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Weak center problem and bifurcation of critical periods for a Z_4-equivariant cubic system

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Abstract

This paper is devoted to study a center problem and a weak center problem for cubic systems in Z_4-equivariant vector fields. By computing the Lyapunov constants and periodic constants carefully, we show that there exit five weak centers of second order, and center conditions and weak center conditions are given for this system. In terms of the problem of multiple weak centers, there are few results studied and thus our work is new and interesting. At the same time, we investigate the critical periodic bifurcation from a weak center.

Keywords: Z_4-equivariant; Lyapunov constants; center condition; weak center

1 Introduction

For the following planar differential system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

the qualitative analysis has attracted much attention from mathematicians who deal with limit cycles bifurcation problems, center and isochronous center problems, critical periods bifurcations problems *etc.* For center-type planar differential systems, the study about the integrability and order of weak centers is worth investigating, because the order number of a weak center is related to critical period bifurcations. Ref. [1] gave the critical period bifurcations theory and proved that a quadratic system can occur in two critical periods at most. Ref. [1, 2] obtained the result that k critical points can bifurcate from a weak center of order k. Ref. [3] studied weak centers and bifurcations of critical periods for a class of homogeneous nonlinear cubic systems. Ref. [4] studied weak centers and bifurcations of critical periods for a class of Kukles systems. Ref. [5] studied weak centers and bifurcations of critical periods in reversible cubic systems.

For weak center-type planar differential systems, the study of the number of orders for weak center and isochronous center problems is a hot topic, and it is significant to compute Lyapunov constants and periodic constants. If all Lyapunov constants of a singular point vanish, then this singular point is a center. At the same time, it is an isochronous center if all periodic constants of a center vanish, and it is a weak center if there exists a kth-order non-zero periodic constant. For the study on an isochronous center problem, many good results have been obtained, for example, articles [6–10] *etc.* At present, in terms of the study of a weak center problem and an isochronous center problem, much attention has been paid to a sole singular point, and there are less results about multiple centers or multiple weak centers. Based on such considerations, we study the weak center problem for a class of Z_4-equivariant cubic systems, and the investigated system is as follows:

$$\begin{cases} \frac{dx}{dt} = 2B_{21}x - A_{10}y - 2B_{21}x^3 - 3A_{10}x^2y - 4A_{21}x^2y + 2B_{21}xy^2 + A_{10}y^3, \\ \frac{dy}{dt} = A_{10}x + 2B_{21}y - A_{10}x^3 + 2B_{21}x^2y + 3A_{10}xy^2 + 4A_{21}xy^2 - 2B_{21}y^3, \end{cases} \tag{1.2}$$

in which $A_{10}, A_{21}, B_{21} \in R$.

Under transformations of rotation $\widetilde{x} = x\cos\frac{\pi}{2} - y\sin\frac{\pi}{2}$, $\widetilde{y} = x\cos\frac{\pi}{2} + y\sin\frac{\pi}{2}$, the form of system (1.2) is invariant, hence system (1.2) lies in Z_4-equivariant vector field. Obviously, system (1.2) has five elementary singular points, *i.e.*, the origin and $(\pm1, 0)$ and $(0, \pm1)$, in which $(\pm1, 0)$ and $(0, \pm1)$ have the same topological structure.

This paper includes three sections. In Section 2, we give the sufficient and necessary conditions for the origin and $(\pm1, 0)$ and $(0, \pm1)$ to become five centers by analyzing the Lyapunov constants and finding their first integral. In Section 3, we give the method to compute periodic constants and obtain that five elementary singular points of system (1.2) become weak centers of order 2 respectively. It is worth mentioning that because there are few results of multiple singular points becoming weak centers at the same time, our work is interesting. Finally, we introduce our definition of critical periodic bifurcation function and investigate the critical periodic bifurcation from each weak center.

2 Center conditions for five elementary singular points of system (1.2)

System (1.2) lies in Z_4-equivariant vector field, $(\pm1, 0)$ and $(0, \pm1)$ have the same topological structure, so we only need to study the center problem of one singular point when the center problem of $(\pm1, 0)$ and $(0, \pm1)$ is investigated. Without loss of generality, we may as well study the case of $(1, 0)$. Hence, next we will consider the center problem of the origin and $(1, 0)$.

2.1 Method to obtain the center condition

Consider the following real system:

$$\begin{cases} \frac{dx}{dt} = \delta x - y + \sum_{k=2}^{\infty} X_k(x, y), \\ \frac{dy}{dt} = x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y), \end{cases} \tag{2.1}$$

where $X_k(x, y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^\alpha y^\beta$, $Y_k(x, y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^\alpha y^\beta$. Under the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, system (2.1) takes the following form:

$$\frac{dr}{d\theta} = r\frac{\delta + \sum_{k=2}^{\infty} r^{k-1}\varphi_{k+1}(\theta)}{1 + \sum_{k=2}^{\infty} r^{k-1}\psi_{k+1}(\theta)}, \tag{2.2}$$

where

$$\begin{aligned} &\varphi_{k+1}(\theta) = \cos\theta X_k(\cos\theta, \sin\theta) + \sin\theta Y_k(\cos\theta, \sin\theta), \\ &\psi_{k+1}(\theta) = \cos\theta Y_k(\cos\theta, \sin\theta) + \sin\theta X_k(\cos\theta, \sin\theta), \\ &\quad k = 2, 3, \ldots. \end{aligned}$$

For sufficiently small h, let

$$d(h) = r(2\pi, h) - h, \quad r = r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m \tag{2.3}$$

be the Poincaré succession function and a solution of Eq. (2.2) which satisfies the initial-value condition $r|_{\theta=0} = h$. It is evident that

$$v_1(\theta) = e^{\delta\theta} > 0, \qquad v_m(0) = 0, \quad m = 2, 3, \ldots.$$

Definition 2.1 For system (2.1), in the expression (2.3), if $v_1(2\pi) \neq 1$, then the origin is called the rough focus (strong focus); if $v_1(2\pi) = 1$ and $v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0$, $v_{2k+1}(2\pi) \neq 0$, then the origin is called fine focus (weak focus) of order k, and the quantity of $v_{2k+1}(2\pi)$, $k = 1, 2, \ldots$ is called the kth focal value (or Lyapunov constant) at the origin; if $v_1(2\pi) = 1$, and for any positive integer k, $v_{2k+1}(2\pi) = 0$, then the origin is called a center.

Remark 1 In fact, it is impossible to show all the expressions of Lyapunov constants, hence we are only able to find the necessary condition to decide whether a singular point can become a center, *i.e.*, finite Lyapunov constants vanish.

Ref. [11, 12] give the necessary condition to decide whether a singular point can become a center and the method to compute the Lyapunov constant. We may as well introduce this kind of method.

Under transformation

$$z = x + iy, \qquad w = x - iy, \qquad T = it, \quad i = \sqrt{-1},$$

system $(2.1)|_{\delta=0}$ can be transformed into the following complex system:

$$\begin{cases} \frac{dz}{dT} = z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w), \\ \frac{dw}{dT} = -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w), \end{cases} \tag{2.4}$$

where z, w, T are complex variables and

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta}, \qquad W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}.$$

Obviously, the coefficients of (2.4) satisfy the conjugate condition, *i.e.*,

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \quad \alpha \geq 0, \beta \geq 0, \alpha + \beta \geq 2.$$

System $(2.1)|_{\delta=0}$ and system (2.4) are called concomitant systems.

Lemma 2.1 (see [11, 12]) *For system* (2.4), *we can derive successively the terms of the following formal series*:

$$M = 1 + \sum_{\alpha+\beta=1}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta},$$

such that

$$\frac{\partial M}{\partial z}Z - \frac{\partial M}{\partial w}W + \left(\frac{\partial Z}{\partial z} - \frac{\partial W}{\partial w}\right)M = \sum_{m=1}^{\infty}(m+1)\mu_m(zw)^m,$$

where $c_{11} = 1$, $c_{20} = c_{02} = 0$, *for all* $c_{kk} \in R$, $k = 2,3,\ldots$, *and to any integer* m, μ_m *is determined by the following formulas*:

$$c_{1,1} = 1, \qquad c_{2,0} = c_{0,2} = 0,$$
$$\textit{if } (\alpha = \beta = 0 \textit{ and } \beta \neq 1) \textit{ or } \alpha < 0, \textit{ or } \beta < 0, \textit{ then } c_{\alpha,\beta} = 0,$$

otherwise

$$c_{\alpha,\beta} = \frac{1}{\beta - \alpha}\sum_{k+j=3}^{\alpha+\beta+2}\left[(\alpha - k + 1)a_{k,j-1} - (\beta - j + 1)b_{j,k-1}\right]c_{\alpha-k+1,\beta-j+1},$$
$$\mu_m = \sum_{k+j=3}^{2m+4}\left[(m - k + 2)a_{k,j-1} - (m - j + 2)b_{j,k-1}\right]c_{m-k+2,m-j+2}.$$

μ_k *in Lemma* 2.1 *is called kth-order singular point value at the origin of system* (2.4).

Lemma 2.2 (see [11]) *For system* (2.1) *and any positive integer* m, *among* $v_{2m}(2\pi)$, $v_k(2\pi)$ *and* $v_k(\pi)$, *there exists the following relation*:

$$v_{2m}(2\pi) = \frac{1}{1 + v_1(\pi)}\left[\xi_m^{(0)}\big(v_1(2\pi) - 1\big) + \sum_{k=1}^{m-1}\xi_m^{(k)}v_{2k+1}(2\pi)\right],$$

where $\xi_m^{(k)}$ *are all polynomials of* $v_1(\pi), v_2(\pi), \ldots, v_m(\pi)$ *and* $v_1(2\pi), v_2(2\pi), \ldots, v_m(2\pi)$ *with rational coefficients*.

Obviously, we can imply that $v_{2m}(2\pi) = 0$ when $v_1(2\pi) = 1$, $v_{2k+1}(2\pi) = 0$, $k = 1,2,\ldots, m-1$.

Lemma 2.3 (see [11]) *For system* $(2.1)|_{\delta=0}$, (2.4) *and any positive integer* m, *the following assertion holds*:

$$v_{2k+1}(2\pi) = i\pi\left(\mu_m + \sum_{k=1}^{m-1}\xi_m^{(k)}\mu_k\right),$$

where $\xi_m^{(k)}$ $(k = 1,2,\ldots,m-1)$, *are polynomial functions of the coefficients of system* (2.4).

According to Lemma 2.2 and Lemma 2.3, we have the following result.

Lemma 2.4 *For system* $(2.1)|_{\delta=0}$ *and* (2.4), *the following relation holds*:

$$v_{2m+1}(2\pi) = i\pi\,\mu_m$$

when $\mu_k = 0$, $k = 1,2,\ldots,m-1$.

From the above analysis, we have the following theorem.

Theorem 2.5 *If the origin of system* $(2.1)|_{\delta=0}$ *is a center, then we obtain* $\mu_k = 0$, $k = 1, 2, \ldots$.

If we want to obtain the sufficient condition that the origin of system $(2.1)|_{\delta=0}$ is a center, then it is valid to find the first integral or integral factors of system $(2.1)|_{\delta=0}$.

2.2 The center condition of four symmetrical singular points and the origin for system (1.2)

At first, we investigate the center problem of four symmetrical singular points. Because system (1.2) lies in Z_4-equivariant vector field, its four symmetrical singular points have the same bifurcation behavior. Without loss of generality, we only need to consider the center problem of $(1, 0)$.

By means of transformation

$$u = x - 1, \qquad v = y, \tag{2.5}$$

system (1.2) is changed into the following form:

$$\begin{cases} \frac{du}{dt} = -4B_{21}u - 6B_{21}u^2 - 2B_{21}u^3 - 4(A_{10} + A_{21})v - 2(3A_{10} + 4A_{21})uv \\ \qquad - (3A_{10} + 4A_{21})u^2v + 2B_{21}v^2 + 2B_{21}uv^2 + A_{10}v^3, \\ \frac{dv}{dt} = -2A_{10}u - 3A_{10}u^2 - A_{10}u^3 + 4B_{21}v + 4B_{21}uv + 2B_{21}u^2v + (3A_{10} + 4A_{21})v^2 \\ \qquad + (3A_{10} + 4A_{21})uv^2 - 2B_{21}v^3. \end{cases} \tag{2.6}$$

Clearly, the equilibrium $(1, 0)$ of system (1.2) becomes the origin of (2.6) correspondingly.

Under the transformation

$$\begin{aligned} u &= \frac{2A_{10}B_{21} - \sqrt{2}\sqrt{-A_{10}^2(A_{10}^2 + A_{10}A_{21} + 2B_{21}^2)}}{A_{10}^2}\xi \\ &\quad + \frac{2A_{10}B_{21} + \sqrt{2}\sqrt{-A_{10}^2(A_{10}^2 + A_{10}A_{21} + 2B_{21}^2)}}{A_{10}^2}\eta, \\ v &= \xi + \eta, \qquad d\tau = \frac{-2\sqrt{2}A_{10}\sqrt{-(A_{10}^2 + A_{10}A_{21} + 2B_{21}^2)}}{|A_{10}|}\,dt, \end{aligned} \tag{2.7}$$

system (2.7) becomes

$$\begin{cases} \frac{d\xi}{d\tau} = -\eta + h.o.t, \\ \frac{d\eta}{d\tau} = \xi + h.o.t. \end{cases} \tag{2.8}$$

Clearly, system (2.8) belongs to the type of $(2.1)|_\delta = 0$. According to the method in the article [11, 12], to compute the Liyapunov constants (or focal values) of the origin of system (2.8), we obtain the Lyapunov constants (or focal values) of the origin of (2.8) (namely the focal values of the equilibrium $(1, 0)$ of model (1.2)) as follows.

Under the following transformation

$$z = \xi + i\eta, \qquad w = \xi - i\eta, \qquad T = i\tau, \quad i = \sqrt{-1},$$

system (2.8) can be transformed into the following complex system:

$$\begin{cases} \frac{dz}{dT} = z + \sum_{k=2}^{3} Z_k(z,w), \\ \frac{dw}{dT} = -w - \sum_{k=2}^{3} W_k(z,w). \end{cases} \tag{2.9}$$

According to the formulas of Lemma 2.1, the singular values of the origin of system (2.9) are obtained by computing carefully.

Theorem 2.6 *The first singular values of the origin of system* (2.9) *are as follows*:

$$\mu_1 = -\frac{4B_{21}i\sqrt{2}[(A_{10}+A_{21})^2+B_{21}^2]\sqrt{-A_{10}^2(A_{10}^2+A_{10}A_{21}+2B_{21}^2)}}{A_{10}(A_{10}^2+A_{10}A_{21}+2B_{21}^2)^2}.$$

From Theorem 2.6, we have the following result.

Theorem 2.7 *The first singular values of the origin of system* (2.9) *vanish if and only if* $B_{21} = 0$.

Obviously, Theorem 2.7 gives the necessary condition that the origin of system (2.6) or (2.8) becomes a center. Moreover, we can obtain that it is also a sufficient condition, namely the following theorem.

Theorem 2.8 *The origin of system* (2.6) *or* (2.8) *is a center if and only if* $B_{21} = 0$.

Proof From Theorem 2.7, $B_{21} = 0$ is the necessary condition for the origin of system (2.6) or (2.8) to become a center. Next we prove it is also a sufficient condition. If $B_{21} = 0$ holds, then system (2.6) becomes

$$\begin{cases} \frac{du}{dt} = -4(A_{10}+A_{21})v - 2(3A_{10}+4A_{21})uv - (3A_{10}+4A_{21})u^2v + A_{10}v^3, \\ \frac{dv}{dt} = -2A_{10}u - 3A_{10}u^2 - A_{10}u^3 + (3A_{10}+4A_{21})v^2 + (3A_{10}+4A_{21})uv^2. \end{cases} \tag{2.10}$$

System (2.10) has the first integral as follows:

$$\begin{aligned} f = {} & 2(A_{10}+A_{21})^5(3A_{10}+4A_{21})u^2v^2 - 4(A_{10}+A_{21})^5u\big(A_{10}u^2 - 3A_{10}v^2 - 4A_{21}v^2\big) \\ & - 4(A_{10}+A_{21})^5\big(A_{10}u^2 - 2A_{10}v^2 - 2A_{21}v^2\big) - A_{10}(A_{10}+A_{21})^5\big(u^4+v^4\big). \end{aligned}$$

Hence, the origin of system (2.6) or (2.8) is a center if and only if $B_{21} = 0$. The proof is completed. □

Obviously, $B_{21} = 0$ is the sufficient and necessary condition for the four symmetrical singular points $(\pm1, 0)$, $(0, \pm1)$ of system (1.2) to become four centers at the same time. Moreover, we can deduce that the origin of system (1.2) is a center if $B_{21} = 0$.

Theorem 2.9 *The origin of system* (1.2) *is a center if* $B_{21} = 0$.

Proof If $B_{21} = 0$, system (1.2) has the first integral as follows:

$$m = 2A_{10}x^2 - A_{10}x^4 + 2A_{10}y^2 + 6A_{10}x^2y^2 + 8A_{21}x^2y^2 - A_{10}y^4.$$

Hence, the origin of system (1.2) is a center if $B_{21} = 0$. The proof is completed. □

From the above analysis, it is clear that the following theorem holds.

Theorem 2.10 *The origin and four symmetrical singular points* $(\pm 1, 0)$, $(0, \pm 1)$ *of system* (1.2) *become five centers if and only if* $B_{21} = 0$.

3 Period constant, the order of weak centers and bifurcation of critical periods

After studying the center conditions of system (1.2), ulteriorly we consider whether the five singular points of system (1.2) can be isochronous centers or weak centers, and investigate bifurcation of critical period problem of weak centers. In order to study this class of problems, at first we introduce a kind of methods to compute periodic constants which is necessary for finding the order number of weak centers.

3.1 Method to compute periodic constant and find the order number of weak centers

In Ref. [1], periodic constants and the order number of a weak center are defined as follows.

Definition 3.1 Suppose that the origin of system (2.1) is a center, $T(\rho)$ is the period of periodic trail passing point $(\rho, 0)$ $(\rho \in (0, a))$, then $T(\rho)$ can be expressed as follows:

$$T(\rho) = 2\pi + \sum_{k=1}^{\infty} p_k \rho^k, \tag{3.1}$$

here p_k is called a periodic constant. At the same time, if

$$p_0 = p_1 = p_2 = \cdots = p_{2k-1} = 0, \quad p_{2k} \neq 0, \tag{3.2}$$

then the origin is called a weak center of order k. If $k = \infty$, then the origin is called an isochronous center.

Next we introduce our method to compute the periodic constant. Considering complex analytic system (2.4) and making the transformation

$$z = re^{i\theta}, \qquad w = re^{-i\theta}, \qquad T = it, \tag{3.3}$$

system (2.4) can be transformed into

$$\begin{aligned} \frac{dr}{dt} &= i\frac{wZ - zW}{2r} = ir \sum_{k=1}^{\infty} \frac{wZ_{k+1} - zW_{k+1}}{2zw} \\ &= \frac{ir}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} - b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m, \\ \frac{d\theta}{dt} &= \frac{wZ + zW}{2zw} = 1 + \sum_{k=1}^{\infty} \frac{wZ_{k+1} + zW_{k+1}}{2zw} \\ &= 1 + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} + b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m. \end{aligned} \tag{3.4}$$

According to the relation between system (2.1) and (2.4), we get

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{3.5}$$

For the complex constant h, $|h| \ll 1$, we write the solution of (3.4) associated with the initial condition $r|_{\theta=0} = h$ as

$$r = \tilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta)h^k \tag{3.6}$$

and denote

$$\begin{aligned}\tau(\varphi, h) &= \int_0^{\varphi} \frac{dt}{d\theta}\,d\theta \\ &= \int_0^{\varphi} \left[1 + \frac{1}{2}\sum_{m=1}^{\infty}\sum_{\alpha+\beta=m+2}(a_{\alpha,\beta-1} + b_{\beta,\alpha-1})e^{i(\alpha-\beta)\theta}\tilde{r}^m(\theta, h)\right]^{-1} d\theta.\end{aligned} \tag{3.7}$$

Evidently, if system (2.1) is a real system, $v_{2k+1}(2\pi)$ $(k = 1, 2, \ldots)$ is the kth focal value of the origin.

Definition 3.2 For a sufficiently small complex constant h, the origin of system (2.4) or (3.4) is called a complex center if $\tilde{r}(2\pi, h) \equiv h$. The origin is a complex isochronous center if

$$\tilde{r}(2\pi, h) \equiv h, \qquad \tau(2\pi, h) \equiv 2\pi. \tag{3.8}$$

Example 1 The origin of the system

$$\frac{dz}{dT} = z, \qquad \frac{dw}{dT} = -w$$

is a complex center and a complex isochronous center.

Lemma 3.1 (see [9]) *For system* (2.4), *we can derive uniquely the formal series*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj}z^k w^j, \qquad \eta = w + \sum_{k+j=2}^{\infty} d_{k,j}w^k z^j, \tag{3.9}$$

where $c_{k+1,k} = d_{k+1,k} = 0$, $k = 1, 2, \ldots$, p_k *and* q_k *are polynomials in* $a_{\alpha\beta}$, $b_{\alpha\beta}$ *with rational coefficients, such that*

$$\frac{d\xi}{dT} = \xi + \sum_{j=1}^{\infty} p_j \xi^{j+1}\eta^j, \qquad \frac{d\eta}{dT} = -\eta - \sum_{j=1}^{\infty} q_j \eta^{j+1}\xi^j. \tag{3.10}$$

Let $\mu_0 = \tau_0 = 0$, $\mu_k = p_k - q_k$, $\tau_k = p_k + q_k$, $k = 1, 2, \ldots$. *As in Definition* 3.2, μ_k *is called the kth singular point value of the origin of system* (2.4).

Definition 3.3 For any positive integer k, we say that $\tau(k) = p_k + q_k$ is the kth complex period constant of the origin of system (2.4) or (3.4).

Theorem 3.1 *Suppose that the origin of system* (2.4) *or* (3.4) *is a complex center* (*i.e.*, $\mu_m = 0$, $m = 1, 2, \ldots$). *If there exists a positive integer k such that* $\tau_0 = \tau_1 = \cdots = \tau_{k-1} = 0$, $\tau_k \neq 0$, *then*

$$\tau(2\pi, h) = \pi\left[2 + \tau_k h^{2k} + o\left(h^{2k}\right)\right]. \tag{3.11}$$

Proof By using the transformations

$$\xi = \rho e^{i\varphi}, \qquad \eta = \rho e^{-i\varphi}, \qquad T = it, \tag{3.12}$$

system (3.4) can be reduced to

$$\frac{d\rho}{dt} = \frac{i}{2}\rho \sum_{j=1}^{\infty} \mu_j \rho^{2j}, \qquad \frac{d\varphi}{dt} = 1 + \frac{1}{2}\sum_{j=1}^{\infty} \tau_j \rho^{2j}. \tag{3.13}$$

Under the assumptions that for all $m > 1$, $\mu_m = 0$ and $\tau_0 = \tau_1 = \cdots = \tau_{k-1} = 0$, $\tau_k \neq 0$, system (3.13) becomes

$$\frac{d\rho}{dt} \equiv 0, \qquad \frac{d\varphi}{dt} = 1 + \frac{1}{2}\tau_k \rho^{2k} + o\left(\rho^{2k}\right). \tag{3.14}$$

For a sufficiently small complex constant h, we see from (3.10), (3.12) and (3.14) that $\rho^2 = h^2 + o(h^2)$, so that we have

$$\begin{aligned}
\tau(2\pi, h) &= \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{dt}{d\varphi}\, d\varphi \\
&= \int_0^{2\pi} \left[1 + \frac{1}{2}\tau_k h^{2k} + o\left(h^{2k}\right)\right]^{-1} d\varphi \\
&= \pi\left[2 + \tau_k h^{2k} + o\left(h^{2k}\right)\right].
\end{aligned}$$

□

From Theorem 3.1, we obtain the following.

Theorem 3.2 *If the coefficients of system* (2.4) *satisfy*

$$b_{\alpha\beta} = \bar{a}_{\alpha\beta}, \quad \alpha \geq 0, \beta \geq 0, \alpha + \beta \geq 2,$$

then we have

$$\tau_k = -\frac{1}{\pi}\rho_{2k},$$

where τ_k *is the first nonzero complex period constant and* ρ_{2k} *is the first nonzero period constant of its associated system* (2.4). *For systems* (2.4) *and* (3.4), *the origin is a complex isochronous center if and only if* $\mu_k = \tau_k = 0$, $k = 1, 2, 3, \ldots$.

We now introduce the method of computation of τ_k in Ref. [9].

Lemma 3.2 (see [9]) *For system* (2.4), *we can derive uniquely the formal series*

$$f(z,w) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j, \qquad g(z,w) = w + \sum_{k+j=2}^{\infty} d'_{k,j} w^k z^j, \tag{3.15}$$

where $c'_{k+1,k} = d'_{k+1,k} = 0$, $k = 1, 2, \ldots$, *such that*

$$\frac{df}{dT} = f(z,w) + \sum_{j=1}^{\infty} p'_j z^{j+1} w^j, \qquad \frac{dg}{dT} = -g(z,w) - \sum_{j=1}^{\infty} q'_j w^{j+1} z^j, \tag{3.16}$$

and when $k - j - 1 \neq 0$, c'_{kj} *and* d'_{kj} *are determined by the recursive formulas*

$$c'_{kj} = \frac{1}{j+1-k} \sum_{\alpha+\beta=3}^{k+j+1} \bigl[(k-\alpha+1)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}\bigr] c'_{k-\alpha+1,j-\beta+1},$$
$$d'_{kj} = \frac{1}{j+1-k} \sum_{\alpha+\beta=3}^{k+j+1} \bigl[(k-\alpha+1)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}\bigr] d'_{k-\alpha+1,j-\beta+1}, \tag{3.17}$$

and for any positive integer j, p'_j *and* q'_j *are determined by the recursive formulas*

$$p'_j = \sum_{\alpha+\beta=3}^{2j+2} \bigl[(j-\alpha+2)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}\bigr] c'_{j-\alpha+2,j-\beta+1},$$
$$q'_j = \sum_{\alpha+\beta=3}^{2j+2} \bigl[(j-\alpha+2)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}\bigr] d'_{j-\alpha+2,j-\beta+1}. \tag{3.18}$$

In (3.17) *and* (3.18), *we have taken* $c'_{1,0} = d'_{1,0} = 1$, $c'_{0,1} = d'_{0,1} = 0$, *and if* $\alpha < 0$ *or* $\beta < 0$, *we take* $a_{\alpha\beta} = b_{\alpha\beta} = c'_{\alpha\beta} = d'_{\alpha\beta} = 0$.

We next consider the relations between p_j, q_j and p'_j, q'_j ($j = 1, 2, \ldots$).

Lemma 3.3 (see [9]) *Let* $p_0 = q_0 = p'_0 = q'_0 = 0$. *If there is a positive integer* m *such that*

$$p_0 = q_0 = p_1 = q_1 = \cdots = p_{m-1} = q_{m-1} = 0, \tag{3.19}$$

then

$$p'_0 = q'_0 = p'_1 = q'_1 = \cdots = p'_{m-1} = q'_{m-1} = 0, \qquad p_m = p'_m, \qquad q_m = q'_m, \tag{3.20}$$

and vice versa.

Actually, the above two lemmas give an algorithm to compute τ_m. For any positive integer m, in order to compute τ_m, we only need to carry out the addition, substraction, multiplication and division of the coefficients of system (2.4). The algorithm is recursive. It avoids some complicated integrating operations and solving equations. In addition, it can be easily realized by computer algebra systems.

Notice that the complex period constants are polynomials of the coefficients of system (2.4) or (3.4). By the Hilbert basis theorem, there exists $m \in \mathbf{N}$ such that all $\tau_k = 0$ ($k = 1, 2, \ldots$) if and only if $\tau_1 = \tau_2 = \cdots = \tau_m = 0$. We say that the set $\{\tau_1, \tau_2, \ldots, \tau_m\}$ is a period constant basis of system (2.4) or (3.4).

From Theorem 3.2, the definitions of periodic constants from Definition 3.1 and Definition 3.3 are coincident, and their relation is $\tau_k = p_{2k}$ if $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0$. Hence, we can give the following definition.

Definition 3.4 If $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0$, $\tau_k \neq 0$ of (3.11), then τ_k is called the kth periodic constant and the origin is called a weak center of order k. If $k = \infty$, then the origin is called an isochronous center.

3.2 Weak center problem of system (1.2) and bifurcation of critical periods

According to the formulas of Lemma 3.2, we can find the period constants of four symmetrical singular points and the origin for system (1.2), namely the following several theorems.

Theorem 3.3 *If $B_{21} = 0$, then the first two period constants of four symmetrical singular points of system* (1.2) *are as follows*:

$$\tau_1 = 2(3A_{10} + 4A_{21})/A_{10},$$

and if $A_{21} = -\frac{3}{4}A_{10}$, then

$$\tau_2 = -3.$$

Theorem 3.4 *For system* (1.2), *if $B_{21} = 0$, then the first two period constants of the origin are as follows*:

$$\tau_1 = A_{21}/A_{10},$$

and if $A_{21} = 0$, then

$$\tau_2 = -\frac{3}{4}.$$

From Theorem 3.3 and Theorem 3.4, we have the following theorems.

Theorem 3.5 *Four symmetrical singular points of system* (1.2) (*namely* $(\pm 1, 0)$ *and* $(0, \pm 1)$) *are four weak centers of order* 2 *if $B_{21} = 0$, $A_{21} = -\frac{3}{4}A_{10}$.*

Theorem 3.6 *The origin of system* (1.2) *is a weak center of order* 2 *if $B_{21} = 0$, $A_{21} = 0$.*

From Theorem 3.5 and Theorem 3.6, the following two theorems hold.

Theorem 3.7 *The origin of system* (1.2) *is a weak center of order* 2 *and four symmetrical singular points of system* (1.2) (*namely* $(\pm 1, 0)$ *and* $(0, \pm 1)$) *are four weak centers of order* 1 *if* $B_{21} = 0$, $A_{21} = 0$.

Theorem 3.8 *The origin of system* (1.2) *is a weak center of order* 1 *and four symmetrical singular points of system* (1.2) (*namely* $(\pm 1, 0)$ *and* $(0, \pm 1)$) *are four weak centers of order* 2 *if* $B_{21} = 0$, $A_{21} = -\frac{3}{4}A_{10}$.

After considering the weak center problem, ulteriorly we investigate the bifurcation of critical periods from a weak center.

From Definition 3.1, suppose that the origin of system (2.1) is a center, $T(\rho)$ is the period of periodic trail passing point $(\rho, 0)$ $(\rho \in (0, a))$, then $T(\rho)$ can be expressed as follows:

$$T(\rho) = 2\pi + \sum_{k=1}^{\infty} p_k \rho^k, \tag{3.21}$$

in which the first nonzero p_k is an even number.

According to (3.11) and Theorem 3.2, we can obtain

$$T(\rho) = \pi\left[2 + \tau_k \rho^{2k} + o\left(\rho^{2k}\right)\right], \tag{3.22}$$

in which τ_k is the kth periodic constant.

From (3.22), we have

$$T(\rho) - 2\pi = \pi\left[\tau_k \rho^{2k} + o\left(\rho^{2k}\right)\right]. \tag{3.23}$$

If the origin of system (2.1) is a weak center of order k, let coefficients of system (2.1) disturb with small amplitude, then the periodic function will have some critical zero points, namely some critical periods will occur. At the same time, the number of critical points will decide the number of critical periodic bifurcations. Next we discuss the number of critical points from the periodic function problem.

Let λ be the disturbed coefficients' group of system (2.1), parameter $\varepsilon \to 0$, then

$$T(\lambda, \varepsilon\rho) - 2\pi = \pi\left[\tau_1(\lambda)(\varepsilon\rho)^2 + \tau_2(\lambda)(\varepsilon\rho)^4 + \cdots + \tau_k(\lambda)(\varepsilon\rho)^{2k} + o\left((\varepsilon\rho)^{2k}\right)\right]. \tag{3.24}$$

Note that

$$P(\lambda, \varepsilon\rho) = \tau_1(\lambda)(\varepsilon\rho)^2 + \tau_2(\lambda)(\varepsilon\rho)^4 + \cdots + \tau_k(\lambda)(\varepsilon\rho)^{2k} + o\left((\varepsilon\rho)^{2k}\right). \tag{3.25}$$

Obviously, the number of zero points of $P(\lambda, \varepsilon\rho)$ about ρ is equal to the number of critical points from the periodic function problem. We may as well give the following definition.

Definition 3.5 The function expressed by (3.25) is called a critical periodic function of the origin from system (2.1).

Next we investigate the critical periodic bifurcation problem of system $(1.2)|_{B_{21}=0}$. We have the following two theorems.

Theorem 3.9 *If the four symmetrical singular points of system* (1.2) (*namely* $(\pm 1, 0)$ *and* $(0, \pm 1)$) *are four weak centers of order* 2, *then each one of the four symmetrical singular points of system* $(1.2)|_{B_{21}=0}$ *can bifurcate one critical periodic when the coefficient groups* (A_{21}, A_{10}) *of system* (1.2) *are disturbed via an appropriate way.*

Proof Let coefficient group A_{21}, A_{10} be disturbed in the following way:

$$A_{10} = a + k\varepsilon^2 + o(\varepsilon^2), \qquad A_{21} = -\frac{3}{4}a + \frac{3}{8}(a - 2k)\varepsilon^2 + o(\varepsilon^2), \tag{3.26}$$

in which ε is a small amplitude disturbance, $\varepsilon \to 0$.

Under (3.26), the critical periodic function of each one of the four symmetrical singular points of system $(1.2)|_{B_{21}=0}$ is as follows:

$$P(\lambda, \varepsilon\rho) = 3\varepsilon^4\rho^2(1 - \rho^2) + o((\varepsilon\rho)^4). \tag{3.27}$$

Obviously, equation (3.27) has a positive zero point about ρ near 1.

Hence, each one of the four symmetrical singular points of system $(1.2)|_{B_{21}=0}$ can occur one critical periodic bifurcation if the coefficient groups A_{21}, A_{10} of system (1.2) are disturbed as (3.26). The proof is completed. □

Theorem 3.10 *If the origin of system* (1.2) *is a weak center of order* 2, *then one critical periodic bifurcation can occur at the origin of system* $(1.2)|_{B_{21}=0}$ *if the coefficient groups* A_{21}, A_{10} *are disturbed via an appropriate way.*

Proof Let the coefficient groups A_{21}, A_{10} be disturbed in the following way

$$A_{10} = a + k\varepsilon^2 + o(\varepsilon^2), \qquad A_{21} = \frac{3}{4}a\varepsilon^2 + o(\varepsilon^2), \tag{3.28}$$

in which ε is a small amplitude disturbance, $\varepsilon \to 0$.

Under (3.28), the critical periodic function of the origin of system $(1.2)|_{B_{21}=0}$ is as follows:

$$P(\lambda, \varepsilon\rho) = \frac{3}{4}\varepsilon^4\rho^2(1 - \rho^2) + o((\varepsilon\rho)^4). \tag{3.29}$$

Obviously, equation (3.29) has a positive zero point about ρ near 1.

Hence, one critical periodic bifurcation can occur at the origin of system $(1.2)|_{B_{21}=0}$ if the coefficient groups A_{21}, A_{10} of system (1.2) are disturbed as (3.28). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, CD, YL and CL, contributed to each part of this study equally and read and approved the final version of the manuscript.

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