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Existence of positive solutions of advanced differential equations

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Abstract

In this paper, we study the advanced differential equations

$$\left[r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right]'+\sum_{i=1}^{n}p_i(t)\left|x\left(t+\tau_i(t)\right)\right|^{\alpha-1}x\left(t+\tau_i(t)\right)=0$$

and

$$\left[r(t)\left(y(t)-P(t)y(t-\tau)\right)'\right]'+\sum_{i=1}^{n}p_i(t)f\left(y(t+\sigma)\right)=0.$$

By using the generalized Riccati transformation and the Schauder-Tyichonoff theorem, we establish the conditions for the existence of positive solutions of the above equations.

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1 Introduction

In the last years, oscillation and nonoscillation of differential equations attracted a considerable attention. Many results have been obtained, and we refer the reader to the papers [1–20].

In 2008, Luo *et al.* [11] investigated the existence of positive periodic solutions of the following two kinds of neutral functional differential equations:

$$\left(x(t)-cx\left(t-\tau(t)\right)\right)'=-a(t)x(t)+f\left(t,x\left(t-\tau(t)\right)\right)$$

and

$$\left(x(t)-c\int_{-\infty}^{0}Q(r)x(t+r)\,dr\right)'=-a(t)x(t)+b(t)\int_{-\infty}^{0}Q(r)f\left(t,x(t+r)\right)dr,$$

where $a,b\in C(R,(0,\infty))$, $\tau\in C(R,R)$, $f\in C(R\times R,R)$, and $a(t)$, $b(t)$, $\tau(t)$, $f(t,x)$ are ω-periodic functions, $\omega>0$, $Q(r)\in C((-\infty,0],[0,\infty))$, $\int_{-\infty}^{0}Q(r)\,dr=1$, and $\omega,|c|<1$ are constants.

Péics *et al.* [15] obtained the existence of positive solutions of half-linear delay differential equations

$$\left[\left|x'(t)\right|^{\alpha-1}x'(t)\right]' + \sum_{i=1}^{n} p_i(t)\left|x\left(t-\tau_i(t)\right)\right|^{\alpha-1}x\left(t-\tau_i(t)\right) = 0,$$

where $t \geq t_0$ and $\alpha > 0$, $\tau_i(t) \leq t$.

Zhang *et al.* [19] obtained the existence of nonoscillatory solutions of the first-order linear neutral delay differential equation

$$\left[x(t) + P(t)x(t-\tau)\right]' + Q_1(t)x(t-\sigma_1) - Q_2(t)x(t-\sigma_2) = 0,$$

where $P \in C([t_0,\infty),R)$, $\tau \in (0,\infty)$, $\sigma_1,\sigma_2 \in [0,\infty)$, $Q_1, Q_2 > 0$.

In this paper, we consider the advanced differential equation

$$\left[r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right]' + \sum_{i=1}^{n} p_i(t)\left|x\left(t+\tau_i(t)\right)\right|^{\alpha-1}x\left(t+\tau_i(t)\right) = 0, \tag{1.1}$$

where $t \geq t_0$ and $\alpha > 0$.

Throughout this work, we always assume that the following conditions hold:

(H_1) $p_i \in C([t_0,\infty),R)$, $i = 1,2,3,\ldots,n$;
(H_2) $\tau_i \in C([t_0,\infty),R^+)$, $i = 1,2,3,\ldots,n$, and $0 < r(t) \leq k$.

For convenience, we introduce the notation

$$\eta^{\alpha^*} = |\eta|^{\alpha-1}\eta, \quad \alpha > 0. \tag{1.2}$$

It is convenient to rewrite (1.1) in the form

$$\left[r(t)\left|x'(t)\right|^{\alpha^*}\right]' + \sum_{i=1}^{n} p_i(t)\left|x\left(t+\tau_i(t)\right)\right|^{\alpha^*} = 0. \tag{1.3}$$

Definition 1.1 A function x is said to be a solution of Eq. (1.1) if $x \in C^1([T,\infty),R)$, $T \geq t_0$, which has the property $|x'|^{\alpha-1}x' \in C^1([T,\infty),R)$ and it satisfies Eq. (1.1) for $t \geq T$. We say that a solution of Eq. (1.1) is oscillatory if it has arbitrarily large zeros. Otherwise, it is nonoscillatory.

One of the most important methods of the study of nonoscillation is the method of generalized characteristic equation [6]. The method was applied to second-order half-linear equations without delay, for example, in [8, 9]. Concerning cases with advanced, let us apply the Riccati-transformation

$$x(t) = \exp\left(\int_{t_0}^{t} \left(\omega(s)\right)^{\left(\frac{1}{\alpha}\right)^*} ds\right). \tag{1.4}$$

By (1.4), we have

$$x'(t) = \left(\exp\left(\int_{t_0}^{t} \omega(s)^{\left(\frac{1}{\alpha}\right)^*} ds\right)\right)' = \omega(t)^{\left(\frac{1}{\alpha}\right)^*} \exp\left(\int_{t_0}^{t} \omega(s)^{\left(\frac{1}{\alpha}\right)^*} ds\right),$$

$$\bigl(x\bigl(t+\tau_i(t)\bigr)\bigr)^{\alpha^*}=\exp\biggl(\alpha^*\int_{t_0}^{t+\tau_i(t)}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr).$$

From (1.3), we obtain

$$\biggl[r(t)\omega(t)\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)\biggr]'+\sum_{i=1}^{n}p_i(t)\exp\biggl(\alpha^*\int_{t_0}^{t+\tau_i(t)}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)=0. \tag{1.5}$$

Since

$$\begin{aligned}
&\biggl[r(t)\omega(t)\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)\biggr]'\\
&\quad=\bigl(r(t)\omega(t)\bigr)'\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)+r(t)\omega(t)\biggl(\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)\biggr)'\\
&\quad=\bigl(r'(t)\omega(t)+r(t)\omega'(t)\bigr)\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)+\alpha^*r(t)\bigl|\omega(t)\bigr|^{1+\frac{1}{\alpha}}\\
&\qquad\times\exp\biggl(\alpha^*\int_{t_0}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr),
\end{aligned}$$

it is convenient to rewrite (1.5) in the form

$$r'(t)\omega(t)+r(t)\omega'(t)+\alpha r(t)\bigl|\omega(t)\bigr|^{1+\frac{1}{\alpha}}+\sum_{i=1}^{n}p_i(t)\exp\biggl(\alpha\int_{t}^{t+\tau_i(t)}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr)=0. \tag{1.6}$$

2 Preliminaries

Lemma 2.1 *Suppose that* (H_1) *and* (H_2) *hold. Then the following statements are equivalent*:

(i) *Eq.* (1.1) *has an eventually positive solution*;

(ii) *There is a function* $\omega\in C^1([T,\infty),R)$, $T\ge t_0$, *such that* ω *solves the Riccati equation* (1.6).

Proof (i) $\Rightarrow$ (ii). Let x be an eventually positive solution of Eq. (1.1) such that $x(t)>0$ for $t\ge T\ge t_0$. The function ω defined by

$$\omega(t)=\biggl(\frac{x'(t)}{x(t)}\biggr)^{\alpha^*},\quad t\ge T,$$

is continuous.

We will show that it is a solution of (1.6) on $[T,\infty)$. By (1.2) and observing that

$$\begin{aligned}
&\omega(t)=\biggl(\frac{x'(t)}{x(t)}\biggr)^{\alpha^*}=\biggl|\frac{x'(t)}{x(t)}\biggr|^{\alpha-1}\frac{x'(t)}{x(t)},\\
&\frac{x'(t)}{x(t)}=\bigl|\omega(t)\bigr|^{\frac{1}{\alpha}-1}\omega(t)=\omega(t)^{(\frac{1}{\alpha})^*},
\end{aligned}$$

it follows that

$$x(t)=x(T)\exp\biggl(\int_{T}^{t}\omega(s)^{(\frac{1}{\alpha})^*}\,ds\biggr).$$

Dividing both sides of (1.1) by $|x(t)|^{\alpha-1}x(t)$ gives that

$$\frac{[r(t)|x'(t)|^{\alpha-1}x'(t)]'}{|x(t)|^{\alpha-1}x(t)} + \sum_{i=1}^{n} p_i(t)\frac{|x(t+\tau_i(t))|^{\alpha-1}x(t+\tau_i(t))}{|x(t)|^{\alpha-1}x(t)} = 0. \tag{2.1}$$

From the definition of ω, we obtain

$$\left|x'(t)\right|^{\alpha-1}x'(t) = \omega(t)\left|x(t)\right|^{\alpha-1}x(t) = \omega(t)x^{\alpha}(t).$$

Further

$$\begin{aligned}\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' &= \left(r(t)\omega(t)x^{\alpha}(t)\right)' \\ &= r'(t)\omega(t)x^{\alpha}(t) + r(t)\omega'(t)x^{\alpha}(t) + \alpha r(t)\omega(t)x^{\alpha-1}(t)x'(t)\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\frac{x(t+\tau_i(t))}{x(t)} = \exp\left(\int_t^{t+\tau_i(t)} \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right) > 0,\\ &\left|\frac{x(t+\tau_i(t))}{x(t)}\right|^{\alpha^*} = \exp\left(\alpha\int_t^{t+\tau_i(t)} \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right).\end{aligned} \tag{2.3}$$

By substituting (2.2), (2.3) into (2.1), we get

$$\begin{aligned}&\frac{[r(t)|x'(t)|^{\alpha-1}x'(t)]'}{|x(t)|^{\alpha-1}x(t)} + \sum_{i=1}^{n} p_i(t)\frac{|x(t+\tau_i(t))|^{\alpha-1}x(t+\tau_i(t))}{|x(t)|^{\alpha-1}x(t)}\\ &\quad= \frac{r'(t)\omega(t)x^{\alpha}(t) + r(t)\omega'(t)x^{\alpha}(t) + \alpha r(t)\omega(t)x^{\alpha-1}(t)x'(t)}{x^{\alpha}(t)}\\ &\qquad+ \sum_{i=1}^{n} p_i(t)\exp\left(\alpha\int_t^{t+\tau_i(t)} \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right)\\ &\quad= 0.\end{aligned}$$

We obtain (1.6), and the proof of (i) $\Rightarrow$ (ii) is complete.

(ii) $\Rightarrow$ (i). Let ω be a continuously differentiable solution of Eq. (1.6) for $t \geq T \geq t_0$.

We show that a function x defined by

$$x(t) = \exp\left(\int_T^t \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right)$$

is the solution of Eq. (1.1).

Since

$$\begin{aligned}&\frac{x'(t)}{x(t)} = \omega(t)^{(\frac{1}{\alpha})^*},\\ &\left(x'(t)\right)^{\alpha^*} = \left(x(t)\right)^{\alpha^*}\omega(t) = x^{\alpha}(t)\omega(t).\end{aligned}$$

By (1.6), we obtain

$$\begin{aligned}
\big[r(t)\big(x'(t)\big)^{\alpha^*}\big]' &= \big(r(t)\omega(t)x^\alpha(t)\big)' \\
&= r'(t)\omega(t)x^\alpha(t) + r(t)\omega'(t)x^\alpha(t) + \alpha r(t)\omega(t)x^{\alpha-1}(t)x'(t) \\
&= r'(t)\omega(t)x^\alpha(t) + r(t)\omega'(t)x^\alpha(t) + \alpha r(t)x^\alpha(t)\big|\omega(t)\big|^{1+\frac{1}{\alpha}} \\
&= -\sum_{i=1}^n p_i(t)\exp\left(\alpha\int_t^{t+\tau_i(t)} \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right)\exp\left(\alpha\int_T^t \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right) \\
&= -\sum_{i=1}^n p_i(t)\exp\left(\alpha\int_T^{t+\tau_i(t)} \omega(s)^{(\frac{1}{\alpha})^*}\,ds\right) \\
&= -\sum_{i=1}^n p_i(t)x^\alpha\big(t+\tau_i(t)\big),
\end{aligned}$$

thus,

$$\big[r(t)\big(x'(t)\big)^{\alpha^*}\big]' + \sum_{i=1}^n p_i(t)x^\alpha\big(t+\tau_i(t)\big) = 0, \quad t \ge T.$$

The proof of (ii) $\Rightarrow$ (i) is complete. The proof is complete. □

Lemma 2.2 *Suppose that* (H_1) *and* (H_2) *hold. The following statements are equivalent*:

(a) *There is a solution* $\omega \in C^1([T,\infty),R)$ *of the Riccati equation* (1.6) *for some* $T \ge t_0$ *such that*

$$\int_t^\infty \left[\sum_{i=1}^n p_i(s)\exp\left(\alpha\int_s^{s+\tau_i(s)} \omega(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds < \infty. \tag{2.4}$$

(b) *There is a function* $u \in C([T,\infty),R)$ *for some* $T \ge t_0$ *such that*

$$\begin{aligned}
u(t) = \frac{1}{r(t)}\Bigg\{&\alpha\int_t^\infty r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds \\
&+ \int_t^\infty \left[\sum_{i=1}^n p_i(s)\exp\left(\alpha\int_s^{s+\tau_i(s)} \omega(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\Bigg\}.
\end{aligned} \tag{2.5}$$

Proof (a) $\Rightarrow$ (b). Let $\omega = u$ be a solution of Eq. (1.6) for $t \ge T \ge t_0$ and with the property (2.4). Let $t_1 \ge t \ge T$ be fixed arbitrarily and integrate (1.6) over $[t,t_1]$:

$$\begin{aligned}
u(t_1)r(t_1) - u(t)r(t) = {}&-\alpha\int_t^{t_1} r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds \\
&- \int_t^{t_1}\left[\sum_{i=1}^n p_i(s)\exp\left(\alpha\int_s^{s+\tau_i(s)} \big|u(\xi)\big|^{\frac{1}{\alpha}-1}u(\xi)\,d\xi\right)\right]ds.
\end{aligned} \tag{2.6}$$

We claim that

$$\int_t^\infty r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds < \infty. \tag{2.7}$$

Assuming the contrary, if $\int_t^\infty r(s)|u(s)|^{1+\frac{1}{\alpha}}\,ds = \infty$, then in view of (2.6) there is $T_1 \geq t$ such that

$$\begin{aligned} &u(t_1)r(t_1) + \alpha \int_{T_1}^{t_1} r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds \\ &\quad= u(t)r(t) - \alpha \int_t^{T_1} r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds \\ &\qquad - \int_t^{t_1} \left[\sum_{i=1}^n p_i(s) \exp\left(\alpha \int_s^{s+\tau_i(s)} \big|u(\xi)\big|^{\frac{1}{\alpha}-1} u(\xi)\,d\xi\right)\right] ds \\ &\quad\leq -1 \end{aligned}$$

for $t_1 \geq T_1 \geq t$, or equivalently,

$$-u(t_1)r(t_1) \geq 1 + \alpha \int_{T_1}^{t_1} r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds, \quad t_1 \geq T_1. \tag{2.8}$$

Then we have

$$u(t_1) \leq 0.$$

From $u(t) = (\frac{x'(t)}{x(t)})^{\alpha^*}$, it follows that $x'(t_1) < 0$, $t_1 \geq T_1$. Dividing both sides of (2.8) by $1 + \alpha \int_{T_1}^{t_1} r(s)|u(s)|^{1+\frac{1}{\alpha}}\,ds > 0$ gives that

$$\frac{|u(t_1)|^{1+\frac{1}{\alpha}} r(t_1)}{1 + \alpha \int_{T_1}^{t_1} r(s)|u(s)|^{1+\frac{1}{\alpha}}\,ds} \geq \big(-u(t_1)\big)^{\frac{1}{\alpha}} = -\frac{x'(t_1)}{x(t_1)}, \quad t_1 \geq T_1. \tag{2.9}$$

Integrating the above inequality over $[T_1, t_1]$ then yields

$$\frac{1}{\alpha} \ln\left(1 + \alpha \int_{T_1}^{t_1} r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds\right) \geq \ln\left(\frac{x(T_1)}{x(t_1)}\right).$$

Combining with (2.8), we have

$$\big(-r(t_1)u(t_1)\big)^{\frac{1}{\alpha}} \geq \frac{x(T_1)}{x(t_1)}, \quad t_1 \geq T_1$$

and

$$-r^{\frac{1}{\alpha}}(t_1)x'(t_1) \geq x(T_1).$$

Integrating the last inequality and using $0 < r(t) \leq k$, we see that $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the assumption that $x(t)$ is eventually positive. Therefore (2.7) must hold.

Let $t_1 \to \infty$ in (2.6). Using (2.4) and (2.7), we get $\lim_{t_1\to\infty} r(t_1)u(t_1) = 0$. So,

$$u(t) = \frac{1}{r(t)} \left\{\alpha \int_t^\infty r(s)\big|u(s)\big|^{1+\frac{1}{\alpha}}\,ds + \int_t^\infty \left[\sum_{i=1}^n p_i(s) \exp\left(\alpha \int_s^{s+\tau_i(s)} u(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right] ds\right\}$$

must hold.

(b) $\Rightarrow$ (a). Assume that there is a function $u(t)$ satisfying Eq. (2.5) on $[T,\infty)$. Differentiation of (2.5) then shows that $u = \omega$ is a solution of (1.6) for $t \geq T$, and it satisfies (2.4). The proof of (b) $\Rightarrow$ (a) is complete. □

3 Main results

Theorem 3.1 *Assume that there exist $T \geq t_0$ and functions $\beta, \gamma \in C([T,\infty), R)$ such that $\beta(t) \leq \gamma(t)$,*

$$\int_t^{\infty}\left[\sum_{i=1}^{n}|p_i(s)|\exp\left(\alpha\int_s^{s+\tau_i(s)}\gamma(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds < \infty. \tag{3.1}$$

$$\beta(t) \leq \nu(t) \leq \gamma(t) \text{ implies that } S\nu \text{ is defined and } \beta(t) \leq (S\nu)(t) \leq \gamma(t) \tag{3.2}$$

for every function $\nu \in C([T,\infty), R)$, where

$$\begin{aligned}(S\nu)(t) = \frac{1}{r(t)}\Bigg\{\alpha\int_t^{\infty} r(s)\big|\nu(s)\big|^{1+\frac{1}{\alpha}}\,ds \\ + \int_t^{\infty}\left[\sum_{i=1}^{n}p_i(s)\exp\left(\alpha\int_s^{s+\tau_i(s)}\nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\Bigg\}.\end{aligned} \tag{3.3}$$

Then there exists a continuous solution $u(t)$ of Eq. (2.5) *which satisfies the inequality $\beta(t) \leq u(t) \leq \gamma(t)$.*

Proof Let T_1 and T_2 be real numbers such that $T \leq T_1 \leq T_2 < \infty$. Then $[T_1, T_2]$ is an arbitrary compact subinterval of $[T,\infty)$ and set

$$L = \max_{T_1\leq t\leq T_2}\big\{\max\big\{|\beta(t)|, |\gamma(t)|\big\}\big\}, \qquad \tau = \max_{T_1\leq t\leq T_2}\Big\{\max_{1\leq i\leq n}\tau_i(t)\Big\},$$

$$L_1 = L^{\frac{1}{\alpha}-1}e^{\alpha\tau L^{\frac{1}{\alpha}}}, \qquad N = \min_{T_1\leq t\leq T_2} r(t),$$

$$M = \max_{T_1\leq t\leq T_2}\sum_{i=1}^{n}|p_i(t)|, \qquad c = \frac{k(\alpha+1)L^{\frac{1}{\alpha}} + ML_1\tau}{N}.$$

Define

$$F = \big\{\nu \in C\big([T,\infty), R\big) \mid \beta(t) \leq \nu(t) \leq \gamma(t), t \in [T,\infty)\big\}.$$

It follows from (3.1) and (3.2), that the operator S is defined for $\nu \in F$ and satisfies

$$\int_t^{\infty} r(\zeta)\big|\nu(\zeta)\big|^{1+\frac{1}{\alpha}}\,d\zeta < \infty. \tag{3.4}$$

By (3.2), we see that the functions in the image set SF are uniformly bounded on any finite interval of $[T,\infty)$.

To prove that the functions in SF are equicontinuous on any finite interval of $[T,\infty)$, we choose the finite interval $[T_1, T_2]$ as before, and let t_1 and t_2 be two arbitrary numbers from $[T_1, T_2]$. Since $\frac{1}{r(t)}$ is continuous on $[T_1, T_2]$, $\forall\varepsilon > 0$, $\exists\delta_1 > 0$, such that for $|t_1 - t_2| < \delta_1$, we have

$$\left|\frac{1}{r(t_1)} - \frac{1}{r(t_2)}\right| < \frac{\varepsilon}{2} \quad\Rightarrow\quad \frac{1}{r(t_1)} < \frac{1}{r(t_2)} + \frac{\varepsilon}{2}.$$

Further,

$$\begin{aligned}
&\left|S\nu(t_1) - S\nu(t_2)\right| \\
&\quad= \left|\frac{1}{r(t_1)}\left\{\alpha\int_{t_1}^{\infty} r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds + \int_{t_1}^{\infty}\left[\sum_{i=1}^{n} p_i(s)\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right\}\right. \\
&\qquad- \frac{1}{r(t_2)}\left\{\alpha\int_{t_2}^{\infty} r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds\right. \\
&\qquad\left.\left.+ \int_{t_2}^{\infty}\left[\sum_{i=1}^{n} p_i(s)\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right\}\right| \\
&\quad\le \left|\frac{1}{r(t_2)}\left\{\alpha\int_{t_1}^{t_2} r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds + \int_{t_1}^{t_2}\left[\sum_{i=1}^{n}|p_i(s)|\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right\}\right| \\
&\qquad+ \frac{\varepsilon}{2}\left|\left\{\alpha\int_{t_1}^{\infty} r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds + \int_{t_1}^{\infty}\left[\sum_{i=1}^{n}|p_i(s)|\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right\}\right| \\
&\quad\le \frac{1}{N}\left(\alpha kL^{1+\frac{1}{\alpha}} + Me^{\alpha\tau L^{\frac{1}{\alpha}}}\right)|t_1 - t_2| \\
&\qquad+ \frac{\varepsilon}{2}\left|\left\{\alpha\int_{t_1}^{\infty} r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds + \int_{t_1}^{\infty}\left[\sum_{i=1}^{n}|p_i(s)|\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right\}\right|.
\end{aligned}$$

Due to (3.1) and (3.4), there exists δ_2 such that for $|t_1 - t_2| < \delta_2$, $|S\nu(t_1) - S\nu(t_2)| < \varepsilon$, hence SF is equicontinuous.

Let the sequence $\{\nu_n(t)\} \in F$ tend to $\nu(t)$ uniformly on any finite interval ($n \to \infty$). In particular, the convergence is uniform on the interval $[T_1, T_2]$. Using the mean value theorem, we have

$$\left|r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}} - r(s)\left|\nu_n(s)\right|^{1+\frac{1}{\alpha}}\right| = \left(1 + \frac{1}{\alpha}\right)r(s)\left|\nu(s) - \nu_n(s)\right|\left|\sigma(s)\right|^{\frac{1}{\alpha}},$$

where $|\sigma(s)|$ is between $|\nu(s)|$ and $|\nu_n(s)|$, and similarly

$$\begin{aligned}
&\exp\left(\alpha\int_{s}^{s+\tau_i(s)} \left|\nu_n(\xi)\right|^{\frac{1}{\alpha}-1}\nu_n(\xi)\,d\xi\right) - \exp\left(\alpha\int_{s}^{s+\tau_i(s)} \left|\nu(\xi)\right|^{\frac{1}{\alpha}-1}\nu(\xi)\,d\xi\right) \\
&\quad= \alpha e^{\sigma_i(s)}\int_{s}^{s+\tau_i(s)} \left(\left|\nu_n(\xi)\right|^{\frac{1}{\alpha}-1}\nu_n(\xi) - \left|\nu(\xi)\right|^{\frac{1}{\alpha}-1}\nu(\xi)\right)d\xi
\end{aligned}$$

for every $i = 1, 2, 3, \ldots, n$ and $T_1 \le s \le T_2$, where $\sigma_i(s)$ is between $\alpha\int_{s}^{s+\tau_i(s)} |\nu(\xi)|^{\frac{1}{\alpha}-1}\nu(\xi)\,d\xi$ and $\alpha\int_{s}^{s+\tau_i(s)} |\nu_n(\xi)|^{\frac{1}{\alpha}-1}\nu_n(\xi)\,d\xi$.

Since $|\sigma_i(s)| \le \alpha\tau L^{\frac{1}{\alpha}}$ for $T_1 \le t \le T_2$, we obtain

$$\begin{aligned}
&\left|\exp\left(\alpha\int_s^{s+\tau_i(s)}\left|\nu_n(\xi)\right|^{\frac{1}{\alpha}-1}\nu_n(\xi)\,d\xi\right)-\exp\left(\alpha\int_s^{s+\tau_i(s)}\left|\nu(\xi)\right|^{\frac{1}{\alpha}-1}\nu(\xi)\,d\xi\right)\right| \\
&\quad\le \alpha e^{\alpha\tau L^{\frac{1}{\alpha}}}\int_s^{s+\tau_i(s)}\left|\left|\nu_n(\xi)\right|^{\frac{1}{\alpha}-1}\nu_n(\xi)-\left|\nu(\xi)\right|^{\frac{1}{\alpha}-1}\nu(\xi)\right|d\xi \\
&\quad\le L^{\frac{1}{\alpha}-1}e^{\alpha\tau L^{\frac{1}{\alpha}}}\int_s^{s+\tau_i(s)}\left|\nu_n(\xi)-\nu(\xi)\right|d\xi.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|S\nu(t)-S\nu_n(t)\right| \\
&\quad\le \lim_{T_2\to\infty}\frac{1}{r(t)}\left\{\alpha\int_t^{T_2}\left|r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}-r(s)\left|\nu_n(s)\right|^{1+\frac{1}{\alpha}}\right|ds+\int_t^{T_2}\left[\sum_{i=1}^n\left|p_i(s)\right|\right.\right. \\
&\qquad\left.\left.\times\left|\exp\left(\alpha\int_s^{s+\tau_i(s)}\left|\nu_n(\xi)\right|^{\frac{1}{\alpha}-1}\nu_n(\xi)\,d\xi\right)-\exp\left(\alpha\int_s^{s+\tau_i(s)}\left|\nu(\xi)\right|^{\frac{1}{\alpha}-1}\nu(\xi)\,d\xi\right)\right|ds\right]\right\} \\
&\quad\le \frac{1}{r(t)}\left[\lim_{T_2\to\infty}(\alpha+1)L^{\frac{1}{\alpha}}\int_t^{T_2}r(s)\left|\nu(s)-\nu_n(s)\right|ds\right. \\
&\qquad\left.+ML_1\int_t^{T_2}\int_s^{s+\tau}\left|\nu(s)-\nu_n(s)\right|d\xi\,ds\right].
\end{aligned}$$

The uniform convergence $\nu_n(t)-\nu(t)\to 0$ on any finite interval of $[T,\infty)$ implies that if n is sufficiently large,

$$\left|\nu(t)-\nu_n(t)\right|<\delta,\quad T_1\le t\le T_2,$$

where $\delta=\frac{\varepsilon}{T_2}$, and hence we obtain

$$\begin{aligned}
\left|S\nu(t)-S\nu_n(t)\right| &\le \frac{1}{r(t)}\left[\lim_{T_2\to\infty}(\alpha+1)L^{\frac{1}{\alpha}}k\delta(T_2-t)+ML_1\tau\delta(T_2-t)\right] \\
&\le \lim_{T_2\to\infty}\frac{1}{N}\left[(\alpha+1)L^{\frac{1}{\alpha}}k+ML_1\tau\right]\delta T_2 \\
&\le \lim_{T_2\to\infty}\frac{1}{N}\left[(\alpha+1)L^{\frac{1}{\alpha}}k+ML_1\tau\right]\varepsilon \\
&= c\varepsilon
\end{aligned}$$

for $T_1\le t\le T_2$. Thus, $S\nu_n(t)\to S\nu(t)$ uniformly on a finite interval.

We obtained that the conditions of the Schauder-Tychonoff theorem are satisfied, hence the mapping S has at least one fixed point ν in F, and because $\nu(t)=(S\nu)(t)$ for $t\ge T$, ν is the continuous solution of Eq. (2.5). □

Theorem 3.2 *Assume that* (H_1), (H_2) *hold and there exists a positive function* $\mu(t)$ *for* $t\ge T\ge t_0$ *such that*

$$\frac{1}{r(t)}\int_t^\infty\left[\alpha r(s)\mu^{1+\frac{1}{\alpha}}(s)+\sum_{i=1}^n\left|p_i(s)\right|\exp\left(\alpha\int_s^{s+\tau_i(s)}\mu^{\frac{1}{\alpha}}(\xi)\,d\xi\right)\right]ds\le\mu(t)\tag{3.5}$$

holds for t large enough. Then Eq. (1.1) *has a positive solution* $x(t)$ *with the property* $|\frac{x'(t)}{x(t)}| \le \mu^{\frac{1}{\alpha}}(t)$.

Proof Let $\mu(t)$ be given such that the conditions of the theorem hold. We show that the conditions of Theorem 3.1 are satisfied with $\beta(t) = -\mu(t)$ and $\gamma(t) = \mu(t)$ for t large enough.

Let $\nu(t)$ be a continuous function such that $|\nu(t)| \le \mu(t)$. It follows from (3.5) that

$$\begin{aligned}\left|S\nu(t)\right| &= \frac{1}{r(t)}\left|\alpha\int_t^\infty r(s)\left|\nu(s)\right|^{1+\frac{1}{\alpha}}\,ds + \int_t^\infty\left[\sum_{i=1}^n p_i(s)\exp\left(\alpha\int_s^{s+\tau_i(s)}\nu(\xi)^{(\frac{1}{\alpha})^*}\,d\xi\right)\right]ds\right| \\ &\le \frac{1}{r(t)}\left[\alpha\int_t^\infty r(s)\mu^{1+\frac{1}{\alpha}}(s)\,ds + \int_t^\infty\sum_{i=1}^n\left|p_i(s)\right|\exp\left(\alpha\int_s^{s+\tau_i(s)}\mu^{(\frac{1}{\alpha})^*}(\xi)\,d\xi\right)ds\right] \\ &\le \mu(t).\end{aligned}$$

Therefore, by Theorem 3.1, Lemma 2.1 and Lemma 2.2, Eq. (1.1) has a positive solution, and the proof is complete. □

Next, we consider neutral differential equations of the form

$$\left[r(t)\left(y(t) - P(t)y(t-\tau)\right)'\right]' + \sum_{i=1}^n p_i(t)f\left(y(t+\sigma)\right) = 0, \quad t \ge t_0. \tag{3.6}$$

We assume that:

(i) $\tau > 0$, $\sigma \ge 0$;

(ii) $r, P, p_i \in C([t_0,\infty),(0,\infty))$, $i = 1,2,\ldots,n$;

(iii) f is nondecreasing continuous function and $xf(x) > 0$, $x \ne 0$.

The following fixed point theorem will be used to prove the main results.

Lemma 3.1 (Schauder's fixed point theorem) *Let* Ω *be a closed, convex and nonempty subset of a Banach space* X. *Let* $T : \Omega \to \Omega$ *be a continuous mapping such that* $T\Omega$ *is a relatively compact subset of* X. *Then* T *has at least one fixed point in* Ω. *That is, there exists an* $x \in \Omega$ *such that* $Tx = x$.

Theorem 3.3 *Suppose that*

$$\int_{t_0}^\infty\sum_{i=1}^n p_i(t)\,dt = \infty \tag{3.7}$$

and there exist $\zeta \ge 0$, $0 < k_1 \le k_2$ *such that*

$$\frac{k_2}{k_1}\exp\left[(k_1 - k_2)\int_{t_0-\zeta}^{t_0}\sum_{i=1}^n p_i(t)\,dt\right] \le 1, \tag{3.8}$$

$$\begin{aligned}&\exp\left(-k_2\int_{t-\tau}^t\sum_{i=1}^n p_i(t)\,dt\right) + \exp\left(k_2\int_{t_0-\zeta}^{t-\tau}\sum_{i=1}^n p_i(s)\,ds\right)\int_t^\infty\frac{1}{r(s)}\int_s^\infty\sum_{i=1}^n p_i(\xi) \\ &\quad\times f\left(\exp\left(-k_1\int_{t_0-\zeta}^{\xi+\sigma}\sum_{i=1}^n p_i(z)\,dz\right)\right)d\xi\,ds\end{aligned}$$

$$\begin{aligned}
&\le P(t) \\
&\le \exp\left(-k_1 \int_{t-\tau}^{t} \sum_{i=1}^{n} p_i(t)\,dt\right) + \exp\left(k_1 \int_{t_0-\zeta}^{t-\tau} \sum_{i=1}^{n} p_i(s)\,ds\right) \int_t^\infty \frac{1}{r(s)} \int_s^\infty \sum_{i=1}^{n} p_i(\xi) \\
&\quad \times f\left(\exp\left(-k_2 \int_{t_0-\zeta}^{\xi+\sigma} \sum_{i=1}^{n} p_i(z)\,dz\right)\right) d\xi\,ds, \quad t \ge t_0.
\end{aligned}$$

Then Eq. (3.6) *has a positive solution which tends to zero.*

Proof First: Choose $\widetilde{T} \ge t_0 + \tau$,

$$u(t) = \exp\left(-k_2 \int_{t_0-\zeta}^{t} \sum_{i=1}^{n} p_i(t)\,dt\right), \qquad v(t) = \exp\left(-k_1 \int_{t_0-\zeta}^{t} \sum_{i=1}^{n} p_i(t)\,dt\right), \quad t \ge t_0.$$

Let $C([t_0,\infty),R)$ be the set of all continuous functions with the norm

$$\|y(t)\| = \sup_{t\ge t_0} |y(t)| < \infty.$$

Then $C([t_0,\infty),R)$ is a Banach space. We define a closed, bounded convex subset Ω of $C([t_0,\infty),R)$ as follows:

$$\Omega = \{y \mid y \in C([t_0,\infty),R) : u(t) \le y(t) \le v(t), t \ge t_0\}.$$

Define the map $\mathrm{T} : \Omega \to C([t_0,\infty),R)$:

$$(\mathrm{T}y)(t) = \begin{cases} P(t)y(t-\tau) - \int_t^\infty \frac{1}{r(s)} \int_s^\infty \sum_{i=1}^{n} p_i(\xi) f(y(\xi+\sigma))\,d\xi\,ds, & t \ge \widetilde{T}, \\ (\mathrm{T}y)(\widetilde{T}) + v(t) - v(\widetilde{T}), & t_0 \le t \le \widetilde{T}. \end{cases}$$

We can show that for any $y \in \Omega$, $\mathrm{T}y \in \Omega$.
Second: We prove that T is continuous.
Third: We show that $\mathrm{T}\Omega$ is relatively compact.
The proof is similar to Theorem 2.1 of [2], we omitted it. □

Corollary 3.1 *Suppose that* $k > 0$, (3.7) *holds and*

$$\begin{aligned}
P(t) = {}& \exp\left(-k \int_{t-\tau}^{t} \sum_{i=1}^{n} p_i(t)\,dt\right) + \exp\left(k \int_{t-\zeta}^{t-\tau} \sum_{i=1}^{n} p_i(s)\,ds\right) \int_t^\infty \frac{1}{r(s)} \int_s^\infty \sum_{i=1}^{n} p_i(\xi) \\
& \times f\left(\exp\left(-k \int_{t_0-\zeta}^{\xi+\sigma} \sum_{i=1}^{n} p_i(z)\,dz\right)\right) d\xi\,ds, \quad t \ge t_0.
\end{aligned}$$

Then Eq. (3.6) *has a solution*

$$y(t) = \exp\left(-k \int_{t_0}^{t} \sum_{i=1}^{n} p_i(t)\,dt\right), \quad t \ge t_0.$$

Example 3.1 Consider the advanced differential equations

$$\big(x'(t)\big)' + \sum_{i=1}^{n} p_i(t)x(2t) = 0, \quad t \geq 2, \tag{3.9}$$

where $p_i \in C([t_0,\infty),R)$ and $\sum_{i=1}^{n} |p_i(t)| = \frac{1}{8\sqrt{2}t^2}$. Choose $\mu(t) = \frac{1}{2t}$,

$$\int_t^\infty \left(\frac{1}{4s^2} + \frac{1}{8\sqrt{2}s^2}\exp\left(\frac{1}{2}\int_s^{2s}\frac{1}{\xi}\,d\xi\right)\right)ds = \int_t^\infty \frac{3}{8s^2}\,ds \leq \frac{3}{8t} \leq \frac{1}{2t}.$$

All the conditions of Theorem 3.2 are satisfied. Equation (3.9) has a positive solution and $|\frac{x'(t)}{x(t)}| \leq \frac{1}{2t}$. In fact, we can choose $\mu(t) = 1/(\eta t)$, $\eta \in (4 - 2\sqrt{2}, 4 + 2\sqrt{2})$, Eq. (3.9) has a positive solution with $|\frac{x'(t)}{x(t)}| \leq \mu(t)$, and the solution satisfies $x(2)\cdot 2^{1/\eta}\cdot t^{-1/\eta} \leq x(t) \leq x(2)\cdot 2^{-1/\eta}\cdot t^{1/\eta}$.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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