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Existence and uniqueness of solutions for nonlinear impulsive differential equations with two-point and integral boundary conditions

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Abstract
In the present paper, a system of nonlinear impulsive differential equations with two-point and integral boundary conditions is investigated. Theorems on the existence and uniqueness of a solution are established under some sufficient conditions on nonlinear terms. A simple example of application of the main result of this paper is presented.

1 Introduction

The theory of impulsive differential equations is an important branch of differential equations which has an extensive physical background. Impulsive differential equations arise frequently in the modeling of many physical systems whose states are subject to sudden change at certain moments. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments (see, for instance, the monographs [1–4] and the references therein).

Many of the physical systems can better be described by integral boundary conditions. Integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering and cellular systems. Moreover, boundary value problems with integral conditions constitute a very interesting and important class of problems. They include two-point, three-point, multipoint and nonlocal boundary value problems as special cases. For boundary value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to the papers [5–18] and the references therein.

In the present paper, we study the existence and uniqueness of the system of nonlinear impulsive differential equations of the type

$$\dot{x}(t) = f\bigl(t, x(t)\bigr) \quad \text{for } t \neq t_i, i = 1, 2, \ldots, p, t \in [0, T], \tag{1}$$

subject to two-point and integral boundary conditions

$$Ax(0) + Bx(T) = \int_0^T g\bigl(s, x(s)\bigr)\, ds \tag{2}$$

and impulsive conditions

$$x(t_i^+) - x(t_i) = I_i\big(x(t_i)\big), \quad i = 1, 2, \ldots, p, t \in [0, T],$$
$$0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T, \tag{3}$$

where $A, B \in R^{n\times n}$ are given matrices and $\det(A + B) \neq 0$; $f, g : [0, T] \times R^n \to R^n$ and $I_i : R^n \to R^n$ are given functions;

$$\Delta x(t_i) = x\big(t_i^+\big) - x\big(t_i^-\big),$$

where

$$x\big(t_i^+\big) = \lim_{h\to 0+} x(t_i + h), \qquad x\big(t_i^-\big) = \lim_{h\to 0+} x(t_i - h) = x(t_i)$$

are the right- and left-hand limits of $x(t)$ at $t = t_i$, respectively.

The organization of the present paper is as follows. First, we provide the necessary background. Second, theorems on the existence and uniqueness of a solution of problem (1), (2), (3) are established under some sufficient conditions on the nonlinear terms. Third, a simple example of application of the main result of this paper is presented.

2 Preliminaries

In this section, we present some basic definitions and preliminary facts which are used throughout the paper. By $C([0, T], R^n)$, we denote the Banach space of all vector continuous functions $x(t)$ from $[0, T]$ into R^n with the norm

$$\|x\| = \max\big\{\big|x(t)\big| : t \in [0, T]\big\},$$

where $|\cdot|$ is the norm in the space R^n.

We consider the linear space

$$PC\big([0, T], R^n\big) = \big\{x : [0, T] \to R^n : x(t) \in C\big((t_i, t_{i+1}], R^n\big), i = 0, 1, \ldots, p,$$
$$x\big(t_i^-\big) \text{ and } x\big(t_i^+\big) \text{ exist } i = 1, \ldots, p \text{ and } x\big(t_i^-\big) = x(t_i)\big\}.$$

$PC([0, T], R^n)$ is a Banach space with the norm

$$\|x\|_{PC} = \max\big\{\|x\|_{(t_i,t_{i+1}]}, i = 0, 1, \ldots, p\big\}.$$

We define a solution of problem (1), (2) and (3) as follows.

Definition 2.1 A function $x \in PC([0, T], R^n)$ is said to be a solution of problem (1), (2) and (3) if

$$\dot{x}(t) = f\big(t, x(t)\big)$$

for each $t \in [0, T]$, $t \neq t_i$, $i = 1, 2, \ldots, p$, and for each

$$i = 1, 2, \ldots, p, \quad x\big(t_i^+\big) - x(t_i) = I_i\big(x(t_i)\big), \quad 0 < t_1 < t_2 < \cdots < t_p < T$$

and boundary condition (2) are satisfied.

Lemma 2.1 *Let* $y, g \in C([0,T], R^n)$ *and* $a_i \in R^n$, $i = 1, 2, \ldots, p$. *Then the boundary value problem for the impulsive differential equation*

$$\dot{x}(t) = y(t), \quad t \in [0,T], t \neq t_i, i = 1, 2, \ldots, p, \tag{4}$$

$$x(t_i^+) - x(t_i) = a_i, \quad i = 1, 2, \ldots, p, 0 < t_1 < t_2 < \cdots < t_p < T, \tag{5}$$

$$Ax(0) + Bx(T) = \int_0^T g(s)\,ds \tag{6}$$

has a unique solution $x(t) \in PC([0,T], R^n)$ *given by*

$$x(t) = C + \int_0^T K(t,\tau)y(\tau)\,d\tau + \sum_{0<t_k<T} K(t_i, t_k)a_k$$

for $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, p$, *where*

$$K(t,\tau) = \begin{cases} (A+B)^{-1}A, & 0 \leq \tau < t, \\ -(A+B)^{-1}B, & t \leq \tau \leq T, \end{cases}$$

$$C = (A+B)^{-1} \int_0^T g(s)\,ds.$$

Proof Assume that $x(t)$ is a solution of boundary value problem (4)-(6), then integrating equation (4) for $t \in (0, t_{j+1})$, we get

$$\begin{aligned} \int_0^t y(s)\,ds &= \int_0^t x'(s)\,ds \\ &= \left[x(t_1) - x(0^+)\right] + \left[x(t_2) - x(t_1^+)\right] + \cdots + \left[x(t) - x(t_j^+)\right] \\ &= -x(0) - \left[x(t_1^+) - x(t_1)\right] - \left[x(t_2^+) - x(t_2)\right] - \cdots - \left[x(t_j^+) - x(t_j)\right] + x(t). \end{aligned}$$

Using this formula and condition (5), we can write

$$x(t) = x(0) + \int_0^t y(s)\,ds + \sum_{0<t_j<t} a_j. \tag{7}$$

Applying formula (7) and condition (6), we get

$$(A+B)x(0) = \int_0^T g(t)\,dt - B\int_0^T y(t)\,d\tau\,dt - B\sum_{0<t_k<T} a_k. \tag{8}$$

Hence, we obtain

$$x(0) = C - (A+B)^{-1}B\int_0^T y(t)\,dt - (A+B)^{-1}B\sum_{0<t_k<T} a_k. \tag{9}$$

From formulas (7) and (9), it follows

$$\begin{aligned}
x(t) &= C - (A+B)^{-1}B\int_0^T y(t)\,dt - (A+B)^{-1}B\sum_{0<t_k<T} a_k + \int_0^t y(s)\,ds + \sum_{0<t_j<t} a_j \\
&= C + \int_0^T K(t,\tau)y(\tau)\,d\tau + \sum_{0<t_k<T} K(t_j,t_k)a_k.
\end{aligned}$$

Therefore we can state that

$$x(t) = C + \int_0^T K(t,\tau)y(\tau)\,d\tau + \sum_{0<t_k<T} K(t_i,t_k)a_k \quad \text{for } t \in (t_i,t_{i+1}], i = 0,1,\dots,p. \tag{10}$$

Lemma 2.1 is established. □

Remark 2.1 Note that for solution (7) we have that:

i. $C = (A+B)^{-1}\int_0^T g(s)\,ds$ is the solution of $\dot{x}(t) = 0$ with nonlocal boundary condition (6);
ii. The function $\int_0^T K(t,\tau)y(\tau)\,d\tau$ is the solution of $\dot{x}(t) = y(t)$ with the nonlocal boundary condition $Ax(0) + Bx(T) = 0$. Here $K(t,\tau)$ is Green's function of this problem;
iii. The functions $\sum_{0<t_k<T} K(t_i,t_k)a_k$, $i = 1,2,\dots,p$, are the solution of $\dot{x}(t) = 0$ with the nonlocal boundary condition $Ax(0) + Bx(T) = 0$ and are jumps (5).

Lemma 2.2 *Assume that $f, g \in C([0,T] \times R^n, R^n)$ and $I_k(x) \in C(R^n)$, then the function $x(t)$ is a solution of impulsive boundary value problem* (1), (2) *and* (3) *if and only if $x(t)$ is a solution of the impulsive integral equation*

$$x(t) = \int_0^T K(t,s)f\bigl(s,x(s)\bigr)\,ds + (A+B)^{-1}\int_0^T g\bigl(s,x(s)\bigr)\,ds + \sum_{k=1}^{p} K(t_i,t_k)I_k\bigl(x(t_k)\bigr) \tag{11}$$

for $t \in (t_i,t_{i+1}]$, $i = 0,1,\dots,p$.

Proof Let $x(t)$ be a solution of boundary value problem (1), (2) and (3), then in the same way as in Lemma 2.1, we can prove that it is a solution of impulsive integral equation (11). By direct verification we can show that the solution of impulsive integral equation (11) also satisfies equation (1) and nonlocal boundary condition (3). Also, it is easy to verify that it satisfies condition (2). Lemma 2.2 is proved. □

3 Main results

The first main statement of the present study is the existence and uniqueness of boundary value problem (1), (2) and (3), a result that is based on a Banach fixed point theorem.

Theorem 3.1 *Assume that*:

(H1) *There exists a constant $N > 0$ such that*

$$\bigl|f(t,x) - f(t,y)\bigr| \le N|x-y|$$

for any $t \in [0,T]$ and all $x, y \in R^n$.

(H2) *There exists a constant* $M > 0$ *such that*

$$\left|g(t,x) - g(t,y)\right| \le M|x-y|$$

for any $t \in [0,T]$ *and all* $x, y \in R^n$.

(H3) *There exist constants* $l_i > 0$, $i = 1,2,\ldots,p$, *such that*

$$\left|I_i(x) - I_i(y)\right| \le l_i|x-y|$$

for all $x, y \in R^n$.

If

$$L = \left[S\left(NT + \sum_{k=1}^{p} l_k\right) + MT\left\|(A+B)^{-1}\right\|\right] < 1, \tag{12}$$

then boundary value problem (1), (2) *and* (3) *has a unique solution on* $[0,T]$. *Here*

$$S = \max\left\{\left\|(A+B)^{-1}A\right\|, \left\|(A+B)^{-1}B\right\|\right\}.$$

Proof We will transform problem (1), (2) and (3) into a fixed point problem. Consider the operator

$$F : PC\left([0,T],R^n\right) \to PC\left([0,T],R^n\right),$$

defined by

$$F(x)(t) = \int_0^T K(t,s)f\left(s,x(s)\right)ds + (A+B)^{-1}\int_0^T g\left(s,x(s)\right)ds + \sum_{k=1}^{p} K(t_i,t_k)I_k\left(x(t_k)\right) \tag{13}$$

for $t \in (t_i, t_{i+1}]$, $i = 0,1,\ldots,p$.

Clearly, the fixed points of the operator F are solutions of problem (1), (2) and (3). We will use the Banach contraction principle to prove that F defined by (13) has a fixed point. We will show that F is a contraction.

Let $x, y \in PC([0,T],R^n)$. Then, for each $t \in (t_i, t_{i+1}]$, we have that

$$\begin{aligned}\left|F(x)(t) - F(y)(t)\right| \le{} & \int_0^T \left|K(t,s)\right|\left|f\left(s,x(s)\right) - f\left(s,y(s)\right)\right|ds \\ & + \left\|(A+B)^{-1}\right\|\int_0^T \left|g\left(s,x(s)\right) - g\left(s,y(s)\right)\right|ds \\ & + \sum_{k=1}^{p}\left|K(t_i,t_k)\right|\left|I_k\left(x(t_k)\right) - I_k\left(y(t_k)\right)\right| \\ \le{} & SNT\|x-y\| + M\left\|(A+B)^{-1}\right\|\|x-y\| + S\sum_{k=1}^{p} l_k\left|x(t_k) - y(t_k)\right| \\ \le{} & \left[S\left(NT + \sum_{k=1}^{p} l_k\right) + MT\left\|(A+B)^{-1}\right\|\right]\|x-y\|_{PC}.\end{aligned}$$

Thus

$$\left\|F(x)(t) - F(y)(t)\right\|_{PC} \le L\|x - y\|_{PC}.$$

Consequently, by assumption (12) the operator F is a contraction. As a consequence of the Banach fixed point theorem, we deduce that the operator F has a fixed point which is a solution of problem (1), (2) and (3). Theorem 3.1 is established. □

The second main statement of the present study is an existence result for boundary value problem (1), (2) and (3) that is based on the Schaefer fixed point theorem.

Theorem 3.2 *Assume that*:

(H4) *The function* $f : [0, T] \times R^n \to R^n$ *is continuous.*

(H5) *There exists a constant* $N_1 > 0$ *such that*

$$\left|f(t,x)\right| \le N_1 \quad \textit{for any } t \in [0,T] \textit{ and all } x \in R^n.$$

(H6) *The function* $g : [0, T] \times R^n \to R^n$ *is continuous.*

(H7) *There exists a constant* $N_2 > 0$ *such that*

$$\left|g(t,x)\right| \le N_2 \quad \textit{for any } t \in [0,T] \textit{ and all } x \in R^n.$$

(H8) *The functions* $I_k(x)$, $x \in R^n$, $k = 1, 2, \dots, p$, *are continuous and there exists a constant* $N_3 > 0$ *such that*

$$\max_{k\in\{1,\dots,p\}} \left|I_k(x)\right| \le N_3$$

for all $x \in R^n$.

Then boundary value problem (1), (2) *and* (3) *has at least one solution on* $[0, T]$.

Proof We will divide the proof into four main steps in which we will show that under the assumptions of the theorem, the operator F has a fixed point.

Step 1. The operator F under the assumptions of the theorem is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC([0, T], R^n)$. Then, for any $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, p$,

$$\begin{aligned}
&\left|F(x_n)(t) - F(x)(t)\right| \\
&\quad\le \int_0^T \left|K(t,s)\right|\left|f\bigl(s,x_n(s)\bigr) - f\bigl(s,x(s)\bigr)\right| ds \\
&\qquad + \left\|(A+B)^{-1}\right\| \int_0^T \left|g\bigl(s,x_n(s)\bigr) - g\bigl(s,x(s)\bigr)\right| ds + \sum_{k=1}^{p} \left|K(t_i,t_k)\right|\left|I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x(t_k)\bigr)\right| \\
&\quad\le ST \max_{s\in[0,T]} \left|f\bigl(s,x_n(s)\bigr) - f\bigl(s,x(s)\bigr)\right| \\
&\qquad + \left\|(A+B)^{-1}\right\| T \max_{s\in[0,T]} \left|g\bigl(s,x_n(s)\bigr) - g\bigl(s,x(s)\bigr)\right| + S\sum_{k=1}^{p} \left|I_k\bigl(x_n(t_k)\bigr) - I_k\bigl(x(t_k)\bigr)\right|.
\end{aligned}$$

Since f, g and I_k, $k = 1, 2, \dots, p$, are continuous functions, we have

$$\left\| F(x_n)(t) - F(x)(t) \right\|_{PC} \to 0$$

as $n \to \infty$.

Step 2. The operator F maps bounded sets in bounded sets in $PC([0, T], R^n)$. Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l such that for any $x \in B_\eta = \{x \in PC([0, T], R^n) : \|x\| \le \eta\}$, we have $\|F(x(\cdot))\| \le l$. Applying the triangle inequality, assumptions (H5), (H7) and (H8), for $t \in (t_i, t_{i+1}]$, we obtain

$$\begin{aligned} \left|F(x)(t)\right| \le{}& \int_0^T \left|K(t,s)\right| \left|f\bigl(s, x(s)\bigr)\right| ds + \left\|(A+B)^{-1}\right\| \int_0^T \left|g\bigl(s, x(s)\bigr)\right| ds \\ &+ \sum_{k=1}^{p} \left|K(t_i, t_k)\right| \left|I_k(t_k)\right| \end{aligned}$$

for any $t \in [0, T]$. Hence,

$$\left|F(x)(t)\right| \le STN_1 + N_2 T \left\|(A+B)^{-1}\right\| + SpN_3.$$

Thus

$$\left\|F(x)(t)\right\|_{PC} \le S[TN_1 + pN_3] + N_2 T \left\|(A+B)^{-1}\right\| = l.$$

Step 3. The operator F maps bounded sets into equicontinuous sets of $PC([0, T], R^n)$.

Let $\tau_1, \tau_2 \in (t_i, t_{i+1}]$, $\tau_1 < \tau_2$, B_η be a bounded set of $PC([0, T], R^n)$ as in Step 2, and let $x \in B_\eta$. We have that

$$\begin{aligned} &F(x)(\tau_2) - F(x)(\tau_1) \\ &\quad = (A+B)^{-1} A \int_0^{\tau_2} f\bigl(s, x(s)\bigr) ds - (A+B)^{-1} B \int_{\tau_2}^{T} f\bigl(s, x(s)\bigr) ds \\ &\qquad - (A+B)^{-1} A \int_0^{\tau_1} f\bigl(s, x(s)\bigr) ds + (A+B)^{-1} B \int_{\tau_1}^{T} f\bigl(s, x(s)\bigr) ds \\ &\quad = (A+B)^{-1} A \int_{\tau_1}^{\tau_2} f\bigl(s, x(s)\bigr) ds + (A+B)^{-1} B \int_{\tau_1}^{\tau_2} f\bigl(s, x(s)\bigr) ds. \end{aligned}$$

Then, applying the triangle inequality, assumptions (H5), (H7) and (H8), we obtain

$$\begin{aligned} &\left|F(x)(\tau_2) - F(x)(\tau_1)\right| \\ &\quad \le \left\|(A+B)^{-1} A\right\| \int_{\tau_1}^{\tau_2} \left|f\bigl(s, x(s)\bigr)\right| ds + \left\|(A+B)^{-1} B\right\| \int_{\tau_1}^{\tau_2} \left|f\bigl(s, x(s)\bigr)\right| ds \\ &\quad \le 2S \int_{\tau_1}^{\tau_2} \left|f\bigl(s, x(s)\bigr)\right| ds. \end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that the operator $F : PC([0, T], R^n) \to PC([0, T], R^n)$ is completely continuous.

Step 4. *A priori* bounds. Now, it remains to show that the set

$$\Delta = \{x \in PC([0,T], R^n) : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $x = \lambda(Fx)$ for some $0 < \lambda < 1$. Then, for any $t \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, p$, we have

$$x(t) = \lambda\left[\int_0^T K(t,s) f(s, x(s))\,ds + (A+B)^{-1}\int_0^T g(s, x(s))\,ds + \sum_{k=1}^{p} K(t_i, t_k) I_k(x(t_k))\right].$$

This implies by (H5), (H7) and (H8) (as in Step 2) that for any $t \in [0,T]$, we have

$$|F(x)(t)| \le [N_1 T + pN_3]S + N_2 T\|(A+B)^{-1}\|.$$

Therefore, for every $t \in [0,T]$, we have that

$$\|x\|_{PC} \le [N_1 T + pN_3]S + N_2 T\|(A+B)^{-1}\| = R.$$

This shows that the set Δ is bounded. As a consequence of the Schaefer fixed point theorem, we deduce that F has a fixed point which is a solution of problem (1), (2) and (3). Theorem 3.2 is established. □

In the following theorem, we give an existence result for problem (1), (2) and (3) by means of an application of a Leray-Schauder type nonlinear alternative, where the conditions (H5), (H7) and (H8) are weakened.

Theorem 3.3 *Assume that* (H4), (H6) *and the following conditions hold*:

(H9) *There exist* $\theta_f \in L^1([0,T], R^+)$ *and a continuous and nondecreasing* $\psi_f : [0,\infty) \to [0,\infty)$ *such that*

$$|f(t,x)| \le \theta_f(t)\psi_f(|x|) \quad \textit{for any } t \in [0,T] \textit{ and all } x \in R^n.$$

(H10) *There exist* $\theta_g \in L^1([0,T], R^+)$ *and a continuous nondecreasing* $\psi_g : [0,\infty) \to [0,\infty)$ *such that*

$$|g(t,x)| \le \theta_g(t)\psi_g(|x|) \quad \textit{for every } t \in [0,T] \textit{ and all } x \in R^n.$$

(H11) *There exist* $\psi : [0,\infty) \to [0,\infty)$ *and a continuous and nondecreasing function such that*

$$|I_k(x)| \le \psi(|x|)$$

for all $x \in R^n$.

(H12) *There exists a number* $K > 0$ *such that*

$$\frac{K}{S\psi_f(K)\int_0^T \theta_f(t)\,dt + \|(A+B)^{-1}\|\psi_g(K)\int_0^T \theta_g(t)\,dt + Sp\psi(K)} > 1.$$

Then boundary value problem (1), (2) *and* (3) *has at least one solution on* $[0,T]$.

Proof Consider the operator F defined in Theorems 3.2 and 3.3. It can be easily shown that F is continuous and completely continuous. For $\lambda \in [0,1]$ let x be such that for each $t \in (t_i, t_{i+1}]$, $i = 0,1,\dots,p$, we have $x(t) = \lambda(Fx)(t)$. Then from (H9)-(H11) we have

$$\begin{aligned} |x(t)| &\le \int_0^T |K(t,s)| \left| f\left(s, x(s)\right)\right| ds + \left\|(A+B)^{-1}\right\| \int_0^T \left| g\left(s, x(s)\right)\right| ds \\ &\quad + \sum_{k=1}^{p} |K(t_i, t_k)| \left| I_k\left(x(t_k)\right)\right| \\ &\le S\psi_f\left(\|x\|\right) \int_0^T \theta_f(t)\,dt + \left\|(A+B)^{-1}\right\| \psi_g\left(\|x\|\right) \int_0^T \theta_g(t)\,dt + Sp\psi\left(\|x\|\right) \end{aligned}$$

for each $t \in (t_i, t_{i+1}]$, $i = 0,1,\dots,p$. Therefore,

$$\frac{\|x\|_{PC}}{S\psi_f(\|x\|_{PC}) \int_0^T \theta_f(t)\,dt + \|(A+B)^{-1}\| \psi_g(\|x\|_{PC}) \int_0^T \theta_g(t)\,dt + Sp\psi(\|x\|_{PC})} \le 1.$$

Then, by condition (H12), there exists K such that $\|x\| \ne K$.

Let

$$U = \left\{x \in PC\left([0,T], R^n\right) : \|x\| < K\right\}.$$

The operator $F : \overline{U} \to PC([0,T], R)$ is continuous and completely continuous. By the choice of U, there exists no $x \in \partial U$ such that $x = \lambda F(x)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [19], we deduce that F has a fixed point x in $\overline{U}$, which is a solution of problem (1), (2) and (3). Theorem 3.2 is proved. □

4 An example

Now, we give an example to illustrate the usefulness of our main results. Let us consider the following nonlocal boundary value problem for a system of impulsive differential equations:

$$\begin{cases} \dot{x}_1(t) = \frac{1}{5}\cos x_2(t), & t \in (0,1), \\ \dot{x}_2(t) = \frac{1}{5}\sin x_1(t), & t \in (0,1), \\ x_1(0) = \int_0^1 \cos(0.1x_2(t))\,dt, & x_2(1) = 1, \\ \Delta x_1(0.2) = \frac{1}{10}x_2(0.2). \end{cases} \tag{14}$$

Evidently, $T = p = 1$, $A + B = E$, $\|(A+B)^{-1}\| = 1$ and $S = 1$. Hence, the conditions (H1)-(H2) hold with $N = M = 0.2$; $l_1 = 0.1$. We can easily see that condition (4) is satisfied. Indeed,

$$L = \left[S\left(NT + \sum_{i=1}^{p} l_i\right) + MT\left\|(A+B)^{-1}\right\|\right] = 0.2 + 0.1 + 0.2 = 0.5 < 1, \tag{15}$$

then by Theorem 3.1 boundary value problem (14) has a unique solution on $[0,1]$.

5 Conclusion

In this work, some existence and uniqueness of a solution results have been established for the system of nonlinear impulsive differential equations with two-point and integral boundary conditions under some sufficient conditions on the nonlinear terms. These statements without proof are formulated in [20]. Of course, such type of existence and uniqueness results hold under the same sufficient conditions on the nonlinear terms for the system of nonlinear impulsive differential equations (1), subject to multipoint nonlocal and integral boundary conditions

$$Ex(0) + \sum_{j=1}^{J} B_j x(\lambda_j) = \int_0^T g\bigl(s, x(s)\bigr)\, ds \tag{16}$$

and impulsive conditions

$$x\bigl(t_i^+\bigr) - x(t_i) = I_i\bigl(x(t_i)\bigr), \quad i = 1, 2, \ldots, p, 0 < t_1 < t_2 < \cdots < t_p < T, \tag{17}$$

where $B_j \in R^{n\times n}$ are given matrices and $\sum_{j=1}^{J} \|B_j\| < 1$. Here, $0 < \lambda_1 < \cdots < \lambda_J \le T$.

Moreover, applying the result of the paper [21], the single-step difference schemes for the numerical solution of nonlocal boundary value problem (1), (16) and (17) can be presented. Of course, such type of existence and uniqueness results hold under some sufficient conditions on the nonlinear terms for the solution of the system of these difference schemes.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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