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Some results about functions that share functions with their derivative of higher order

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Abstract
In this paper, we investigate the growth of some functions that share functions with their derivative of higher order. The first main theorem is an improvement of the result obtained by Lü (Bull. Korean Math. Soc. 48: 951-957, 2011), two examples are given to show that the conclusion is sharp. The second main theorem is of estimating, more exactly, the order of an entire function sharing polynomial, which extends the related result of Lü, Xu and Chen (Arch. Math. 92: 593-601, 2009).
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1 Introduction and main results

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We assume that the reader is familiar with the basic notions of Nevanlinna theory. In the Nevanlinna theory, the order and the hyper-order of a meromorphic function are two important concepts. So, it is meaningful to discuss the properties of the order and the hyper-order for a meromorphic function. Let us recall the definitions of the order and the hyper-order of a meromorphic function f, which are respectively defined as (see [1])

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$
$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

In addition, we say that two meromorphic functions $f(z)$ and $g(z)$ share a finite value a IM (ignoring multiplicities) when $f(z) - a$ and $g(z) - a$ have the same zeros. And we say that $f(z)$ and $g(z)$ share a finite value a CM (counting multiplicities) when $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities. $\deg P(z)$ denotes the degree of the polynomial $P(z)$. If $R(z) = \frac{P_1(z)}{P_2(z)}$ is a rational function (where $P_2(z) \not\equiv 0$ and $P_1(z), P_2(z)$ are two coprime polynomials), then we indicate $\deg R(z) = \max\{\deg P_1(z), \deg P_2(z)\}$ to denote the degree of the rational function.

The subject on sharing values between entire functions and their derivatives was first studied by Rubel and Yang [2]. In 1977, they proved the result that if a nonconstant entire function f and its first derivative f' share two distinct finite numbers a, b CM, then $f \equiv f'$. Since then, shared value problems have been studied by many authors and a number of profound results have been obtained (see, *e.g.*, [3, 4]).

In 1982, Bank and Laine [5] investigated the complex oscillation theory of differential equations and obtained the following main result.

Theorem A *Let $A(z)$ be a nonconstant polynomial of degree n, and let f_1 and f_2 be two linearly independent solutions of the equation $f'' + A(z)f = 0$. Then at least one of f_1 and f_2 has the property that the exponent of convergence of its zero-sequence is $\frac{n+2}{2}$.*

Since then, to study properties of the exponent of convergence, the order and the hyper-order for the solutions of some differential equations becomes a hot topic and is discussed by many experts.

In 2008, Li and Gao [6] deduced the following result.

Theorem B *Let Q_1 and Q_2 be two nonzero polynomials, and let P be a polynomial. If f is a nonconstant solution of the equation*

$$f^{(k)} - Q_1 = e^P(f - Q_2),$$

then $\sigma(f) = n$, where, and in the sequel, n denotes the degree of P.

Recently, Lü [7] obtained the result.

Theorem C *Let f be a transcendental meromorphic function with finitely many poles, let $n \geq 2$ be an integer, and let $\alpha = Pe^Q$ ($\not\equiv \alpha'$) be an entire function such that the order of α is less than that of f, where P, Q are two polynomials. If f^n and $(f^n)'$ share α CM, then $f = Ae^{\frac{1}{n}z}$, where A is a nonzero constant.*

From Theorem C, we see that f^n and $(f^n)'$ share a function with finite order. So, it is natural to ask what will happen if they share functions with infinite order and also what will happen if $(f^n)'$ is replaced by $(f^n)^{(k)}$. In this work, we discuss these problems and derive the following result.

Theorem 1.1 *Let f be a meromorphic function with finitely many poles, let R be a rational function, γ be an entire function. If all zeros of f have multiplicity at least $k + 1$ and*

$$f^{(k)}(z) = \alpha(z) \quad \Rightarrow \quad f(z) = \alpha(z),$$

where $\alpha = Re^{\gamma}$, then $\sigma(f) \leq \sigma(\alpha) = \rho(\gamma)$.

Remark 1 The following examples show that the conclusion $\sigma(f) \leq \rho(\gamma)$ is sharp.

Example 1 Let $f(z) = Ae^z$, where A is a nonzero constant. Let $\alpha(z) = e^{e^z+z^2}$. Noting that $f \equiv f^{(k)}$, we have

$$f^{(k)}(z) = \alpha(z) \quad \Rightarrow \quad f(z) = \alpha(z).$$

Thus, $\sigma(f) = 0 < \sigma(\alpha) = 1$.

Example 2 Let $f(z) = ze^{z^2} + e^{z/2}$, $\alpha(z) = (4z^2 - z + 2)e^{z^2}$, and $\gamma(z) = z^2$. Then

$$\frac{f - \alpha}{f' - \alpha} = 2.$$

Thus, $\sigma(f) = \rho(\gamma) = 0$.

Remark 2 If γ is a polynomial, then the above condition obviously holds.

Remark 3 In Theorem 1.1, if the order of γ is zero, for example, γ is a polynomial, then $\sigma(f) = 0$.

In 2009, Lü, Xu and Chen [8] obtained the following result.

Theorem D *Let $f(z)$ be a nonconstant meromorphic function with finitely many poles, and let Q_1, Q_2 ($\not\equiv Q_1$) be two polynomials. If*

$$f(z) = Q_1(z) \quad\Rightarrow\quad f'(z) = Q_1(z) \quad \text{and} \quad f(z) = Q_2(z) \quad\Rightarrow\quad f'(z) = Q_2(z),$$

then $f(z)$ is of finite order.

In this paper, we get the following results which improve Theorem D.

Theorem 1.2 *Let $f(z)$ be a nonconstant meromorphic function with finitely many poles, and let Q_1, Q_2 ($\not\equiv Q_1$) be two polynomials. If*

$$f(z) = Q_1(z) \quad\Rightarrow\quad f'(z) = Q_1(z) \quad \text{and} \quad f(z) = Q_2(z) \quad\Rightarrow\quad f'(z) = Q_2(z),$$

then $\rho(f) \leq 2 + 2\max\{\deg R_1(z), \deg R_2(z)\}$, where $R_1 = \frac{Q_2 - Q_1'}{Q_2 - Q_1}$, $R_2 = \frac{Q_1 - Q_1'}{Q_2 - Q_1}$ are rational functions.

Corollary 1.3 *Let $f(z)$ be a nonconstant entire function and let Q_1, Q_2 ($\not\equiv Q_1$) be two polynomials. If $f(z) = Q_1(z) \Rightarrow f'(z) = Q_1(z)$ and $f(z) = Q_2(z) \Rightarrow f'(z) = Q_2(z)$, then $\rho(f) \leq 1 + \max\{\deg R_1(z), \deg R_2(z)\}$, where $R_1 = \frac{Q_2 - Q_1'}{Q_2 - Q_1}$, $R_2 = \frac{Q_1 - Q_1'}{Q_2 - Q_1}$ are rational functions.*

Very recently, Li, Gao and Zhang [9] proved the following result.

Theorem E *Let f be a nonconstant entire function. If f and f' share the value 1 CM, and if $N(r, \frac{1}{f'}) < \alpha T(r,f)$, where $\alpha \in [0, \frac{1}{4}]$, then $f' - 1 = c(f - 1)$ for some nonzero constant c.*

By the same method of Li, Gao and Zhang [9], we also consider the kth derivative and improve the above result as follows.

Theorem 1.4 *Let f be a nonconstant entire function, and let k be a positive integer. If f and $f^{(k)}$ share the value 1 CM and if $N(r, \frac{1}{f^{(k)}}) < \alpha T(r,f)$, where $\alpha \in [0, \frac{1}{4})$, then $f^{(k)} - 1 = c(f - 1)$ for some nonzero constant c.*

2 Some lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1 [3] *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disc $\triangle$ with the property that for each $f(z) \in \mathcal{F}$, all zeros of $f(z)$ have multiplicity at least $k+1$. If k is a positive integer and $a_n \to a$, $|a| < 1$ and $f_n^\sharp(a_n) \to \infty$, there exist*

1. *a subsequence of functions $f_n \in \mathcal{F}$ (also denoted by f_n);*
2. *a sequence of complex numbers $z_n \to z_0$, $|z_0| < 1$;*
3. *a positive sequence $\rho_n \to 0$;*
4. *$\frac{f_n(z_n+\rho_n\xi)}{\rho_n^k} = g_n(\xi) \to g(\xi)$, here g is a nonconstant meromorphic (entire) function satisfying $g^\sharp(\xi) \le g^\sharp(0) = k+1$ and*

$$\rho_n \le \frac{M}{\sqrt[k+1]{f_n^\sharp(a_n)}},$$

here M, n are respective positive numbers.

With a similar method to that in [3, Lemma 2], we obtain the following Lemma 2.3, which plays an important part in the proof of Theorem 1.1. For the sake of convenience, the detailed proof will be given after Lemma 2.3.

Lemma 2.2 *Let f be a meromorphic function of hyper-order $\sigma(f) > 0$. Then, for any $\epsilon > 0$, there exists a sequence $z_n \to \infty$ such that $f^\sharp(z_n) > e^{|z_n|^{\sigma(f)-\epsilon}}$ if n is large enough.*

Proof On the contrary, there exist $\epsilon > 0$ and $R > 0$ such that $\epsilon < \sigma(f)$ and for all z, $|z| \ge R$ satisfying $f^\sharp(z) \le e^{|z|^{\sigma(f)-\epsilon}}$. Thus,

$$\begin{aligned} S(r,f) &= \frac{1}{\pi}\iint_{|z|<r} f^\sharp(z)^2\,d\sigma = \frac{1}{\pi}\iint_{R\le|z|<r} f^\sharp(z)^2\,d\sigma + O(1) \\ &\le \frac{1}{\pi}\iint_{R\le|z|<r} e^{2|z|^{\sigma(f)-\epsilon}}\,d\sigma + O(1) = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_R^r e^{2|z|^{\sigma(f)-\epsilon}}\,dt + O(1) \\ &\le 2re^{2r^{\sigma(f)-\epsilon}}\bigl[1+o(1)\bigr]. \end{aligned}$$

By the definition of Ahlfors characteristic of f, we have

$$T(r,f) = \int_0^r \frac{S(t,f)}{t}\,dt \le 2re^{2r^{\sigma(f)-\epsilon}}\bigl[1+o(1)\bigr].$$

Then the hyper-order of f is

$$\sigma(f) = \limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r} \le \limsup_{r\to\infty}\frac{(\sigma(f)-\epsilon)\log r}{\log r} = \sigma(f) - \epsilon,$$

a contradiction. Thus, the proof is completed. □

Lemma 2.3 [9] *Let f be a nonconstant entire function and let $k \ge 1$ be positive. Suppose that f and $f^{(k)}$ share the value 1 CM. Then $f-1$ has infinitely many zeros such that each* *zero of $f-1$ is of order at most k, and*

$$N\left(r,\frac{1}{f-1}\right)\geq\left(\frac{1}{2}-\epsilon\right)T(r,f),\qquad\overline{N}\left(r,\frac{1}{f-1}\right)\geq\left(\frac{1}{k+1}-\epsilon\right)T(r,f),$$

for any $\epsilon>0$ and large enough r, and

$$\delta(1,f)\leq\frac{1}{2},\qquad\Theta(1,f)\leq\frac{k}{1+k}.$$

Remark 4 There exist some mistakes in Theorem 2.2 and its proof and the proof of Theorem 3.1 in [9], we will correct them in another paper.

Lemma 2.4 [1] *Let f be a meromorphic function and $k\geq 1$ be positive. Then*

$$N\left(r,\frac{1}{f^{(k)}}\right)\leq N\left(r,\frac{1}{f}\right)+k\overline{N}(r,f)+S(r,f).$$

Lemma 2.5 [4] *Let $f(z)$ be a meromorphic function in the complex plane, $\rho(f)>2$, then for each $0<\mu<\frac{\rho(f)-2}{2}$, there exist points $a_n\to\infty$ $(n\to\infty)$ such that*

$$\lim_{n\to\infty}\frac{f^{\sharp}(a_n)}{|a_n|^{\mu}}=+\infty.$$

Lemma 2.6 [10] *Let $f(z)$ be an entire function in the complex plane, $\rho(f)>1$, then for each $0<\mu<\rho(f)-1$, there exist points $a_n\to\infty$ $(n\to\infty)$ such that*

$$\lim_{n\to\infty}\frac{f^{\sharp}(a_n)}{|a_n|^{\mu}}=+\infty.$$

3 The proof of Theorem 1.1

The proof of Theorem 1.1 In a similar way to that of [8, 11], we prove Theorem 1.1 as follows.

Noting that f and R have at most finitely many poles, there exists a positive number r such that f and R have no poles in $D=\{z:|z|\geq r\}$. Then f and R are holomorphic in D.

Noting that $\alpha=Re^{\gamma}$, we have $\sigma(\alpha)=\rho(\gamma)$, and then we just need to prove $\sigma(f)\leq\rho(\gamma)$.

On the contrary, suppose that $\sigma(f)=d>c=\rho(\gamma)$. Set $F=\frac{f}{\alpha}$, obviously, $\sigma(F)=\sigma(f)=d$. Then, for $0<\epsilon<\frac{d-c}{2}$, by Lemma 2.2, there exists a sequence $w_n\to\infty$ as $n\to\infty$ such that

$$F^{\sharp}(w_n)>e^{|w_n|^{\sigma(F)-\epsilon}}=e^{|w_n|^{d-\epsilon}}.$$

In view of $w_n\to\infty$ as $n\to\infty$, without loss of generality, we may assume that $|w_n|\geq r+1$ for all n. Define $D_1=\{z:|z|<1\}$ and

$$F_n(z)=F(w_n+z)=\frac{f(w_n+z)}{\alpha(w_n+z)}.\tag{3.1}$$

Then all $F_n(z)$ are holomorphic in D_1.

Thus, we structure a family of holomorphic functions $(F_n)_n$. Moreover,

$$F_n^{\sharp}(0)=F^{\sharp}(w_n)\to\infty\quad\text{as }n\to\infty.$$

It follows from Marty's criterion that $(F_n)_n$ is not normal at $z=0$.

Therefore, applying Lemma 2.1 with $\alpha = k$ and choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n \in D_1$ and $(\rho_n)_n$ such that $\rho_n \to 0$ and

$$g_n(\zeta) = \rho_n^{-k} F_n(z_n + \rho_n\zeta) = \rho_n^{-k} \frac{f(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{3.2}$$

locally uniformly in $\mathbb{C}$, where g is a nonconstant entire function of order at most 1, all zeros of g have multiplicity at least $k + 1$, and

$$\rho_n \le \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \le Me^{-|w_n|^{d-\epsilon}} \tag{3.3}$$

for a positive number M.

We claim that

$$\frac{f^{(l)}(w_n + z_n + \rho_n\zeta)}{\rho_n^{k-l}\alpha(w_n + z_n + \rho_n\zeta)} \to g^{(l)}(\zeta) \quad (0 \le l \le k) \tag{3.4}$$

locally uniformly in $\mathbb{C}$. Obviously, the claim is correct if $l = 0$. Next, we will prove that the claim holds by mathematical induction. We may assume that (3.4) holds for $l = s$, $0 \le s < k$, and then

$$H_n(\zeta) = \frac{f^{(s)}(w_n + z_n + \rho_n\zeta)}{\rho_n^{k-s}\alpha(w_n + z_n + \rho_n\zeta)} \to g^{(s)}(\zeta). \tag{3.5}$$

Now we will prove that (3.4) still holds for $l = s + 1$. Differentiating (3.5), we deduce

$$\frac{f^{(s+1)}(w_n + z_n + \rho_n\zeta)}{\rho_n^{k-s-1}\alpha(w_n + z_n + \rho_n\zeta)} = H_n'(\zeta) + \frac{\rho_n H_n(\zeta)\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)}. \tag{3.6}$$

In view of the definition of order, we have

$$\begin{aligned}\left|\frac{\alpha'}{\alpha}\bigg|_{z=w_n+z_n+\rho_n\zeta}\right| &= \left|\frac{R' + R\gamma'}{R}\bigg|_{z=w_n+z_n+\rho_n\zeta}\right| \\ &\le |w_n|^q M\bigl(|w_n + z_n + \rho_n\zeta|, \gamma'\bigr) \\ &\le |w_n|^q M\bigl(2|w_n|, \gamma'\bigr) \\ &\le |w_n|^q e^{A|w_n|^{c+\epsilon}},\end{aligned} \tag{3.7}$$

where A is a positive constant and q is an integer.

Noting that $0 < \epsilon < \frac{d-c}{2}$, we have $d - \epsilon > c + \epsilon$. Then combining (3.3) and (3.7) yields

$$\begin{aligned}\frac{\rho_n H_n(\zeta)\alpha'(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} &= \frac{\rho_n H_n(\zeta)[R' + R\gamma'](w_n + z_n + \rho_n\zeta)}{R(w_n + z_n + \rho_n\zeta)} \\ &\le M\bigl|g_n(\zeta)\bigr||w_n|^q e^{A|w_n|^{c+\epsilon} - |w_n|^{d-\epsilon}} \to 0\end{aligned} \tag{3.8}$$

as $n \to \infty$. By (3.6) and (3.8), we obtain

$$\frac{f^{(s+1)}(w_n + z_n + \rho_n\zeta)}{\rho_n^{k-s-1}\alpha(w_n + z_n + \rho_n\zeta)} \to g^{(s+1)}(\zeta). \tag{3.9}$$

So, (3.4) holds. Set $l = k$, then

$$\frac{f^{(k)}(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} \to g^{(k)}(\zeta). \tag{3.10}$$

Now, if $g^{(k)}(\zeta) \equiv 1$, then we obtain that $g(\zeta)$ is a polynomial of degree at most k. It contradicts with all zeros of g having multiplicity at least $k+1$. Suppose that $g^{(k)}(\zeta_0) = 1$, then by Hurwitz's theorem there exist ζ_n, $\zeta_n \to \zeta_0$ such that (for sufficiently large n)

$$\frac{f^{(k)}(w_n + z_n + \rho_n\zeta)}{\alpha(w_n + z_n + \rho_n\zeta)} = 1.$$

Noting that $f^{(k)}(z) = \alpha(z) \Rightarrow f(z) = \alpha(z)$ and (3.2), we obtain $g(\zeta_0) = \infty$. It contradicts with the assumption that $g^{(k)}(\zeta_0) = 1$. So, $g^{(k)}(\zeta) \neq 1$.

Next, by the famous Hayman inequality for $g(\zeta)$, it is easy to obtain contradiction.

Hence, we complete the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Let $H = f - Q_1$, then we have

$$H = 0 \quad \Rightarrow \quad H' = Q_1 - Q_1', \qquad H = Q_2 - Q_1 \quad \Rightarrow \quad H' = Q_2 - Q_1'.$$

Let $P = Q_2 - Q_1$ and $P_2 = Q_2 - Q_1'$. Then $P \not\equiv 0$. We consider the function $F = \frac{H}{P}$, obviously, $\rho(F) = \rho(f)$

If $\rho(F) > 2 + 2\max\{\deg(\frac{Q_2 - Q_1'}{Q_2 - Q_1}), \deg(\frac{Q_1 - Q_1'}{Q_2 - Q_1})\}$. By Lemma 2.5, for each $0 < \mu < \frac{\rho(f) - 2}{2}$, there exist $w_n \to \infty$ such that for $n \to \infty$,

$$\lim_{n\to\infty} \frac{F^\sharp(w_n)}{|w_n|^\mu} = +\infty. \tag{4.1}$$

Since P is a polynomial, we know that for any $\epsilon > 0$, there exists an $r_1 > 0$ such that

$$\left|\frac{P'(z)}{P(z)}\right| < \epsilon \quad \text{and} \quad P(z) \neq 0$$

for all $z \in \mathbb{C}$ satisfying $|z| \geq r_1$. Note that $f(z)$ has only finitely many poles, hence $H(z)$ has only finitely many poles. Thus, there is an $r_2 > 0$ such that $H(z)$ is holomorphic in $|z| \geq r_2$. Let $r = \max\{r_1, r_2\}$ and $D = \{z : |z| \geq r\}$, then F is holomorphic in D. Without loss of generality, we may assume that $|w_n| \geq r + 1$ for all n. We define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z).$$

Then all $F_n(z)$ are holomorphic in D_1 and $F_n^\sharp(0) = F^\sharp(w_n) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, by using Lemma 2.1 and choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ with $|z_n| < r < 1$ and $\rho_n \to 0$ such that the sequences $(g_n)_n$ are defined by

$$g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{H(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{4.2}$$

locally uniformly in $\mathbb{C}$ with a nonconstant entire function g and

$$\rho_n \le \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \tag{4.3}$$

for a positive number M. Let $G_n(\zeta) = \rho_n \frac{H'(w_n+z_n+\rho_n\zeta)}{P(w_n+z_n+\rho_n\zeta)}$, then from (4.2) and $|\frac{P'(w_n+z_n+\rho_n\zeta)}{P(w_n+z_n+\rho_n\zeta)}| < 1$ as $n \to \infty$, we get

$$G_n(\zeta) = g_n'(\zeta) + \frac{\rho_n g_n(\zeta)P'(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} \to g'(\zeta) \tag{4.4}$$

locally uniformly in $\mathbb{C}$.

Suppose that $g(\zeta_0) = 1$, then by Hurwitz's theorem there exist ζ_n, $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$g_n(\zeta_n) = \frac{H(w_n + z_n + \rho_n\zeta_n)}{P(w_n + z_n + \rho_n\zeta_n)} = 1.$$

By the assumption of Theorem 1.2, we have

$$H'(w_n + z_n + \rho_n\zeta_n) = P_2(w_n + z_n + \rho_n\zeta_n). \tag{4.5}$$

From (4.1) and (4.3), we deduce that

$$\lim_{n\to\infty} w_n^{l_1}\rho_n = 0 \tag{4.6}$$

for any fixed constant $0 \le l_1 < \frac{\rho(f)-2}{2}$. Meanwhile, we have

$$\frac{P_2(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} = O\bigl(|w_n|^{l_1}\bigr),$$

here $\deg R_1 = \deg(\frac{P_2}{P}) = \deg(\frac{Q_2-Q_1'}{Q_2-Q_1}) = l_1$. By (4.6), we deduce that

$$\rho_n \frac{P_2(w_n + z_n + \rho_n\zeta)}{P(w_n + z_n + \rho_n\zeta)} = O\bigl(\rho_n w_n^{l_1}\bigr) \to 0 \quad (n \to \infty). \tag{4.7}$$

By (4.4), (4.5) and (4.7), we obtain that

$$g'(\zeta_0) = \lim_{n\to\infty} \rho_n \frac{H'(w_n + z_n + \rho_n\zeta_n)}{P(w_n + z_n + \rho_n\zeta_n)} = \lim_{n\to\infty} \rho_n \frac{P_2(w_n + z_n + \rho_n\zeta_n)}{P(w_n + z_n + \rho_n\zeta_n)} = 0.$$

Thus $g(\zeta) = 1 \Rightarrow g'(\zeta) = 0$, which yields that the zeros of $g - 1$ are of multiplicity at least 2. Similarly, we can prove that the zeros of g are of multiplicity at least 2.

Noting that $Q_1(z) \not\equiv Q_2(z)$, without loss of generality, we assume that $Q_1(z) \not\equiv 0$. Next, we shall prove that $g(\zeta) \ne 0$. Suppose that ξ_0 is a zero of $g(\zeta)$ with multiplicity m (≥ 2), then $g^{(m)}(\xi_0) \ne 0$. Thus there exists a positive number δ such that

$$g(\zeta) \ne 0, \qquad g'(\zeta) \ne 0, \qquad g^{(m)}(\zeta) \ne 0 \tag{4.8}$$

on $D_\delta^o = \{z : 0 < |\zeta - \xi_0| < \delta\}$.

Noting that $g(\zeta) \not\equiv 0$, by Rouché's theorem there exist $\zeta_{n,j}$ $(j = 1, 2, \ldots, m)$ on $D_{\delta/2} = \{\xi : |\zeta - \xi_0| < \delta/2\}$ such that

$$g_n(\zeta_{n,j}) = H(w_n + z_n + \rho_n \zeta_{n,j}) = 0 \quad (j = 1, \ldots, m).$$

Note that, for n large enough,

$$H'(w_n + z_n + \rho_n \zeta_{n,j}) = Q_1(w_n + z_n + \rho_n \zeta_{n,j}) - Q_1'(w_n + z_n + \rho_n \zeta_{n,j}) \neq 0,$$

so each $\zeta_{n,j}$ is a simple zero of $H(w_n + z_n + \rho_n \zeta)$, that is, $\zeta_{n,j} \neq \zeta_{n,i}$ $(1 \leq i \neq j \leq m)$.

Likewise (4.7), we have

$$\rho_n \frac{Q_1(w_n + z_n + \rho_n \zeta) - Q_1'(w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} = O\bigl(\rho_n w_n^{l_2}\bigr) \to 0 \quad (n \to \infty), \tag{4.9}$$

here $l_2 = \deg R_2 = \deg \frac{Q_1 - Q_1'}{Q_2 - Q_1}$.

Note that by (4.4) and (4.9) we have

$$K_n(\zeta) = G_n(\zeta) - \rho_n \frac{Q_1(w_n + z_n + \rho_n \zeta) - Q_1'(w_n + z_n + \rho_n \zeta)}{P(w_n + z_n + \rho_n \zeta)} \to g'(\zeta), \tag{4.10}$$

and $K_n(\zeta_{n,j}) = 0$ $(j = 1, \ldots, m)$. From (4.8) we have

$$\lim_{n \to \infty} \zeta_{n,j} = \xi_0 \quad (j = 1, 2, \ldots, m).$$

Noting (4.8), (4.10) and that $K_n(\zeta)$ has m zeros $\zeta_{n,j}$ $(j = 1, 2, \ldots, m)$ in $D_{\delta/2}$, we obtain from Hurwitz's theorem that ξ_0 is a zero of $g'(\zeta)$ with multiplicity m, and thus $g^{(m)}(\xi_0) = 0$. This is a contradiction. Hence $g(\zeta) \neq 0$.

We have shown that g is a nonvanishing entire function that takes the value 1 always with multiplicity at least 2. But this contradicts Nevanlinna's second fundamental theorem that the sum of the defects is at most 2.

This completes the proof of Theorem 1.2.

5 The proof of Corollary 1.3

By Lemma 2.6, the reader could give the proof of Corollary 1.3 with almost the same argument as that in the proof of Theorem 1.2. Here we omit it.

6 The proof of Theorem 1.4

By a similar way to that of Li, Gao and Zhang [9], we prove Theorem 1.4 as follows.

Since f is an entire function, by Lemma 2.4 we have

$$N\left(r, \frac{1}{f^{(k+1)}}\right) \leq N\left(r, \frac{1}{f^{(k)}}\right) + S(r,f) < \alpha T(r,f) + S(r,f). \tag{6.1}$$

Set

$$F = \frac{f^{(k+2)}}{f^{(k+1)}} - \frac{f^{(k+1)}}{f^{(k)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1} + 2\frac{f'}{f - 1}. \tag{6.2}$$

Then F is a meromorphic function and hence

$$m(r,F) = S(r,f) \tag{6.3}$$

from the fundamental estimate of the logarithmic derivative. Since the poles of F appear to the zeros of $f^{(k)}$ and $f^{(k+1)}$, by assumption and (6.1), we have

$$N(r,F) \le N\left(r, \frac{1}{f^{(k)}}\right) + N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \tag{6.4}$$

Combining (6.3) with (6.4), we see that $T(r,F) < 2\alpha T(r,f) + S(r,f)$.

We assume that $F \not\equiv 0$. Notice that all zeros of $f-1$ and $f^{(k)}-1$ are k multiples. Let z_0 be a common zero of $f-1$ and $f^{(k)}-1$. Then $f^{(k+1)}(z_0) \neq 0$, and it is easy to see that F is holomorphic at z_0, and $F(z_0) = 0$. Thus we have

$$\begin{aligned} N\left(r, \frac{1}{f-1}\right) &\le N\left(r, \frac{1}{F}\right) \le T(r,F) + O(1) \\ &\le 2\alpha T(r,f) + S(r,f) < 2\left(\alpha + \frac{1-4\alpha}{8}\right) T(r,f) \\ &= \left(\alpha + \frac{1}{4}\right) T(r,f), \end{aligned} \tag{6.5}$$

for large enough r. However, by Lemma 2.3, we have

$$N\left(r, \frac{1}{f-1}\right) \ge \left(\frac{1}{2} - \left(\frac{1}{4} - \alpha\right)\right) T(r,f) = \left(\alpha + \frac{1}{4}\right) T(r,f), \tag{6.6}$$

for large enough r. (6.6) contradicts to (6.5).

Hence $F \equiv 0$. Integration of (6.2) yields

$$A\frac{f^{(k+1)}}{f^{(k)}} = \left(\frac{f^{(k)}-1}{f-1}\right)^2, \tag{6.7}$$

where A is a nonzero constant. Since f and $f^{(k)}$ share 1 CM, for any point z_1 satisfying that $f(z_1) = f^{(k)}(z_1) = 1$, we obtain that $f'(z_1)f^{(k+1)}(z_1) \neq 0$, and then $A = \frac{f^{(k+1)}(z_1)}{[f'(z_1)]^2}$. Thus if we assume that $\frac{f^{(k+1)}}{f^{(k)}}$ is not a constant function, we see from (6.7) that $f^{(k)}(z) \neq 0$ and $f^{(k+1)}(z) \neq 0$. Noting that $f(z)$ is an entire function, we see that $\frac{f^{(k+1)}}{f^{(k)}}$ has no both pole and zero. So $A\frac{f^{(k+1)}}{f^{(k)}} =: e^{h(z)}$ is a small function of $f^{(k)}$, where $h(z)$ is an entire function. Now we may change (6.7) into

$$f(z) = \left(f^{(k)}(z) - 1\right)e^{-\frac{1}{2}h(z)} + 1. \tag{6.8}$$

By the second main theorem, (6.5) and (6.8), we have

$$\begin{aligned} T(r,f) &= T\left(r,f^{(k)}\right) + S\left(r,f^{(k)}\right) \\ &\le N\left(r,f^{(k)}\right) + N\left(r, \frac{1}{f^{(k)}}\right) + N\left(r, \frac{1}{f^{(k)}-1}\right) + S\left(r,f^{(k)}\right) \end{aligned}$$

$$\leq N\left(r, \frac{1}{f-1}\right) + S(r,f)$$
$$\leq \frac{1}{2} T(r,f) + S(r,f),$$

a contradiction.

Therefore, $\frac{f^{(k+1)}}{f^{(k)}}$ is a constant function and hence there exists a nonzero constant c such that

$$\frac{f^{(k+1)}(z) - 1}{f(z) - 1} = c.$$

This completes the proof of Theorem 1.4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JQ carried out the main part of this manuscript. FL and WY participated discussion and corrected the main theorem. All authors read and approved the final manuscript.

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