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Some results for high-order generalized neutral differential equation

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Abstract

In this paper, we consider the following high-order p-Laplacian generalized neutral differential equation

$$(\varphi_p(x(t)-cx(t-\delta(t)))')^{(n-1)}+g(t,x(t),x(t-\tau(t)),x'(t))=e(t),$$

where $p\geq 2$, $\varphi_p(x)=|x|^{p-2}x$ for $x\neq 0$ and $\varphi_p(0)=0$; $g:\mathbb{R}^4\to\mathbb{R}$ is a continuous periodic function with $g(t+T,\cdot,\cdot,\cdot)\equiv g(t,\cdot,\cdot,\cdot)$, and $g(t,a,a,0)-e(t)\not\equiv 0$ for all $a\in\mathbb{R}$. $e:\mathbb{R}\to\mathbb{R}$ is a continuous periodic function with $e(t+T)\equiv e(t)$ and $\int_0^T e(t)\,dt=0$, c is a constant and $|c|\neq 1$, $\delta\in C^1(\mathbb{R},\mathbb{R})$ and δ is a T-periodic function, T is a positive constant; n is a positive integer. By applications of coincidence degree theory and some analysis skills, sufficient conditions for the existence of periodic solutions are established.

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Keywords: periodic solution; high-order; generalized neutral differential equation

1 Introduction

In recent years, there has been a good amount of work on periodic solutions for neutral differential equations (see [1–9] and the references cited therein). For example, in [1], Cao and He investigated a class of high-order neutral differential equations

$$\bigl(x^{(p)}(t)+b_px^{(p)}(t-h_p)\bigr)+\sum_{i=0}^{p-1}\bigl(a_ix^{(i)}+b_ix^{(i)}(t-h_i)\bigr)=f(t). \tag{1.1}$$

By using the Fourier series method and inequality technique, they obtained the existence of a periodic solution for (1.1). In [8], applying Mawhin's continuation theorem, Wang and Lu studied the existence of a periodic solution for a high-order neutral functional differential equation with distributed delay as follows:

$$\bigl(x(t)-cx(t-\sigma)\bigr)^{(n)}+f\bigl(x(t)\bigr)x'(t)+g\left(\int_{-r}^0 x(t+s)\,d\alpha(s)\right)=p(t), \tag{1.2}$$

here $|c|\neq 1$. Recently, in [5] and [6], Ren *et al.* observed the high-order p-Laplacian neutral differential equation

$$\bigl(\varphi_p\bigl(x(t)-cx(t-\sigma)\bigr)^{(l)}\bigr)^{(n-l)}=F\bigl(t,x(t),x'(t),\dots,x^{(l-1)}(t)\bigr) \tag{1.3}$$

and presented sufficient conditions for the existence of periodic solutions for (1.3) in the critical case (*i.e.*, $|c| = 1$) and in the general case (*i.e.*, $|c| \neq 1$), respectively.

In this paper, we consider the following high-order p-Laplacian generalized neutral differential equation

$$\bigl(\varphi_p\bigl(x(t) - cx\bigl(t - \delta(t)\bigr)\bigr)'\bigr)^{(n-1)} + g\bigl(t, x(t), x\bigl(t - \tau(t)\bigr), x'(t)\bigr) = e(t), \tag{1.4}$$

where $p \geq 2$, $\varphi_p(x) = |x|^{p-2}x$ for $x \neq 0$ and $\varphi_p(0) = 0$; $g : \mathbb{R}^4 \to \mathbb{R}$ is a continuous periodic function with $g(t + T, \cdot, \cdot, \cdot) \equiv g(t, \cdot, \cdot, \cdot)$, and $g(t, a, a, 0) - e(t) \not\equiv 0$ for all $a \in \mathbb{R}$. $e : \mathbb{R} \to \mathbb{R}$ is a continuous periodic function with $e(t + T) \equiv e(t)$ and $\int_0^T e(t)\,dt = 0$, c is a constant and $|c| \neq 1$, $\delta \in C^1(\mathbb{R}, \mathbb{R})$ and δ is a T-periodic function, T is a positive constant; n is a positive integer.

In (1.4), the neutral operator $A = x(t) - cx(t - \delta(t))$ is a natural generalization of the operator $A_1 = x(t) - cx(t - \delta)$, which typically possesses a more complicated nonlinearity than A_1. For example, A_1 is homogeneous in the following sense $(A_1x)'(t) = (A_1x')(t)$, whereas A in general is inhomogeneous. As a consequence, many of the new results for differential equations with the neutral operator A will not be a direct extension of known theorems for neutral differential equations.

The paper is organized as follows. In Section 2, we first give qualitative properties of the neutral operator A which will be helpful for further studies of differential equations with this neutral operator; in Section 3, by applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions for (1.4), an example is also given to illustrate our results.

2 Lemmas

Let $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + T) \equiv \phi(t)\}$ with the norm $|\phi|_\infty = \max_{t\in[0,T]} |\phi(t)|$. Define difference operators A and B as follows:

$$A : C_T \to C_T, \quad (Ax)(t) = x(t) - cx\bigl(t - \delta(t)\bigr); \qquad B : C_T \to C_T,$$
$$(Bx)(t) = c\bigl(t - \delta(t)\bigr).$$

Lemma 2.1 (see [10]) *If $|c| \neq 1$, then the operator A has a continuous inverse A^{-1} on C_T, satisfying*

$$(1) \quad \bigl(A^{-1}f\bigr)(t) = \begin{cases} f(t) + \sum_{j=1}^{\infty} c^j f(s - \sum_{i=1}^{j-1}\delta(D_i)) & \text{for } |c| < 1, \forall f \in C_T, \\ -\frac{f(t+\delta(t))}{c} - \sum_{j=1}^{\infty} \frac{1}{c^{j+1}} f(s + \delta(t) + \sum_{i=1}^{j-1}\delta(D_i)) & \text{for } |c| > 1, \forall f \in C_T. \end{cases}$$

$$(2) \quad \bigl|\bigl(A^{-1}f\bigr)(t)\bigr| \leq \frac{\|f\|}{|1 - |c||}, \quad \forall f \in C_T.$$

$$(3) \quad \int_0^T \bigl|\bigl(A^{-1}f\bigr)(t)\bigr|\,dt \leq \frac{1}{|1 - |c||} \int_0^T \bigl|f(t)\bigr|\,dt, \quad \forall f \in C_T.$$

Let X and Y be real Banach spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1 of X, Y respectively such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$

denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$ and so the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 (Gaines and Mawhin [11]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;

(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.3 (see [12]) *If $x \in C^n(\mathbb{R}, \mathbb{R})$ and $x(t + T) \equiv x(t)$, then*

$$\int_0^T |x'(t)|^p \, dt \le \left(\frac{T}{\pi_p}\right)^{p(n-1)} \int_0^T |x^{(n)}(t)|^p \, dt, \tag{2.1}$$

where $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$, *and p is a fixed real number with $p > 1$.*

Remark 2.1 When $p = 2$, $\pi_2 = 2\int_0^{(2-1)/2} \frac{ds}{(1-\frac{s^2}{2-1})^{1/2}} = \frac{2\pi(2-1)^{1/2}}{2\sin(\pi/2)} = \pi$, then (2.1) is transformed into $\int_0^T |x'(t)|^2 \, dt \le (\frac{T}{\pi})^{2(n-1)} \int_0^T |x^{(n)}(t)|^2 \, dt$.

In order to apply Mawhin's continuation degree theorem, we rewrite (1.4) in the form

$$\begin{cases} (Ax_1)'(t) = \varphi_q(x_2(t)) \\ x_2^{(n-1)}(t) = -g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) + e(t), \end{cases} \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.2), then $x_1(t)$ must be a T-periodic solution to (1.4). Thus, the problem of finding a T-periodic solution for (1.4) reduces to finding one for (2.2).

Now, set $X = \{x = (x_1(t), x_2(t)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$ with the norm $|x|_\infty = \max\{|x_1|_\infty, |x_2|_\infty\}$; $Y = \{x = (x_1(t), x_2(t)) \in C^1(\mathbb{R}, \mathbb{R}^2) : x(t + T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$. Clearly, X and Y are both Banach spaces. Meanwhile, define

$$L : D(L) = \{x \in C^n(\mathbb{R}, \mathbb{R}^2) : x(t + T) = x(t), t \in \mathbb{R}\} \subset X \to Y$$

by

$$(Lx)(t) = \begin{pmatrix} (Ax_1)'(t) \\ x_2^{(n-1)}(t) \end{pmatrix}$$

and $N : X \to Y$ by

$$(Nx)(t) = \begin{pmatrix} \varphi_q(x_2(t)) \\ -g(t, x_1(t), x_1(t-\tau(t)), x_1'(t)) + e(t) \end{pmatrix}. \tag{2.3}$$

Then (2.2) can be converted into the abstract equation $Lx = Nx$. From the definition of L, one can easily see that

$$\operatorname{Ker} L \cong \mathbb{R}^2, \qquad \operatorname{Im} L = \left\{ y \in Y : \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

So, L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \begin{pmatrix} (Ax_1)(0) \\ x_2(0) \end{pmatrix}; \qquad Qy = \frac{1}{T} \int_0^T \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} ds,$$

then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$. Setting $L_P = L|_{D(L)\cap \operatorname{Ker} P}$ and $L_P^{-1} : \operatorname{Im} L \to D(L)$ denotes the inverse of L_P, then

$$\begin{aligned} \left[L_P^{-1} y\right](t) &= \begin{pmatrix} (A^{-1}Gy_1)(t) \\ (Gy_2)(t) \end{pmatrix}, \\ [Gy_1](t) &= \int_0^t y_1(s)\, ds, \\ [Gy_2](t) &= \sum_{j=1}^{n-2} \frac{1}{j!} x_2^{(j)}(0) t^j + \frac{1}{(n-2)!} \int_0^t (t-s)^{n-2} y_2(s)\, ds, \end{aligned} \tag{2.4}$$

where $x_2^{(j)}(0)$ $(j = 1, 2, \ldots, n-2)$ are defined by the following

$$\begin{aligned} B &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & 1 & 0 & \cdots & 0 & 0 \\ b_2 & b_1 & 1 & \cdots & 0 & 0 \\ \cdots & & & & & \\ b_{n-3} & b_{n-4} & b_{n-5} & \cdots & 1 & 0 \\ b_{n-2} & b_{n-3} & b_{n-4} & \cdots & b_1 & 0 \end{pmatrix}_{(n-2)\times(n-2)}, \\ X^\top &= \left(x^{(n-2)}(0), \ldots, x''(0), x'(0)\right), \\ C^\top &= (C_1, C_2, \ldots, C_{n-2}), \\ C_j &= -\frac{1}{j!T} \int_0^T (T-s)^j y_2(s)\, ds, \\ b_k &= \frac{T^k}{(k+1)!}, \quad k = 1, 2, \ldots, n-3. \end{aligned}$$

From (2.3) and (2.4), it is clear that QN and $K(I-Q)N$ are continuous, $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$ which means N is L-compact on $\bar{\Omega}$.

3 Existence of periodic solutions for (1.4)

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) There exists a constant $D > 0$ such that

$$v_1 g(t, v_1, v_2, v_3) > 0 \quad \forall (t, v_1, v_2, v_3) \in [0, T] \times \mathbb{R}^3 \text{ with } |v_1| > D;$$

(H_2) There exists a constant $D_1 > 0$ such that

$$v_1 g(t, v_1, v_2, v_3) < 0 \quad \forall (t, v_1, v_2, v_3) \in [0, T] \times \mathbb{R}^3 \text{ with } |v_1| > D_1;$$

(H_3) There exist non-negative constants α_1, α_2, α_3, m such that

$$\left|g(t, v_1, v_2, v_3)\right| \le \alpha_1 |v_1|^{p-1} + \alpha_2 |v_2|^{p-1} + \alpha_3 |v_3|^{p-1} + m \quad \forall (t, v_1, v_2, v_3) \in [0, T] \times \mathbb{R}^3;$$

(H_4) There exist non-negative constants γ_1, γ_2, γ_3 such that

$$\left|g(t, u_1, u_2, u_3) - g(t, v_1, v_2, v_3)\right| \le \gamma_1 |u_1 - v_1| + \gamma_2 |u_2 - v_2| + \gamma_3 |u_3 - v_3|$$

for all $(t, u_1, u_2, u_3), (t, v_1, v_2, v_3) \in [0, T] \times \mathbb{R}^3$.

Theorem 3.1 *Assume that* (H_1) *and* (H_3) *hold, then* (1.4) *has at least one non-constant T-periodic solution if* $|1 - |c|| - |c|\delta_1 > 0$ *and* $\frac{[(\alpha_1+\alpha_2)T^{p+1}+2^{p-1}\alpha_3 T^2]}{2^{p+1}(|1-|c||-|c|\delta_1)^{p-1}} \cdot (\frac{T}{\pi})^{2(n-3)} < 1$, *here* $\delta_1 = \max_{t\in[0,T]} |\delta'(t)|$.

Proof Consider the equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1).$$

Set $\Omega_1 = \{x : Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x(t) = (x_1(t), x_2(t))^\top \in \Omega_1$, then

$$\begin{cases} (Ax_1)'(t) = \lambda \varphi_q(x_2(t)) \\ x_2^{(n-1)}(t) = -\lambda g(t, x_1(t), x_1(t - \tau(t)), x_1'(t)) + \lambda e(t). \end{cases} \tag{3.1}$$

Substituting $x_2(t) = \lambda^{1-p}\varphi_p[(Ax_1)'(t)]$ into the second equation of (3.1), we get

$$\left(\varphi_p(Ax_1)'(t)\right)^{(n-1)} + \lambda^p g\left(t, x_1(t), x_1\left(t - \tau(t)\right), x_1'(t)\right) = \lambda^p e(t). \tag{3.2}$$

Integrating both sides of (3.2) from 0 to T, we have

$$\int_0^T g\left(t, x_1(t), x_1\left(t - \tau(t)\right), x_1'(t)\right) dt = 0. \tag{3.3}$$

From (3.3), there exists a point $\xi \in [0, T]$ such that

$$g\left(\xi, x_1(\xi), x_1\left(\xi - \tau(\xi)\right), x_1'(\xi)\right) = 0.$$

In view of (H_1), we obtain

$$\left|x_1(\xi)\right| \le D.$$

Then we have

$$\left|x_1(t)\right| = \left|x_1(\xi) + \int_\xi^t x_1'(s)\,ds\right| \le D + \int_\xi^t \left|x_1'(s)\right|ds, \quad t \in [\xi, \xi + T],$$

and

$$\begin{aligned}\left|x_1(t)\right| &= \left|x_1(t-T)\right| \\ &= \left|x_1(\xi) - \int_{t-T}^\xi x_1'(s)\,ds\right| \le D + \int_{t-T}^\xi \left|x_1'(s)\right|ds, \quad t \in [\xi, \xi + T].\end{aligned}$$

Combining the above two inequalities, we obtain

$$\begin{aligned}|x_1|_\infty &= \max_{t\in[0,T]}\left|x_1(t)\right| = \max_{t\in[\xi,\xi+T]}\left|x_1(t)\right| \\ &\le \max_{t\in[\xi,\xi+T]}\left\{D + \frac{1}{2}\left(\int_\xi^t \left|x_1'(s)\right|ds + \int_{t-T}^\xi \left|x_1'(s)\right|ds\right)\right\} \\ &\le D + \frac{1}{2}\int_0^T \left|x_1'(s)\right|ds. \qquad (3.4)\end{aligned}$$

Since $(Ax_1)(t) = x_1(t) - cx_1(t - \delta(t))$, we have

$$\begin{aligned}(Ax_1)'(t) &= \bigl(x_1(t) - cx_1\bigl(t-\delta(t)\bigr)\bigr)' \\ &= x_1'(t) - cx_1'\bigl(t-\delta(t)\bigr) + cx_1'\bigl(t-\delta(t)\bigr)\delta'(t) \\ &= \bigl(Ax_1'\bigr)(t) + cx_1'\bigl(t-\delta(t)\bigr)\delta'(t),\end{aligned}$$

and

$$\bigl(Ax_1'\bigr)(t) = (Ax_1)'(t) - cx_1'\bigl(t-\delta(t)\bigr)\delta'(t).$$

By applying Lemma 2.1, we have

$$\begin{aligned}\left|x_1'\right|_\infty &= \max_{t\in[0,T]}\left|A^{-1}Ax_1'(t)\right| \\ &\le \frac{\max_{t\in[0,T]}|(Ax_1)'(t) - cx_1'(t-\delta(t))\delta'(t)|}{|1-|c||} \\ &\le \frac{\varphi_q(|x_2|_\infty) + |c|\delta_1|x_1'|_\infty}{|1-|c||},\end{aligned}$$

where $\delta_1 = \max_{t\in[0,T]}|\delta'(t)|$. Since $|1-|c|| - |c|\delta_1 > 0$, then

$$\left|x_1'\right|_\infty \le \frac{\varphi_q(|x_2|_\infty)}{|1-|c|| - |c|\delta_1}. \qquad (3.5)$$

On the other hand, from $x_2^{(n-3)}(0) = x_2^{(n-3)}(T)$, there exists a point $t_1 \in [0, T]$ such that $x_2^{(n-2)}(t_1) = 0$, which together with the integration of the second equation of (3.1) on interval $[0, T]$ yields

$$\begin{aligned}
2\left|x_2^{(n-2)}(t)\right| &\le 2\left(x_2^{(n-2)}(t_1) + \frac{1}{2}\int_0^T \left|x_2^{(n-1)}(t)\right| dt\right) \\
&= \lambda \int_0^T \left|-g\left(t, x_1(t), x_1\left(t-\tau(t)\right), x_1'(t)\right) + e(t)\right| dt \\
&\le \alpha_1 \int_0^T \left|x_1(t)\right|^{p-1} dt + \alpha_2 \int_0^T \left|x_1\left(t-\tau(t)\right)\right|^{p-1} dt \\
&\quad + \alpha_3 \int_0^T \left|x_1'(t)\right|^{p-1} dt + \left(m + |e|_\infty\right)T \\
&\le (\alpha_1+\alpha_2)T\left(D + \frac{1}{2}\int_0^T \left|x_1'(t)\right| dt\right)^{p-1} \\
&\quad + \alpha_3 \int_0^T \left|x_1'(t)\right|^{p-1} dt + \left(m + |e|_\infty\right)T \\
&= \frac{(\alpha_1+\alpha_2)T}{2^{p-1}}\left(\frac{2D}{\int_0^T |x_1'(t)|\,dt} + 1\right)^{p-1}\left(\int_0^T \left|x_1'(t)\right| dt\right)^{p-1} \\
&\quad + \alpha_3 \int_0^T \left|x_1'(t)\right|^{p-1} dt + \left(m + |e|_\infty\right)T. \qquad (3.6)
\end{aligned}$$

For a given constant $\delta > 0$, which is only dependent on $k > 0$, we have

$$(1+x)^k \le 1 + (1+k)x \quad \text{for } x \in [0, \delta].$$

From (3.5) and (3.6), we have

$$\begin{aligned}
2\left|x_2^{(n-2)}(t)\right| &\le \frac{(\alpha_1+\alpha_2)T}{2^{p-1}}\left(\frac{2D}{\int_0^T |x_1'(t)|\,dt} + 1\right)^{p-1}\left(\int_0^T \left|x_1'(t)\right| dt\right)^{p-1} \\
&\quad + \alpha_3 \int_0^T \left|x_1'(t)\right|^{p-1} dt + \left(m + |e|_\infty\right)T \\
&\le \frac{(\alpha_1+\alpha_2)T}{2^{p-1}}\left(1 + \frac{2Dp}{\int_0^T |x_1'(t)|\,dt}\right)\left(\int_0^T \left|x_1'(t)\right| dt\right)^{p-1} \\
&\quad + \alpha_3 \int_0^T \left|x_1'(t)\right|^{p-1} dt + \left(m + |e|_\infty\right)T \\
&\le \frac{(\alpha_1+\alpha_2)T}{2^{p-1}} \cdot T^{p-1}\left|x_1'\right|_\infty^{p-1} + \frac{(\alpha_1+\alpha_2)TDp}{2^{p-2}} T^{p-2}\left|x_1'\right|_\infty^{p-2} \\
&\quad + \alpha_3\left|x_1'\right|_\infty^{p-1} T + \left(m + |e|_\infty\right)T \\
&\le \left(\frac{(\alpha_1+\alpha_2)T^p}{2^{p-1}} + \alpha_3 T\right)\frac{(\varphi_q|x_2|_\infty)^{p-1}}{(|1-|c|| - |c|\delta_1)^{p-1}} \\
&\quad + \frac{(\alpha_1+\alpha_2)T^{p-1}Dp}{2^{p-2}}\frac{(\varphi_q|x_2|_\infty)^{p-2}}{(|1-|c|| - |c|\delta_1)^{p-2}} \\
&\quad + \left(m + |e|_\infty\right)T \\
&= \left(\frac{(\alpha_1+\alpha_2)T^p}{2^{p-1}} + \alpha_3 T\right)\frac{|x_2|_\infty}{(|1-|c|| - |c|\delta_1)^{p-1}}
\end{aligned}$$

$$+ \frac{(\alpha_1+\alpha_2)T^{p-1}Dp}{2^{p-2}} \frac{|x_2|_\infty^{2-q}}{(|1-|c||-|c|\delta_1)^{p-2}} + \bigl(m+|e|_\infty\bigr)T. \tag{3.7}$$

Since $\int_0^T \varphi_q(x_2(t))\,dt = \int_0^T (Ax_1)'(t)\,dt = 0$, there exists a point $t_2 \in [0,T]$ such that $x_2(t_2) = 0$. From (3.4) and Remark 2.1, we can easily get

$$\begin{aligned}
|x_2|_\infty &\le \frac{1}{2}\int_0^T \bigl|x_2'(t)\bigr|\,dt \le \frac{\sqrt{T}}{2}\left(\int_0^T \bigl|x_2'(t)\bigr|^2\,dt\right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{T}}{2}\left(\frac{T}{\pi}\right)^{2(n-3)}\left(\int_0^T \bigl|x_2^{(n-2)}(t)\bigr|^2\,dt\right)^{\frac{1}{2}} \\
&\le \frac{T}{2}\left(\frac{T}{\pi}\right)^{2(n-3)} \bigl|x_2^{(n-2)}\bigr|_\infty.
\end{aligned} \tag{3.8}$$

Combination of (3.7) and (3.8) implies

$$\begin{aligned}
|x_2|_\infty &\le \frac{T}{2}\left(\frac{T}{\pi}\right)^{2(n-3)} \bigl|x_2^{(n-2)}\bigr|_\infty \\
&\le \frac{T}{2^2}\left(\frac{T}{\pi}\right)^{2(n-3)}\left[\left(\frac{(\alpha_1+\alpha_2)T^p}{2^{p-1}} + \alpha_3 T\right)\frac{|x_2|_\infty}{(|1-|c||-|c|\delta_1)^{p-1}} \right.\\
&\quad \left. + \frac{(\alpha_1+\alpha_2)T^{p-1}Dp}{2^{p-2}}\frac{|x_2|_\infty^{2-q}}{(|1-|c||-|c|\delta_1)^{p-2}}\right] \\
&\quad + \frac{T}{2^2}\left(\frac{T}{\pi}\right)^{2(n-3)}\bigl(m+|e|_\infty\bigr)T.
\end{aligned}$$

Since $p \ge 2$ and $\frac{[(\alpha_1+\alpha_2)T^{p+1}+2^{p-1}\alpha_3 T^2]}{2^{p+1}(|1-|c||-|c|\delta_1)^{p-1}} \cdot (\frac{T}{\pi})^{2(n-3)} < 1$, there exists a positive constant M_1 (independent of λ) such that

$$|x_2|_\infty \le M_1. \tag{3.9}$$

From (3.5) and (3.9), we obtain that

$$\bigl|x_1'\bigr|_\infty \le \frac{\varphi_q(|x_2|_\infty)}{|1-|c||-|c|\delta_1} \le \frac{M_1^{q-1}}{|1-|c||-|c|\delta_1} := M_2.$$

Hence

$$|x_1|_\infty \le D + \frac{1}{2}\int_0^T \bigl|x_1'(t)\bigr|\,dt \le D + \frac{TM_2}{2} := M_3.$$

From (3.6), we know

$$\begin{aligned}
\bigl|x_2^{(n-2)}\bigr|_\infty &\le \frac{1}{2}\max\left|\int_0^T x_2^{(n-1)}(t)\,dt\right| \\
&\le \frac{1}{2}\int_0^T \bigl|g\bigl(t,x_1(t),x_1\bigl(t-\tau(t)\bigr),x_1'(t)\bigr)+e(t)\bigr|\,dt
\end{aligned}$$

$$\leq \frac{1}{2}\left[(\alpha_1+\alpha_2)T|x_1|_\infty^{p-1}+\alpha_3 T\left|x_1'\right|_\infty^{p-1}+\left(m+|e|_\infty\right)T\right]$$
$$\leq \frac{1}{2}\left[(\alpha_1+\alpha_2)TM_3^{p-1}+\alpha_3 TM_2^{p-1}+\left(m+|e|_\infty\right)T\right]:=M_{n-2}.$$

From (3.8), we can get

$$\left|x_2'\right|_\infty \leq \frac{T}{2}\left(\frac{T}{\pi}\right)^{2(n-4)}\left|x_2^{(n-2)}\right|_\infty \leq \frac{T}{2}\left(\frac{T}{\pi}\right)^{2(n-4)}M_{n-2}:=M_4.$$

Let $M=\max\{M_1,M_2,M_3,M_4\}+1$, $\Omega=\{x=(x_1,x_2)^\top : \|x\|<M\}$ and $\Omega_2=\{x : x\in\partial\Omega\cap \operatorname{Ker} L\}$, then $\forall x\in\partial\Omega\cap\operatorname{Ker} L$

$$QNx=\frac{1}{T}\int_0^T\begin{pmatrix}\varphi_q(x_2(t))\\ -g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))+e(t)\end{pmatrix}dt.$$

If $QNx=0$, then $x_2(t)=0$, $x_1=M$ or $-M$. But if $x_1(t)=M$, we know

$$0=\int_0^T g(t,M,M,0)\,dt.$$

From assumption (H_1), we have $M\leq D$, which yields a contradiction. Similarly, in the case $x_1=-M$, we also have $QNx\neq 0$, that is, $\forall x\in\partial\Omega\cap\operatorname{Ker} L$, $x\notin\operatorname{Im} L$. So, conditions (1) and (2) of Lemma 2.2 are both satisfied. Define the isomorphism $J:\operatorname{Im} Q\to\operatorname{Ker} L$ as follows:

$$J(x_1,x_2)^\top=(x_2,-x_1)^\top.$$

Let $H(\mu,x)=-\mu x+(1-\mu)JQNx$, $(\mu,x)\in[0,1]\times\Omega$, then $\forall(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker} L)$,

$$H(\mu,x)=\begin{pmatrix}-\mu x_1(t)-\frac{1-\mu}{T}\int_0^T[g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))-e(t)]\,dt\\ -\mu x_2(t)-(1-\mu)\varphi_q(x_2(t))\end{pmatrix}.$$

We have $\int_0^T e(t)\,dt=0$ and then

$$H(\mu,x)=\begin{pmatrix}-\mu x_1(t)-\frac{1-\mu}{T}\int_0^T[g(t,x_1(t),x_1(t-\tau(t)),x_1'(t))]\,dt\\ -\mu x_2(t)-(1-\mu)\varphi_q(x_2(t))\end{pmatrix},$$
$$\forall(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker} L).$$

From (H_1), it is obvious that $x^\top H(\mu,x)<0$, $\forall(\mu,x)\in(0,1)\times(\partial\Omega\cap\operatorname{Ker} L)$. Hence,

$$\begin{aligned}\deg\{JQN,\Omega\cap\operatorname{Ker} L,0\}&=\deg\{H(0,x),\Omega\cap\operatorname{Ker} L,0\}\\&=\deg\{H(1,x),\Omega\cap\operatorname{Ker} L,0\}\\&=\deg\{I,\Omega\cap\operatorname{Ker} L,0\}\neq 0.\end{aligned}$$

So, condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation $Lx=Nx$ has a solution $x=(x_1,x_2)^\top$ on $\bar{\Omega}\cap D(L)$, *i.e.*, (2.2) has a T-periodic solution $x_1(t)$.

Finally, observe that $y_1^*(t)$ is not a constant. For if $y_1^* \equiv a$ (constant), then from (1.4) we have $g(t,a,a,0) - e(t) \equiv 0$, which contradicts the assumption that $g(t,a,a,0) - e(t) \not\equiv 0$. The proof is complete. □

Similarly, we can get the following result.

Theorem 3.2 *Assume that* (H_2) *and* (H_3) *hold, then* (1.4) *has at least one non-constant T-periodic solution if* $|1-|c|| - |c|\delta_1 > 0$ *and* $\frac{[(\alpha_1+\alpha_2)T^{p+1}+2^{p-1}\alpha_3 T^2]}{2^{p+1}(|1-|c||-|c|\delta_1)^{p-1}} \cdot (\frac{T}{\pi})^{2(n-3)} < 1$.

We illustrate our results with an example.

Example 3.1 Consider the following neutral functional differential equation

$$\left(\varphi_p\left(x(t) - 15x\left(t - \frac{1}{60}\sin 4t\right)\right)'\right)^{(5)} + \frac{1}{3\pi}x^5(t)$$
$$+ \frac{1}{6\pi}\sin x(t-\cos 4t) + \frac{1}{8\pi}\sin 4t \cos x'(t) = \sin 4t. \tag{3.10}$$

Here $p = 6$. It is clear that $T = \frac{\pi}{2}$, $c = 15$, $\delta(t) = \frac{1}{60}\sin 4t$, $\tau(t) = \cos 4t$, $e(t) = \sin 4t$, $\delta_1 = \max_{t\in[0,T]} |\frac{1}{15}\cos 4t| = \frac{1}{15}$, then we can get $|1-|c|| - |c|\delta_1 = 13 > 0$, $g(t,v_1,v_2,v_3) = \frac{1}{3\pi}v_1^5 + \frac{1}{6\pi}\sin v_2 + \frac{1}{8\pi}\cos v_3 \sin 4t$, and $g(t,a,a,0) - e(t) = \frac{1}{3\pi}a^5 + \frac{1}{6\pi}\sin a - \frac{8\pi-1}{8\pi}\sin 4t \not\equiv 0$. Choose $D = 3\pi$ such that (H_1) holds. Now we consider the assumption (H_3), it is easy to see

$$\left|g(t,z_1,z_2,z_3)\right| \le \frac{1}{3\pi}|z_1|^5 + 1,$$

which means that (H_3) holds with $\alpha_1 = \frac{1}{3\pi}$, $\alpha_2 = 0$, $\alpha_3 = 0$, $m = 1$. Obviously,

$$\begin{aligned}
&\frac{[(\alpha_1+\alpha_2)T^{p+1} + 2^{p-1}\alpha_3 T^2]}{2^{p+1}(|1-|c||-|c|\delta_1)^{p-1}} \cdot \left(\frac{T}{\pi}\right)^{2(n-3)} \\
&\quad = \frac{\frac{1}{3\pi}(\frac{\pi}{2})^7 + 0 + 0}{2^{6+1}(|1-|c|-|c|\delta_1)^{6-1}} \cdot \left(\frac{1}{2}\right)^{2(6-3)} \\
&\quad = \frac{\pi^6}{3\times 2^{20}\times 13^5} < 1.
\end{aligned}$$

By Theorem 3.1, (3.10) has at least one nonconstant $\frac{\pi}{2}$-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CZB and RJL worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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