Xu and Wang *Advances in Difference Equations* 2013, **2013**:184
http://www.advancesindifferenceequations.com/content/2013/1/184

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

The infinite sum of the cubes of reciprocal Pell numbers

Zhefeng Xu and Tingting Wang*

*Correspondence:
tingtingwang126@126.com
Department of Mathematics,
Northwest University, Xi'an, Shaanxi,
P.R. China

Abstract

Given the sequence of Pell numbers $\{P_n\}$, we evaluate the integral part of the reciprocal of the sum $\sum_{k=n}^{\infty}\frac{1}{P_n^3}$ explicitly in terms of the Pell numbers themselves.

MSC: Primary 11B39

Keywords: Pell numbers; floor function; identity

1 Introduction

For any integer $n \geq 0$, the well-known Pell numbers P_n are defined by the second-order linear recurrence sequence $P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$. The Pell-Lucas numbers Q_n are defined by $Q_{n+2} = 2Q_{n+1} + Q_n$, where $Q_0 = 2$ and $Q_1 = 2$. Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Then from the characteristic equations $x^2 - 2x - 1 = 0$, we also have the computational formulae

$$P_n = \frac{1}{2\sqrt{2}}\left(\alpha^n - \beta^n\right) \quad \text{and} \quad Q_n = \alpha^n + \beta^n.$$

For example, the first few values of P_n and Q_n are $P_0 = 1, P_1 = 2, P_2 = 5, P_3 = 12, P_4 = 29, \ldots,$ $Q_0 = 2, Q_1 = 2, Q_2 = 6, Q_3 = 14, Q_4 = 34, Q_5 = 82, \ldots.$

Various properties of the Pell numbers and related sequences have been studied by many authors, see [1–6]. For example, Santos and Sills [3] studied the arithmetic properties of the q-Pell sequence and obtained two identities. Kilic [4] studied the generalized order-k Fibonacci-Pell sequences and gave several congruences. Recently, the authors [7] and [8] studied the infinite sums derived from the Pell numbers and proved the following identities:

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{P_k}\right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ P_{n-1} + P_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{P_k^2}\right)^{-1} \right\rfloor = \begin{cases} 2P_{n-1}P_n - 1 & \text{if } n \text{ is an even number;} \\ 2P_{n-1}P_n & \text{if } n \text{ is an odd number,} \end{cases}$$

where $\lfloor x \rfloor$ is the floor function, that is, it denotes the greatest integer less than or equal to x.

Some related works can also be found in [9] and [10]. Especially in [10], the authors studied a problem, which is little different from (1). That is, they studied the computational problem of the nearest integer function of $(\sum_{k=n}^{\infty}\frac{1}{u_k})^{-1}$ and proved an interesting conclusion:

$$\left\|\left(\sum_{k=n}^{\infty}\frac{1}{u_k}\right)^{-1}\right\| = u_n - u_{n-1} \quad \text{for all } n > n_0,$$

where $\|\cdot\|$ denotes the nearest integer, namely $\|x\| = \lfloor x+\frac{1}{2}\rfloor$, $\{u_n\}_{n\geq 0}$ is an integer sequence satisfying the recurrence formula

$$u_n = au_{n-1} + u_{n-2} + \cdots + u_{n-s} \quad (s \geq 2)$$

with the initial conditions $u_0 \geq 0$, $u_k \in \mathbf{N}$, $1 \leq k \leq s-1$.

Using the method in [10] seems to be very difficult to deal with $(\sum_{k=n}^{\infty}\frac{1}{u_k^s})^{-1}$ for all integers $s \geq 2$.

The main purpose of this paper related to the computing problem of

$$P(s,n) \equiv \left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{P_k^s}\right)^{-1} \right\rfloor \tag{1}$$

for all integers $s \geq 3$. At the end of [7], the authors asked whether there exists a corresponding formula for $P(3,n)$.

In fact, this problem is difficult because it is quite unclear *a priori* what the shape of the result might be. In order to resolve the question, we carefully applied the method of undetermined coefficients and constructed a number of delicate inequalities in order to complete a proof. The result is as follows.

Theorem *For any positive integer $n \geq 1$, we have the identity*

$$P(3,n) = \begin{cases} P_n^2P_{n-1} + 3P_nP_{n-1}^2 + \lfloor -\frac{61}{82}P_n - \frac{91}{82}P_{n-1}\rfloor & \textit{if } n \textit{ is even and } n \geq 2; \\ P_n^2P_{n-1} + 3P_nP_{n-1}^2 + \lfloor \frac{61}{82}P_n + \frac{91}{82}P_{n-1}\rfloor & \textit{if } n \textit{ is odd and } n \geq 1. \end{cases}$$

It remains a difficult problem even to conjecture what might be an analogous expression to the formula for $P(3,n)$ in the theorem for $P(k,n)$ when $k \geq 4$.

2 Proof of the theorem

In this section, we shall prove our theorem directly. First we consider the case that $n = 2m$ is an even number. It is clear that in this case our theorem is equivalent to

$$\begin{aligned} &P_{2m}^2P_{2m-1} + 3P_{2m}P_{2m-1}^2 - \frac{1}{82}(61P_{2m} + 91P_{2m-1}) \\ &\quad < \left(\sum_{k=2m}^{\infty}\frac{1}{P_k^3}\right)^{-1} < P_{2m}^2P_{2m-1} + 3P_{2m}P_{2m-1}^2 - \frac{1}{82}(61P_{2m} + 91P_{2m-1}) + \frac{1}{82} \end{aligned}$$

or

$$\frac{1}{P_{2m}^2P_{2m-1}+3P_{2m}P_{2m-1}^2-\frac{1}{82}(61P_{2m}+91P_{2m-1})+\frac{1}{82}} < \sum_{k=2m}^{\infty}\frac{1}{P_k^3} < \frac{1}{P_{2m}^2P_{2m-1}+3P_{2m}P_{2m-1}^2-\frac{1}{82}(61P_{2m}+91P_{2m-1})}. \tag{2}$$

Now we prove that for all positive integers k, we have the inequality

$$\frac{1}{P_{2k}^3}+\frac{1}{P_{2k+1}^3} < \frac{1}{P_{2k}^2P_{2k-1}+3P_{2k}P_{2k-1}^2-\frac{1}{82}(61P_{2k}+91P_{2k-1})} - \frac{1}{P_{2k+2}^2P_{2k+1}+3P_{2k+2}P_{2k+1}^2-\frac{1}{82}(61P_{2k+2}+91P_{2k+1})}. \tag{3}$$

It is clear that (3) holds for $k=1,2,3$ and 4. So, without loss of generality, we can assume that $k\geq 5$. Note that $P_{2k}^3=\frac{1}{8}(P_{6k}-3P_{2k})$, $P_{2k+1}^3=\frac{1}{8}(P_{6k+3}+3P_{2k+1})$, $P_{2k}^3+P_{2k+1}^3=\frac{1}{8}(P_{6k+3}+P_{6k}+3P_{2k+1}-3P_{2k})$, $P_{2k}^3P_{2k+1}^3=\frac{1}{512}(Q_{12k+3}-6Q_{8k+2}+9Q_{4k+1}+4)$ and

$$P_{2k}^2P_{2k-1}+3P_{2k}P_{2k-1}^2=\frac{1}{8}(P_{6k-1}+3P_{6k-2}+5P_{2k-1}+5P_{2k}),$$

so inequality (3) is equivalent to

$$\frac{8(P_{6k+3}+P_{6k}+3P_{2k+1}-3P_{2k})}{Q_{12k+3}-6Q_{8k+2}+9Q_{4k+1}+4} < \frac{378P_{6k-1}+154P_{6k-2}-\frac{78}{41}P_{2k+1}-\frac{318}{41}P_{2k}}{(P_{6k-1}+3P_{6k-2}-\frac{39}{41}P_{2k}-\frac{159}{41}P_{2k-1})(P_{6k+5}+3P_{6k+4}-\frac{39}{41}P_{2k+2}-\frac{159}{41}P_{2k+1})}. \tag{4}$$

From the definition and properties of P_n and Q_n, we can easily deduce the identities

$$\begin{aligned}
P_nP_k &= \frac{1}{8}Q_{n+k}-\frac{(-1)^k}{8}Q_{n-k}, \quad n\geq k,\\
Q_nQ_k &= Q_{n+k}+(-1)^kQ_{n-k}, \quad n\geq k,\\
P_nQ_k &= P_{n+k}+(-1)^kP_{n-k}, \quad n\geq k.
\end{aligned}$$

So, applying these formulae, we have

$$(P_{6k-1}+3P_{6k-2})(P_{6k+5}+3P_{6k+4})=\frac{1}{8}(8Q_{12k+3}+10Q_{12k+2}-2{,}772)$$

and

$$\begin{aligned}
&\left(P_{6k-1}+3P_{6k-2}-\frac{39}{41}P_{2k}-\frac{159}{41}P_{2k-1}\right)\left(P_{6k+5}+3P_{6k+4}-\frac{39}{41}P_{2k+2}-\frac{159}{41}P_{2k+1}\right)\\
&\quad=\frac{1}{8}\left(8Q_{12k+3}+10Q_{12k+2}-\frac{7{,}344}{41}Q_{8k+1}-\frac{2{,}124}{41}Q_{8k}-\frac{30{,}226{,}500}{1{,}681}Q_{4k-3}\right)\\
&\qquad-\frac{1}{8}\left(\frac{12{,}507{,}840}{1{,}681}Q_{4k-4}+\frac{4{,}442{,}760}{1{,}681}\right).
\end{aligned}$$

From these two identities and (4), we deduce that inequality (3) is equivalent to

$$\begin{aligned}&\frac{P_{6k+3}+P_{6k}+3P_{2k+1}-3P_{2k}}{Q_{12k+3}-6Q_{8k+2}+9Q_{4k+1}+4}\\&\quad<\frac{378P_{6k-1}+154P_{6k-2}-\frac{78}{41}P_{2k+1}-\frac{318}{41}P_{2k}}{8Q_{12k+3}+10Q_{12k+2}-\frac{7{,}344}{41}Q_{8k+1}-\frac{2{,}124}{41}Q_{8k}-\frac{30{,}226{,}500}{1{,}681}Q_{4k-3}-\frac{12{,}507{,}840}{1{,}681}Q_{4k-4}-\frac{4{,}442{,}760}{1{,}681}}.\end{aligned}\tag{5}$$

For convenience, we let

$$\begin{aligned}A&=(P_{6k+3}+P_{6k}+3P_{2k+1}-3P_{2k})\times\biggl(8Q_{12k+3}+10Q_{12k+2}-\frac{7{,}344}{41}Q_{8k+1}\\&\quad-\frac{2{,}124}{41}Q_{8k}-\frac{30{,}226{,}500}{1{,}681}Q_{4k-3}-\frac{12{,}507{,}840}{1{,}681}Q_{4k-4}-\frac{4{,}442{,}760}{1{,}681}\biggr)\end{aligned}$$

and

$$B=(Q_{12k+3}-6Q_{8k+2}+9Q_{4k+1}+4)\biggl(378P_{6k-1}+154P_{6k-2}-\frac{78}{41}P_{2k+1}-\frac{318}{41}P_{2k}\biggr).$$

Then by calculation it follows that

$$\begin{aligned}A&=8(P_{18k+6}+P_{6k})+10(P_{18k+5}+P_{6k-1})-\frac{7{,}344}{41}(P_{14k+4}+P_{2k-2})\\&\quad-\frac{2{,}124}{41}(P_{14k+3}+P_{2k-3})-\frac{30{,}226{,}500}{1{,}681}(P_{10k}-P_{2k+6})-\frac{12{,}507{,}840}{1{,}681}(P_{10k-1}\\&\quad+P_{2k+7})-\frac{4{,}442{,}760}{1{,}681}P_{6k+3}+8(P_{18k+3}-P_{6k+3})+10(P_{18k+2}-P_{6k+2})\\&\quad-\frac{7{,}344}{41}(P_{14k+1}-P_{2k+1})-\frac{2{,}124}{41}(P_{14k}-P_{2k})-\frac{30{,}226{,}500}{1{,}681}(P_{10k-3}-P_{2k+3})\\&\quad-\frac{12{,}507{,}840}{1{,}681}(P_{10k-4}+P_{2k+4})-\frac{4{,}442{,}760}{1{,}681}P_{6k}+3\times8(P_{14k+4}+P_{10k+2})\\&\quad+3\times10(P_{14k+3}+P_{10k+1})-\frac{7{,}344\times3}{41}(P_{10k+2}+P_{6k})-\frac{2{,}124\times3}{41}(P_{10k+1}\\&\quad+P_{6k-1})-\frac{30{,}226{,}500\times3}{1{,}681}(P_{6k-2}+P_{2k-4})-\frac{12{,}507{,}840\times3}{1{,}681}(P_{6k-3}+P_{2k-5})\\&\quad-\frac{4{,}442{,}760}{1{,}681}P_{2k+1}-3\times8(P_{14k+3}-P_{10k+3})-3\times10(P_{14k+2}-P_{10k+2})\\&\quad+\frac{7{,}344\times3}{41}(P_{10k+1}-P_{6k+1})+\frac{2{,}124\times3}{41}(P_{10k}-P_{6k})+\frac{30{,}226{,}500\times3}{1{,}681}(P_{6k-3}\\&\quad-P_{2k-3})+\frac{12{,}507{,}840\times3}{1{,}681}(P_{6k-4}-P_{2k-4})+\frac{4{,}442{,}760}{1{,}681}P_{2k}\\&=154P_{18k+3}+70P_{18k+2}-\frac{95{,}514}{41}P_{14k+1}-\frac{38{,}910}{41}P_{14k}-\frac{486{,}612{,}540}{1{,}681}P_{10k-3}\\&\quad-\frac{201{,}554{,}538}{1{,}681}P_{10k-4}-\frac{977{,}366{,}722}{1{,}681}P_{6k-3}-\frac{344{,}423{,}038}{1{,}681}P_{6k-4}\\&\quad-\frac{285{,}928{,}452}{1{,}681}P_{2k-4}-\frac{118{,}454{,}868}{1{,}681}P_{2k-5},\\B&=378(P_{18k+2}+P_{6k+4})+154(P_{18k+1}-P_{6k+5})-\frac{78}{41}(P_{14k+4}+P_{10k+2})\end{aligned}$$

$$\begin{aligned}
&- \frac{318}{41}(P_{14k+3} - P_{10k+3}) - 6 \times 378(P_{14k+1} + P_{2k+3}) - 6 \times 154(P_{14k} - P_{2k+4}) \\
&+ \frac{78 \times 6}{41}(P_{10k+3} + P_{6k+1}) + \frac{318 \times 6}{41}(P_{10k+2} - P_{6k+2}) + 9 \times 378(P_{10k} - P_{2k-2}) \\
&+ 9 \times 154(P_{10k-1} - P_{2k-3}) - \frac{78 \times 9}{41}(P_{6k+2} + P_{2k}) - \frac{318 \times 9}{41}(P_{6k+1} - P_{2k+1}) \\
&+ 4 \times 378P_{6k-1} + 4 \times 154P_{6k-2} - \frac{78 \times 4}{41}P_{2k+1} - \frac{318 \times 4}{41}P_{2k} \\
= {}& 154P_{18k+3} + 70P_{18k+2} - \frac{95{,}514}{41}P_{14k+1} - \frac{38{,}910}{41}P_{14k} + \frac{158{,}064}{41}P_{10k} \\
&+ \frac{64{,}416}{41}P_{10k-1} + \frac{36{,}496}{41}P_{6k-1} + \frac{14{,}880}{41}P_{6k-2} - \frac{225{,}516}{41}P_{2k-2} - \frac{92{,}796}{41}P_{2k-3}.
\end{aligned}$$

Observe that the major terms of A and B (above those of order P_{10k}) are in total agreement. Note that $P_{n+2} = 2P_{n+1} + P_n$, we have

$$\begin{aligned}
B - A = {}& \frac{41{,}028{,}234}{1{,}681}P_{10k} + \frac{17{,}049{,}300}{1{,}681}P_{10k-1} \\
&+ \frac{348{,}025{,}790}{1{,}681}P_{6k-2} + \frac{290{,}016{,}982}{1{,}681}P_{6k-3} \\
&+ \frac{39{,}772{,}560}{1{,}681}P_{2k-2} + \frac{16{,}612{,}800}{1{,}681}P_{2k-3} > 0
\end{aligned}$$

for all integers $k \geq 1$. So, inequalities (3), (4) and (5) hold for all integers $k \geq 1$.

Now, applying (3) repeatedly, we have

$$\begin{aligned}
\sum_{k=2m}^{\infty} \frac{1}{P_k^3} &= \sum_{k=m}^{\infty} \left(\frac{1}{P_{2k}^3} + \frac{1}{P_{2k+1}^3} \right) \\
&< \sum_{k=m}^{\infty} \frac{1}{P_{2k}^2 P_{2k-1} + 3P_{2k}P_{2k-1}^2 - \frac{1}{82}(61P_{2k} + 91P_{2k-1})} \\
&\quad - \sum_{k=m}^{\infty} \frac{1}{P_{2k+2}^2 P_{2k+1} + 3P_{2k+2}P_{2k+1}^2 - \frac{1}{82}(61P_{2k+2} + 91P_{2k+1})} \\
&= \frac{1}{P_{2m}^2 P_{2m-1} + 3P_{2m}P_{2m-1}^2 - \frac{1}{82}(61P_{2m} + 91P_{2m-1})}.
\end{aligned} \tag{6}$$

On the other hand, we prove the inequality

$$\begin{aligned}
\frac{1}{P_{2k}^3} + \frac{1}{P_{2k+1}^3} > {}& \frac{1}{P_{2k}^2 P_{2k-1} + 3P_{2k}P_{2k-1}^2 - \frac{1}{82}(61P_{2k} + 91P_{2k-1}) + \frac{1}{82}} \\
&- \frac{1}{P_{2k+2}^2 P_{2k+1} + 3P_{2k+2}P_{2k+1}^2 - \frac{1}{82}(61P_{2k+2} + 91P_{2k+1}) + \frac{1}{82}}.
\end{aligned} \tag{7}$$

This inequality is equivalent to

$$\begin{aligned}
&\frac{P_{6k+3} + P_{6k} + 3P_{2k+1} - 3P_{2k}}{Q_{12k+3} - 6Q_{8k+2} + 9Q_{4k+1} + 4} \\
&\quad > \frac{378P_{6k-1} + 154P_{6k-2} - \frac{78}{41}P_{2k+1} - \frac{318}{41}P_{2k}}{(P_{6k-1} + 3P_{6k-2} - \frac{39}{41}P_{2k} - \frac{159}{41}P_{2k-1} + \frac{4}{41})(P_{6k+5} + 3P_{6k+4} - \frac{39}{41}P_{2k+2} - \frac{159}{41}P_{2k+1} + \frac{4}{41})}
\end{aligned}$$

or

$$\frac{4}{41}(P_{6k+3}+P_{6k}+3P_{2k+1}-3P_{2k})\left(60P_{6k+1}+40P_{6k}-\frac{552}{41}P_{2k}-\frac{396}{41}P_{2k-1}+\frac{4}{41}\right)$$
$$>B-A$$

or

$$\begin{aligned}&140Q_{12k+3}+140Q_{12k+2}+\frac{1{,}200}{41}Q_{8k+2}+\frac{5{,}952}{41}Q_{8k+1}+\frac{19{,}116}{41}Q_{4k}+\frac{9{,}444}{41}Q_{4k-1}\\&\quad+\frac{4}{41}P_{6k+3}+\frac{4}{41}P_{6k}+\frac{12}{41}P_{2k+1}-\frac{12}{41}P_{2k}+\frac{47{,}728}{41}>\frac{41}{4}(B-A).\end{aligned}\tag{8}$$

It is clear that inequality (8) holds for all integers $k\geq 5$, so inequality (7) is true. Now, applying (7) repeatedly, we have

$$\begin{aligned}\sum_{k=2m}^{\infty}\frac{1}{P_k^3}&=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k}^3}+\frac{1}{P_{2k+1}^3}\right)\\&>\frac{1}{P_{2m}^2P_{2m-1}+3P_{2m}P_{2m-1}^2-\frac{1}{82}(61P_{2m}+91P_{2m-1})+\frac{1}{82}}.\end{aligned}\tag{9}$$

Combining (6) and (9), we may immediately deduce inequality (2).

Now we consider that $n=2m+1$ is an odd number. It is clear that in this case our theorem is equivalent to

$$\begin{aligned}&P_{2m+1}^2P_{2m}+3P_{2m+1}P_{2m}^2+\frac{1}{82}(61P_{2m+1}+91P_{2m})\\&\quad<\left(\sum_{k=2m+1}^{\infty}\frac{1}{P_k^3}\right)^{-1}<P_{2m+1}^2P_{2m}+3P_{2m+1}P_{2m}^2+\frac{1}{82}(61P_{2m+1}+91P_{2m})+\frac{1}{82}\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{P_{2m+1}^2P_{2m}+3P_{2m+1}P_{2m}^2+\frac{1}{82}(61P_{2m+1}+91P_{2m})+\frac{1}{82}}\\&\quad<\sum_{k=2m+1}^{\infty}\frac{1}{P_k^3}<\frac{1}{P_{2m+1}^2P_{2m}+3P_{2m+1}P_{2m}^2+\frac{1}{82}(61P_{2m+1}+91P_{2m})}.\end{aligned}\tag{10}$$

First we prove the inequality

$$\begin{aligned}\frac{1}{P_{2k+1}^3}+\frac{1}{P_{2k+2}^3}&<\frac{1}{P_{2k+1}^2P_{2k}+3P_{2k+1}P_{2k}^2+\frac{1}{82}(61P_{2k+1}+91P_{2k})}\\&\quad-\frac{1}{P_{2k+1}^2P_{2k}+3P_{2k+1}P_{2k}^2+\frac{1}{82}(61P_{2k+1}+91P_{2k})}.\end{aligned}\tag{11}$$

It is easy to check that inequality (11) is correct for $k=1,2$ and 3. So, we can assume that $k\geq 4$. Note that $P_{2k+1}^3=\frac{1}{8}(P_{6k+3}+3P_{2k+1})$, $P_{2k+2}^3=\frac{1}{8}(P_{6k+6}-3P_{2k+2})$, $P_{2k+1}^3+P_{2k+2}^3=\frac{1}{8}(P_{6k+6}+P_{6k+3}+3P_{2k+1}-3P_{2k+2})$, $P_{2k+1}^3P_{2k+2}^3=\frac{1}{512}(Q_{12k+9}+6Q_{8k+6}+9Q_{4k+3}-4)$, $P_{2k+1}^2P_{2k}+$

$3P_{2k+1}P_{2k}^2 = \frac{1}{8}(P_{6k+2} + 3P_{6k+1} - 5P_{2k+1} - 5P_{2k})$, so inequality (11) is equivalent to the inequality

$$\begin{aligned}&\frac{P_{6k+6} + P_{6k+3} + 3P_{2k+1} - 3P_{2k+2}}{Q_{12k+9} + 6Q_{8k+6} + 9Q_{4k+3} - 4} \\ &\quad < \frac{378P_{6k+2} + 154P_{6k+1} + \frac{78}{41}P_{2k+2} + \frac{318}{41}P_{2k+1}}{(P_{6k+2} + 3P_{6k+1} + \frac{39}{41}P_{2k+1} + \frac{159}{41}P_{2k})(P_{6k+8} + 3P_{6k+7} + \frac{39}{41}P_{2k+3} + \frac{159}{41}P_{2k+2})}.\end{aligned} \tag{12}$$

From the definition and properties of the Pell-Lucas numbers, we have

$$(P_{6k+2} + 3P_{6k+1})(P_{6k+8} + 3P_{6k+7}) = (8Q_{12k+9} + 10Q_{12k+8} + 2{,}772)/8$$

and

$$\begin{aligned}&\left(P_{6k+2} + 3P_{6k+1} + \frac{39}{41}P_{2k+1} + \frac{159}{41}P_{2k}\right)\left(P_{6k+8} + 3P_{6k+7} + \frac{39}{41}P_{2k+3} + \frac{159}{41}P_{2k+2}\right) \\ &\quad = \frac{1}{8}\left(8Q_{12k+9} + 10Q_{12k+8} + \frac{7{,}344}{41}Q_{8k+5} + \frac{2{,}124}{41}Q_{8k+4} - \frac{31{,}844{,}565}{1{,}681}Q_{4k-1}\right. \\ &\quad\quad \left. - \frac{13{,}186{,}923}{1{,}681}Q_{4k-2} + \frac{4{,}442{,}760}{1{,}681}\right).\end{aligned}$$

By these two identities and (12), we deduce that inequality (11) is equivalent to

$$\begin{aligned}&\frac{P_{6k+6} + P_{6k+3} + 3P_{2k+1} - 3P_{2k+2}}{Q_{12k+9} + 6Q_{8k+6} + 9Q_{4k+3} - 4} \\ &\quad < \frac{378P_{6k+2} + 154P_{6k+1} + \frac{78}{41}P_{2k+2} + \frac{318}{41}P_{2k+1}}{8Q_{12k+9} + 10Q_{12k+8} + \frac{7{,}344}{41}Q_{8k+5} + \frac{2{,}124}{41}Q_{8k+4} - \frac{31{,}844{,}565}{1{,}681}Q_{4k-1} - \frac{13{,}186{,}923}{1{,}681}Q_{4k-2} + \frac{4{,}442{,}760}{1{,}681}}.\end{aligned} \tag{13}$$

For convenience, we let

$$\begin{aligned}A' &= \left(8Q_{12k+9} + 10Q_{12k+8} + \frac{7{,}344}{41}Q_{8k+5} + \frac{2{,}124}{41}Q_{8k+4} - \frac{31{,}844{,}565}{1{,}681}Q_{4k-1}\right. \\ &\quad \left. - \frac{13{,}186{,}923}{1{,}681}Q_{4k-2} + \frac{4{,}442{,}760}{1{,}681}\right) \times (P_{6k+6} + P_{6k+3} + 3P_{2k+1} - 3P_{2k+2})\end{aligned}$$

and $B' = (Q_{12k+9} + 6Q_{8k+6} + 9Q_{4k+3} - 4)(378P_{6k+2} + 154P_{6k+1} + \frac{78}{41}P_{2k+2} + \frac{318}{41}P_{2k+1})$. Then we have

$$\begin{aligned}A' &= 8(P_{18k+15} - P_{6k+3}) + 10(P_{18k+14} - P_{6k+2}) + \frac{7{,}344}{41}(P_{14k+11} - P_{2k-1}) \\ &\quad + \frac{2{,}124}{41}(P_{14k+10} - P_{2k-2}) - \frac{31{,}844{,}565}{1{,}681}(P_{10k+5} - P_{2k+7}) - \frac{13{,}186{,}923}{1{,}681}\bigl(P_{10k+4} \\ &\quad + P_{2k+8}\bigr) + \frac{4{,}442{,}760}{1{,}681}P_{6k+6} + 8(P_{18k+12} + P_{6k+6}) + 10(P_{18k+11} + P_{6k+5}) \\ &\quad + \frac{7{,}344}{41}(P_{14k+8} + P_{2k+2}) + \frac{2{,}124}{41}(P_{14k+7} + P_{2k+1}) - \frac{31{,}844{,}565}{1{,}681}(P_{10k+2} - P_{2k+4}) \\ &\quad - \frac{13{,}186{,}923}{1{,}681}(P_{10k+1} + P_{2k+5}) + \frac{4{,}442{,}760}{1{,}681}P_{6k+3} + 3 \times 8(P_{14k+10} + P_{10k+8})\end{aligned}$$

$$
\begin{aligned}
&+ 3\times 10(P_{14k+9}+P_{10k+7}) + \frac{7{,}344\times 3}{41}(P_{10k+6}+P_{6k+4}) + \frac{2{,}124\times 3}{41}\big(P_{10k+5} \\
&+ P_{6k+3}\big) - \frac{31{,}844{,}565\times 3}{1{,}681}(P_{6k}+P_{2k-2}) - \frac{13{,}186{,}923\times 3}{1{,}681}(P_{6k-1}+P_{2k-3}) \\
&+ \frac{4{,}442{,}760\times 3}{1{,}681}P_{2k+1} - 3\times 8(P_{14k+11}-P_{10k+7}) - 3\times 10(P_{14k+10}-P_{10k+6}) \\
&- \frac{7{,}344\times 3}{41}(P_{10k+7}-P_{6k+3}) - \frac{2{,}124\times 3}{41}(P_{10k+6}-P_{6k+2}) + \frac{31{,}844{,}565\times 3}{1{,}681}\big(P_{6k+1} \\
&- P_{2k-3}\big) + \frac{13{,}186{,}923\times 3}{1{,}681}(P_{6k}-P_{2k-4}) - \frac{4{,}442{,}760\times 3}{1{,}681}P_{2k+2} \\
={}& 154P_{18k+12} + 70P_{18k+11} + \frac{95{,}514}{41}P_{14k+8} + \frac{38{,}910}{41}P_{14k+7} - \frac{506{,}756{,}250}{1{,}681}P_{10k+2} \\
&- \frac{209{,}906{,}976}{1{,}681}P_{10k+1} + \frac{983{,}915{,}086}{1{,}681}P_{6k} \\
&+ \frac{407{,}692{,}984}{1{,}681}P_{6k-1} - \frac{771{,}966{,}210}{1{,}681}P_{2k-3} \\
&- \frac{304{,}496{,}118}{1{,}681}P_{2k-4},
\end{aligned}
$$

$$
\begin{aligned}
B' ={}& 378(P_{18k+11}-P_{6k+7}) + 154(P_{18k+10}+P_{6k+8}) + \frac{78}{41}(P_{14k+11}-P_{10k+7}) \\
&+ \frac{318}{41}(P_{14k+10}+P_{10k+8}) + 6\times 378(P_{14k+8}-P_{2k+4}) + 924(P_{14k+7}+P_{2k+5}) \\
&+ \frac{468}{41}(P_{10k+8}-P_{6k+4}) + \frac{1{,}908}{41}(P_{10k+7}+P_{6k+5}) + 9\times 378(P_{10k+5}-P_{2k-1}) \\
&+ 1{,}386(P_{10k+4}-P_{2k-2}) + \frac{702}{41}(P_{6k+5}-P_{2k+1}) + \frac{318\times 9}{41}(P_{6k+4}+P_{2k+2}) \\
&- 4\times 378P_{6k+2} - 4\times 154P_{6k+1} - \frac{78\times 4}{41}P_{2k+2} - \frac{318\times 4}{41}P_{2k+1} \\
={}& 154P_{18k+12} + 70P_{18k+11} + \frac{95{,}514}{41}P_{14k+8} + \frac{38{,}910}{41}P_{14k+7} + \frac{158{,}064}{41}P_{10k+5} \\
&+ \frac{64{,}416}{41}P_{10k+4} - \frac{36{,}496}{41}P_{6k+2} - \frac{14{,}880}{41}P_{6k+1} - \frac{225{,}516}{41}P_{2k-1} - \frac{92{,}796}{41}P_{2k-2}.
\end{aligned}
$$

Note that $P_{n+2} = 2P_{n+1} + P_n$, we have

$$
\begin{aligned}
B' - A' ={}& \frac{597{,}729{,}018}{1{,}681}P_{10k+2} + \frac{247{,}592{,}208}{1{,}681}P_{10k+1} - \frac{992{,}616{,}926}{1{,}681}P_{6k} \\
&- \frac{411{,}295{,}736}{1{,}681}P_{6k-1} + \frac{718{,}126{,}158}{1{,}681}P_{2k-3} + \frac{282{,}199{,}170}{1{,}681}P_{2k-4} \\
={}& \left(\frac{3{,}483{,}829{,}506}{1{,}681}P_{10k} - \frac{992{,}616{,}926}{1{,}681}P_{6k}\right) + \left(\frac{1{,}443{,}050{,}244}{1{,}681}P_{10k-1}\right. \\
&\left.- \frac{411{,}295{,}736}{1{,}681}P_{6k-1}\right) + \frac{718{,}126{,}158}{1{,}681}P_{2k-3} + \frac{282{,}199{,}170}{1{,}681}P_{2k-4} > 0
\end{aligned}
$$

for all integers $k \geq 4$. So, inequalities (11), (12) and (13) hold for all integers $k \geq 4$.

Now, applying (11) repeatedly, we have

$$\begin{aligned}\sum_{k=2m+1}^{\infty}\frac{1}{P_k^3}&=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k+1}^3}+\frac{1}{P_{2k+2}^3}\right)\\&<\sum_{k=m}^{\infty}\frac{1}{P_{2k+1}^2P_{2k}+3P_{2k+1}P_{2k}^2+\frac{1}{82}(61P_{2k+1}+91P_{2k})}\\&\quad-\sum_{k=m}^{\infty}\frac{1}{P_{2k+3}^2P_{2k+2}+3P_{2k+3}P_{2k+2}^2+\frac{1}{82}(61P_{2k+3}+91P_{2k+2})}.\end{aligned}\tag{14}$$

On the other hand, we prove the inequality

$$\begin{aligned}\frac{1}{P_{2k+1}^3}+\frac{1}{P_{2k+2}^3}&>\frac{1}{P_{2k+1}^2P_{2k}+3P_{2k+1}P_{2k}^2+\frac{1}{82}(61P_{2k+1}+91P_{2k})+\frac{1}{82}}\\&\quad-\frac{1}{P_{2k+3}^2P_{2k+2}+3P_{2k+3}P_{2k+2}^2+\frac{1}{82}(61P_{2k+3}+91P_{2k+2})+\frac{1}{82}}.\end{aligned}\tag{15}$$

It is easy to check that inequality (15) is correct for $k = 1, 2$ and 3. So, we can assume that $k \geq 4$. This time, inequality (15) is equivalent to

$$\begin{aligned}&\frac{P_{6k+6}+P_{6k+3}+3P_{2k+1}-3P_{2k+2}}{Q_{12k+9}+6Q_{8k+6}+9Q_{4k+3}-4}\\&\quad>\frac{378P_{6k+2}+154P_{6k+1}+\frac{78}{41}P_{2k+2}+\frac{318}{41}P_{2k+1}}{(P_{6k+2}+3P_{6k+1}+\frac{39}{41}P_{2k+1}+\frac{159}{41}P_{2k}+\frac{4}{41})(P_{6k+8}+3P_{6k+7}+\frac{39}{41}P_{2k+3}+\frac{159}{41}P_{2k+2}+\frac{4}{41})}\end{aligned}$$

or

$$\begin{aligned}&\frac{4}{41}(P_{6k+6}+P_{6k+3}+3P_{2k+1}-3P_{2k+2})\left(60P_{6k+4}+40P_{6k+3}+\frac{552}{41}P_{2k+1}+\frac{396}{41}P_{2k}+\frac{4}{41}\right)\\&\quad>B'-A'\end{aligned}$$

or

$$\begin{aligned}&140Q_{12k+9}+140Q_{12k+8}-\frac{17{,}904}{41}Q_{8k+4}-\frac{8{,}352}{41}Q_{8k+3}+\frac{19{,}116}{41}Q_{4k+2}+\frac{9{,}444}{41}Q_{4k+1}\\&\quad+\frac{4}{41}P_{6k+6}+\frac{4}{41}P_{6k+3}+\frac{12}{41}P_{2k+1}-\frac{12}{41}P_{2k+2}+\frac{21{,}152}{41}>\frac{41}{4}\big(B'-A'\big).\end{aligned}\tag{16}$$

It is clear that inequality (16) holds for all integers $k \geq 4$, so inequality (15) is true. Now, applying (15) repeatedly, we have

$$\begin{aligned}\sum_{k=2m+1}^{\infty}\frac{1}{P_k^3}&=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k+1}^3}+\frac{1}{P_{2k+2}^3}\right)\\&>\frac{1}{P_{2m+1}^2P_{2m}+3P_{2m+1}P_{2m}^2+\frac{1}{82}(61P_{2m+1}+91P_{2m})+\frac{1}{82}}.\end{aligned}\tag{17}$$

Combining (14) and (17), we may immediately deduce inequality (10).

Now our theorem follows from inequalities (2) and (10). This completes the proof of our theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZX drafted the manuscript. TW participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

Acknowledgements
The authors express their gratitude to the referee for his very helpful and detailed comments. This work is supported by the N.S.F. (11001218, 11071194) of P.R. China and the Research Fund for the Doctoral Program of Higher Education (20106101120001) of P.R. China and the G.I.C.F. (YZZ12065) of NWU.

Received: 17 May 2013 Accepted: 4 June 2013 Published: 26 June 2013

References
1. Kilic, E, Altunkaynak, B, Tasci, D: On the computing of the generalized order-k Pell numbers in log time. Appl. Math. Comput. **181**, 511-515 (2006)
2. Hao, P: Arithmetic properties of q-Fibonacci numbers and q-Pell numbers. Discrete Math. **306**, 2118-2127 (2006)
3. Santos, JPO, Sills, AV: q-Pell sequences and two identities of V.A. Lebesgue. Discrete Math. **257**, 125-142 (2002)
4. Kilic, E: The generalized order-k Fibonacci-Pell sequences by matrix methods. J. Comput. Appl. Math. **209**, 133-145 (2007)
5. Egge, ES, Mansour, T: 132-avoiding two-stack sortable permutations, Fibonacci numbers, and Pell numbers. Discrete Appl. Math. **143**, 72-83 (2004)
6. Mansour, T, Shattuck, M: Restricted partitions and q-Pell numbers. Cent. Eur. J. Math. **9**, 346-355 (2011)
7. Zhang, W, Wang, T: The infinite sum of reciprocal Pell numbers. Appl. Math. Comput. **218**, 6164-6167 (2012)
8. Zhang, W, Wang, T: The infinite sum of reciprocal of the square of the Pell numbers. J. Weinan Teach. Univ. **26**, 39-42 (2011)
9. Ohtsuka, H, Nakamura, S: On the sum of reciprocal Fibonacci numbers. Fibonacci Q. **46/47**, 153-159 (2008/2009)
10. Komatsu, T, Laohakosol, V: On the sum of reciprocals of numbers satisfying a recurrence relation of order s. J. Integer Seq. **13**, Article ID 10.5.8 (2010)

doi:10.1186/1687-1847-2013-184
Cite this article as: Xu and Wang: **The infinite sum of the cubes of reciprocal Pell numbers.** *Advances in Difference Equations* 2013 **2013**:184.