mathbb{N}$. Then the *asymptotic density* of K denoted by $\delta(K)$ is defined as $\delta(K) = \lim_n \frac{1}{n}|\{k \le n : k \in K\}|$, where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \le n : |x_k - L| > \epsilon\}$ has asymptotic density zero, *i.e.*,

$$\lim_n \frac{1}{n}\left|\left\{k \le n : |x_k - L|\right\}\right| = 0.$$

In this case, we write $S\text{-}\lim x = L$.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1, \quad \lambda_1 = 0.$$

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) =: \frac{1}{\lambda_n}\sum_{j\in I_n} x_j,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_j)$ is said to be (V,λ)-*summable* to a number L if

$$t_n(x) \to L \quad \text{as } n \to \infty.$$

In this case, L is called the λ-limit of x.

Let $K \subseteq \mathbb{N}$. Then the λ-*density* of K is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n}\bigl|\{n - \lambda_n + 1 \le j \le n : j \in K\}\bigr|.$$

In case $\lambda_n = n$, λ-density reduces to the asymptotic density. Also, since $(\lambda_n/n) \le 1$, $\delta(K) \le \delta_\lambda(K)$ for every $K \subseteq \mathbb{N}$.

A sequence $x = (x_k)$ is said to be λ-*statistically convergent* (see [12]) to L if for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \ge \epsilon\}$ has λ-density zero, *i.e.*, $\delta_\lambda(K_\epsilon) = 0$. That is,

$$\lim_n \frac{1}{\lambda_n}\bigl|\bigl\{k \in I_n : |x_k - L| \ge \epsilon\bigr\}\bigr| = 0.$$

In this case, we write S_λ- $\lim x = L$ and we denote the set of all λ-statistically convergent sequences by S_λ.

2 $\lambda(u)$-Statistical convergence

We consider the infinite matrix of first difference $\Delta = (a_{nm})_{n,m\ge 1}$ defined by $a_{nn} = 1$, $a_{n,n+1} = -1$ and $a_{nm} = 0$ otherwise. Let D_u be the diagonal matrix defined by $[D_u]_{nn} = u_n$ for all n and consider the set U of all sequences such that $u_n \ne 0$ for all n. Then we write $\Delta(u) = D_u\Delta$ for $u \in U$.

From the generalized de la Vallée-Poussin mean defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k \quad \text{for } x = (x_k)_k,$$

we are led to define the following sets:

$$\begin{aligned}
[V,\lambda]_0\bigl(\Delta(u)\bigr) &= \Biggl\{x = (x_k) : \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\bigl|\Delta(u)x_k\bigr| = 0\Biggr\} \\
&= \Biggl\{x = (x_k) : \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\in I_n}\bigl|u_k(x_k - x_{k+1})\bigr| = 0\Biggr\}, \\
[V,\lambda]_\infty\bigl(\Delta(u)\bigr) &= \Biggl\{x = (x_k) : \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}\bigl|\Delta(u)x_k\bigr| < \infty\Biggr\} \\
&= \Biggl\{x = (x_k) : \sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}\bigl|u_k(x_k - x_{k+1})\bigr| = 0\Biggr\}.
\end{aligned}$$

In the case when $\lambda_n = n$, we write the previous sets $[V]_0(\Delta(u))$ and $[V]_\infty(\Delta(u))$, respectively. Now we can state the definition of $\lambda(u)$-statistical convergence to zero.

A sequence $x = (x_k)_{k\geq 1}$ is said to be $\lambda(u)$-*statistically convergent* to zero if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \Delta(u)x_k \right| \geq \varepsilon \right\} \right| = 0.$$

In this case, we write $x_k \to 0S_\lambda(\Delta(u))$. If $\lambda_n = n$ for all n, we then write $x_k \to 0S(\Delta(u))$.

3 Main results

We are ready to prove the following result.

Theorem 1 *Let $u \in U$. Then*

(a) $[V,\lambda]_0(\Delta(u)) \subset S^0_\lambda(\Delta(u))$ *and the inclusion is proper,*
(b) *if $x \in l_\infty$ and $x_k \to 0S_\lambda(\Delta(u))$, then $x \in [V,\lambda]_0(\Delta(u))$,*
(c) $S^0_\lambda(\Delta(u)) \cap l_\infty = [V,\lambda]_0(\Delta(u)) \cap l_\infty$.

Proof (a) Let $\varepsilon > 0$ be given and $x \in [V,\lambda]_0(\Delta(u))$. Then we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left| \Delta(u)x_k \right| \geq \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |x_k - L| \geq \varepsilon}} \left| \Delta(u)x_k \right| \geq \frac{\varepsilon}{\lambda_n} \left| \left\{ k \in I_n : \left| \Delta(u)x_k \right| \geq \varepsilon \right\} \right|.$$

Therefore $x \in S^0_\lambda(\Delta(u))$. The following example shows that the inclusion is proper: Let $x = (x_k)$ be defined by

$$x_k = \begin{cases} \sum_{j=k}^\infty j, & \text{for } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \notin l_\infty$ and for $0 < \varepsilon \leq 1$,

$$\frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \Delta(u)x_k \right| \geq \varepsilon \right\} \right| = \frac{[\sqrt{\lambda_n}]}{\lambda_n} \to 0 \quad (n \to \infty),$$

i.e., $x \in S^0_\lambda(\Delta(u))$. But

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left| \Delta(u)x_k \right| \nrightarrow 0 \quad (n \to \infty),$$

i.e., $x \notin [V,\lambda]_0(\Delta(u))$.

(b) Let $x \in l_\infty$ and $x_k \to 0S_\lambda(\Delta(u))$. Then $|\Delta(u)x_k| \leq M$ for all k, where $M > 0$. For $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k\in I_n} \left| \Delta(u)x_k \right| &= \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |x_k - L| \geq \epsilon}} \left| \Delta(u)x_k \right| + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ |x_k - L| < \epsilon}} \left| \Delta(u)x_k \right| \\ &\leq \frac{M}{\lambda_n} \left| \left\{ k \in I_n : \left| \Delta(u)x_k \right| \geq \varepsilon \right\} \right| + \varepsilon. \end{aligned}$$

Hence, $x \in [V,\lambda]_0(\Delta(u))$.

(c) This immediately follows from (a) and (b).

This completes the proof of the theorem. □

Theorem 2 $S^0(\Delta(u)) \subseteq S^0_\lambda(\Delta(u))$ *if and only if*

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0, \tag{$*$}$$

where by $x \in S^0(\Delta(u))$ (*or* $x \in S^0_\lambda(\Delta(u))$) *we mean* $x_k \to 0S(\Delta(u))$ (*or* $x_k \to 0S_\lambda(\Delta(u))$).

Proof For $\varepsilon > 0$, we have

$$\{k \in I_n : |\Delta(u)x_k| \geq \varepsilon\} \subset \{k \leq n : |\Delta(u)x_k| \geq \varepsilon\}.$$

Therefore

$$\begin{aligned} \frac{1}{n}\bigl|\{k \leq n : |\Delta(u)x_k| \geq \varepsilon\}\bigr| &\geq \frac{1}{n}\bigl|\{k \in I_n : |\Delta(u)x_k| \geq \varepsilon\}\bigr| \\ &\geq \frac{\lambda_n}{n} \cdot \frac{1}{\lambda_n}\bigl|\{k \in I_n : |\Delta(u)x_k| \geq \varepsilon\}\bigr|. \end{aligned}$$

Taking the limit as $n \to \infty$ and using $(*)$, we get the inclusion.

Conversely, suppose that

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} = 0.$$

Choose a subsequence $(n(j))_{j\geq 1}$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$. Define a sequence $x = (x_k)_{k\geq 1}$ such that

$$\Delta x_k = \begin{cases} 1, & \text{for } k \in I_{n(j)}, j = 1, 2, 3, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Delta x \in [C,1]$ and hence, by Theorem 2.1 of [18], $x \in S^0(\Delta(u))$. On the other hand, $x \notin [V,\lambda]_0(\Delta(u))$ and Theorem 1(b) implies that $x \notin S^0_\lambda(\Delta(u))$. Hence, $(*)$ is necessary.

This completes the proof of the theorem. □

Presently, for the reverse inclusion, we have only one way condition.

Theorem 3 *If* $\limsup_n (n - \lambda_n) < \infty$, *then* $S^0_\lambda(\Delta(u)) \subseteq S^0(\Delta(u))$.

Proof Let $\limsup_n (n - \lambda_n) < \infty$. Then there exists $M > 0$ such that $n - \lambda_n \leq M$ for all n. Since $\frac{1}{n} \leq \frac{1}{\lambda_n}$ and $\{1 \leq k \leq n : |\Delta(u)x_k| \geq \varepsilon\} \subseteq \{k \in I_n : |\Delta(u)x_k| \geq \varepsilon\} \cup \{1 \leq k \leq n - \lambda_n : |\Delta(u)x_k| \geq \varepsilon\}$, we have

$$\begin{aligned} &\frac{1}{n}\bigl|\{1 \leq k \leq n : |\Delta(u)x_k| \geq \varepsilon\}\bigr| \\ &\quad\leq \frac{1}{\lambda_n}\bigl|\{1 \leq k \leq n : |\Delta(u)x_k| \geq \varepsilon\}\bigr| \end{aligned}$$

$$\leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\Delta(u)x_k\right| \geq \varepsilon\right\}\right| + \frac{1}{\lambda_n}\left|\left\{k \leq n - \lambda_n : \left|\Delta(u)x_k\right| \geq \varepsilon\right\}\right|$$
$$\leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\Delta(u)x_k\right| \geq \varepsilon\right\}\right| + \frac{M}{\lambda_n}.$$

Now, taking the limit as $n \to \infty$, we get $S_\lambda^0(\Delta(u)) \subseteq S^0(\Delta(u))$.

This completes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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